Two-Queue Systems where Customers of Each Queue Are the Severs of the Other Queue

Efrat Perel[1] and Uri Yechiali[1,2]

[1]Department of Statistics and Operations Research
School of Mathematical Sciences
Tel Aviv University, Tel Aviv 69978, Israel
(naamatie@post.tau.ac.il) (uriy@post.tau.ac.il)

[2]Afeka Academic College of Engineering
Afeka, Tel Aviv 69107, Israel
(uriy@afeka.ac.il)

Abstract

Multi-queue systems where customers also act as servers appear in various real systems such as SETI@home project or file sharing programs. However, only recently a few analytical studies that tackle such models have appeared in the literature (see Perel and Yechiali [5], Senfeld [6]). In those works, two-queue systems are considered, where only the customers of one queue serve the customers of the other queue. In this paper we extend the scope of analysis of such systems by considering models with interlacing queues where customers in both queues act as servers. Namely, customers of each queue are the servers of the other queue. Denoting by L_i the number of customers in queue i, $i=1, 2$, we study two models: In Model 1, queue 1 (Q_1) operates as a multi-server limited-buffer $M/M/\bullet/\bullet$ type queue with arrival and service rate λ_1 and μ_1 respectively, where the L_2 customers present at queue 2 are the potential servers of Q_1. The overall capacity of Q_1 is N customers, such that at any moment the number of active servers at Q_1 is $Min(N, L_2)$. We denote such a queue as $M(\lambda_1)/M(\mu_1)/Min(N, L_2)/N$. At the same time Queue 2 (Q_2) operates as a single-server $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ queue, where its customers are served by the L_1 customers present at the other queue, Q_1. Those L_1 customers join hands together to form a single server with exponentially distributed service time of overall rate $\mu_2 L_1$. In Model 2, Q_1 operates as in Model 1, but Q_2 operates as a multi-server $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system. We present a probabilistic analysis of such systems, applying both Matrix Geometric analysis and Probability Generating Functions (PGFs), derive the stability condition for each model, along with its stationary distribution function. We discover a relationship between the roots of a given matrix, related to the PGFs, and the stability condition of the systems. In addition, we calculate the mean of L_i, i = 1, 2, along with their correlation coefficient, and compare between the models. Numerical examples are presented.

1 Introduction

Consider a system comprised of two connected and dependent queues, where customers of each queue render service to the customers of the other queue as follows: one queue,

Q_1, operates as a multi-server, limited-buffer, $M(\lambda_1)/M(\mu_1)/Min(N, L_2)/N$ system with Poisson arrival rate λ_1, and exponential service time with mean $1/\mu_1$ for each individual customer. The servers at Q_1 are the L_2 customers present in Q_2. That is, each customer present in Q_2 *individually* acts as a server for the customers in Q_1. Q_1 has a limited overall capacity of size N. The other queue, Q_2, is an $M/M/\bullet/\infty$ system and we consider two schemes of service for Q_2:

In Model 1 (Section 2) we assume that Q_2 operates as a single-server infinite-buffer $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ type system with Poisson arrival rate λ_2, but with dynamically changing service rate, $\mu_2 L_1$. That is, the L_1 customers present in Q_1 join hands together and form *a single* server, having a combined service rate of $\mu_2 L_1$, for the customers in Q_2. In other words, the service rate at Q_2 changes according to the queue-size changes of Q_1. We formulate this model as a two-dimensional continuous-time Markov chain and study its steady-state behavior. We use both Matrix Geometric approach, as well as Probability Generating Functions (PGF) method to analyze those systems. We show that the stability condition for Q_2 is $\lambda_2 < \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$, where $E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$ is the mean queue size in Erlang's loss system (see Cooper [2]) and discover a relationship between the roots of a given matrix, related to the PGFs, and this stability condition. Given the generating functions, we calculate numerically the mean total number of customers in Q_2. We further calculate $Cov(L_1, L_2)$, showing that it is non positive, and use this fact to establish an analytic lower bound for $E[L_2]$.

In Model 2 (Section 3) we assume that Q_2 operates as in Model 1, but Q_2 operates as a multi-server, rather than a single-server, unlimited-buffer $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system. That is, each customer present in Q_1 *individually* acts as a server for the customers in Q_2. We show that the stability condition in this case is the <u>same</u> as in Model 1, namely, $\lambda_2 < \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$, and derive the PGFs of the steady-state probabilities of the system state. The mean total number of customers in Q_2, as well as $Cov(L_1, L_2)$ and a lower bound on $E[L_2]$ are also calculated.

Scenarios in which customers in a queue render service elsewhere while waiting for their own service to start or to be completed are quite natural in networks comprised of nodes that can receive and provide service at the same time. An example related to computer networks is presented in Arazi, Ben-Jacob and Yechiali [1]. Another application arises in the field of "file sharing": Once a user activates a file sharing program, he/she operates simultaneously as a server for the other connected users and as a customer searching for a file. An additional application is the SETI@home project. This project, initiated by the Space Sciences Laboratory at the University of California, Berkeley, aims at searching for extraterrestrial intelligence, using radio telescopes. The process of analyzing the vast amount of collected data is assisted by volunteers, who install on their PCs a designated screen saver. Whenever such a PC is idle (for example, when it waits for a user's input), the screen saver is activated and SETI data is processed.

A first step in the analysis of queues where customers act as servers has been only recently presented in [5], where one queue, Q_1, operates as an $M(\lambda_1)/M(\mu_1)/1/N$ system and only the customers of queue 1 act as servers for Q_2. The present work extends the scope of analysis to the case where customers of both queues act as servers, each group serving the other queue.

2 Model 1

Consider two connected queueing systems operating as follows:

One queue, Q_1, operates as an $M(\lambda_1)/M(\mu_1)/Min(N,\ L_2)/N$ system with a Poisson arrival rate λ_1 and service time, for each individual customer, exponentially distributed with parameter μ_1, where L_2 is the number of customers in Q_2. Q_1 has a maximum capacity of size N. The other queue, Q_2, is an $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ system with a Poisson arrival rate λ_2 and service time exponentially distributed with parameter $\mu_2 L_1$, where L_1 is the number of customers in Q_1. That is, the customers present in Q_1 join hands together and form a single server that serves the customers of Q_2, where the service rate at any moment depends on the actual queue length L_1 and equals $\mu_2 L_1$. The arrival processes, as well as the service processes in both queues are mutually independent.

2.1 Balance Equations

Let L_j denote the total number of customers in Q_j, $j=1,\ 2$. Then, the pair (L_1, L_2) defines a non reducible continuous-time Markov process with transition rate diagram as shown in figure 2.1. Let $P_{nm} = P\left(L_1 = n,\ L_2 = m\right)$, $0 \le n \le N$ and $m = 0,\ 1,\ 2, \ldots$ denote the system's stationary probabilities. Then, the set of balance equations is given as follows:

$\underline{n=0}$:

$$\begin{cases} m=1: & (\lambda_1+\lambda_2)P_{01} = \mu_1 P_{11} \\ m\ge 2: & (\lambda_1+\lambda_2)P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m} \end{cases} \tag{2.1}$$

$\underline{n=1}$:

$$\begin{cases} m=0: & (\lambda_1+\lambda_2)P_{10} = \mu_2 P_{11} \\ m=1: & (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{11} = \lambda_1 P_{01} + \lambda_2 P_{10} + \mu_1 P_{21} + \mu_2 P_{12} \\ m\ge 2: & (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + 2\mu_1 P_{2m} + \mu_2 P_{1,m+1} \end{cases} \tag{2.2}$$

$\underline{2\le n\le N-1}$:

$$\begin{cases} m=0: & (\lambda_1+\lambda_2)P_{n0} = \lambda_1 P_{n-1,0} + n\mu_2 P_{n1} \\ 1\le m\le n: & (\lambda_1+\lambda_2+m\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \\ n+1\le m: & (\lambda_1+\lambda_2+n\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \end{cases} \tag{2.3}$$

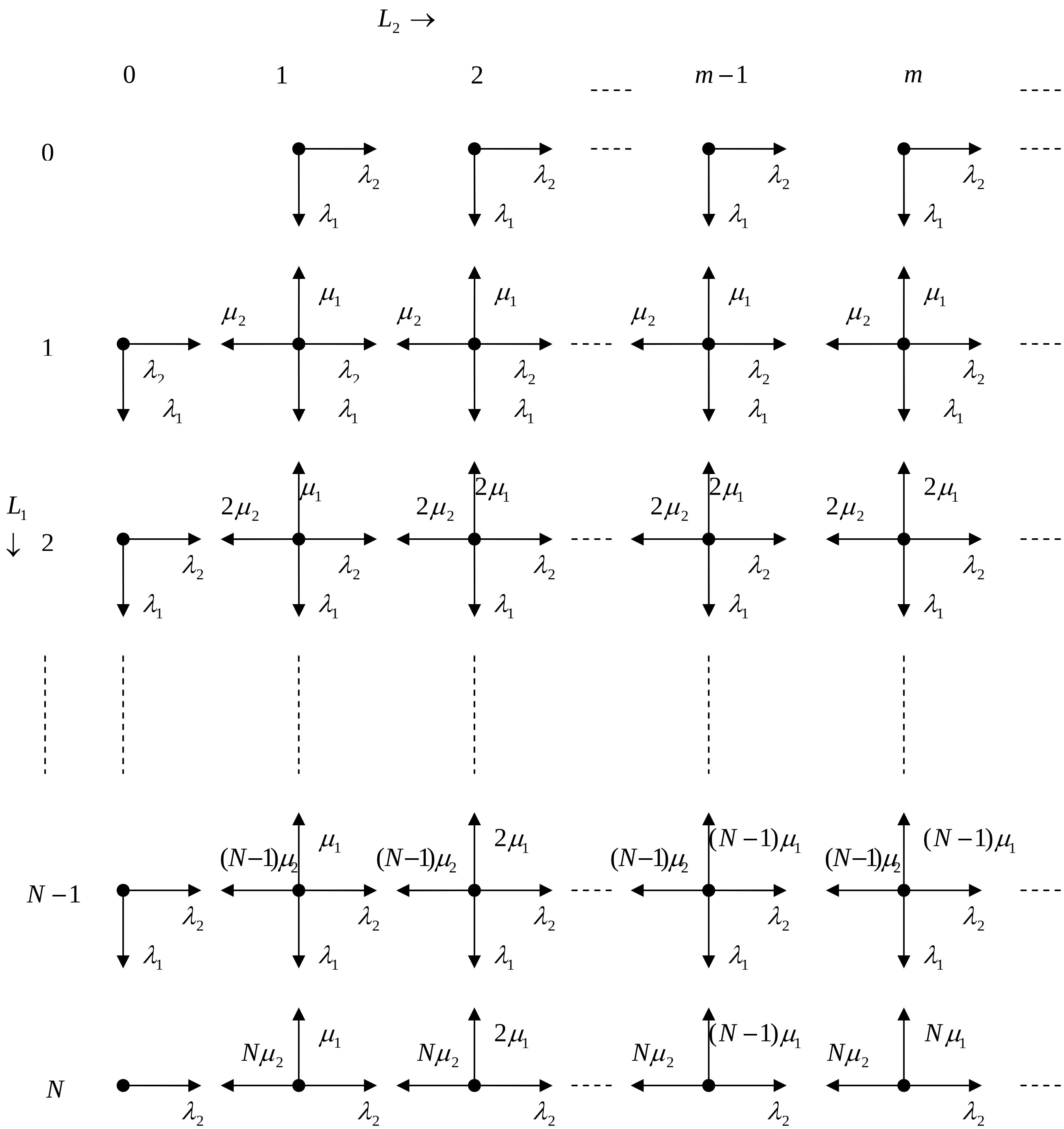

Figure 2.1: Transition-rate diagram of (L_1, L_2) for Model 1.

$\underline{n=N}$:

$$\begin{cases} m=0: \quad \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{N1} \\ 1 \le m \le N: \quad (\lambda_2 + m\mu_1 + N\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \\ N+1 \le m: (\lambda_2 + N\mu_1 + N\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \end{cases} \tag{2.4}$$

Define (where $P_{00} = 0$):

$$P_{n\bullet} = \sum_{m=0}^{\infty} P_{nm} \quad \text{for} \quad 0 \le n \le N$$

$$P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \quad \text{for} \quad m = 0,1,2,\ldots$$

Then, for every $m = 0,\ 1,\ 2,\ldots$, summing equations (2.1) - (2.4) over n yields

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} E[L_1 \big| L_2 = m+1] \tag{2.5}$$

By summing (2.5) over m we get

$$\lambda_2 \sum_{m=0}^{\infty} P_{\bullet m} = \mu_2 \sum_{m=0}^{\infty} P_{\bullet m+1} E[L_1 \mid L_2 = m+1] \tag{2.6}$$

Therefore, $\lambda_2 = \mu_2 \left(E[L_1] - P_{\bullet 0} E[L_1 \mid L_2 = 0] \right) = \mu_2 \left(E[L_1] - \sum_{n=1}^{N} n P_{n0} \right)$.

That is,

$$E[L_1] = \lambda_2 / \mu_2 + \sum_{n=1}^{N} n P_{n0} \tag{2.7}$$

Furthermore, by summing equations (2.1) - (2.4) over m we get, for every $0 \le n \le N-1$,

$$\lambda_1 P_{n\bullet} = (n+1)\mu_1 P_{n+1,\bullet} - \mu_1 \sum_{m=0}^{n} (n+1-m) P_{n+1,m} \tag{2.8}$$

Summing equation (2.8) over *n* yields

$$\sum_{n=0}^{N-1} \lambda_1 P_{n\bullet} = \mu_1 \sum_{n=0}^{N-1} (n+1) P_{n+1,\bullet} - \mu_1 \sum_{n=0}^{N-1} \sum_{m=0}^{n} (n+1-m) P_{n+1,m}$$

$$\lambda_1 (1 - P_{N\bullet}) = \mu_1 E[L_1] - \mu_1 \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m}$$

or,

$$E[L_1] = (1 - P_{N\bullet})\, \lambda_1 / \mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \tag{2.9}$$

By substituting equation (2.7) in equation (2.9) we get

$$E[L_1] = (1 - P_{N\bullet})\, \lambda_1 / \mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} = \lambda_2 / \mu_2 + \sum_{n=1}^{N} n P_{n0} \tag{2.10}$$

Therefore, the probability of blocking at Q_1 is given by

$$P_{N\bullet} = 1 + \frac{\sum_{n=2}^{N}\sum_{m=1}^{n-1}(n-m)P_{n,m} - \lambda_2/\mu_2}{\lambda_1/\mu_1} \tag{2.11}$$

In order to fully solve for the probabilities $\{P_{nm}\}$ we employ in the next subsection probability generating functions.

2.2 Generating Functions

Define, for each $0 \le n \le N$, the probability generating function, $G_n(z) = \sum_{m=0}^{\infty} P_{nm} z^m$.

Multiplying by z^m each equation for m in the sets (2.1) - (2.4), summing over m and rearranging terms we get

$\underline{n = 0:}$

$$\left(\lambda_1 + \lambda_2(1-z)\right)G_0(z) = \mu_1 G_1(z) - \mu_1 P_{10} \tag{2.12}$$

$\underline{1 \le n \le N-1:}$

$$\begin{aligned}\left(\left(\lambda_1 + n\mu_1\right)z + \left(\lambda_2 z - n\mu_2\right)(1-z)\right)G_n(z) &= \lambda_1 z G_{n-1}(z) + (n+1)\mu_1 z G_{n+1}(z) - n\mu_2 P_{n0}(1-z) \\ &\quad + \mu_1 z\left(\sum_{m=0}^{n-1}(n-m)z^m P_{nm} - \sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}\right)\end{aligned} \tag{2.13}$$

$\underline{n = N:}$

$$\left(N\mu_1 z + \left(\lambda_2 z - N\mu_2\right)(1-z)\right)G_N(z) = \lambda_1 z G_{N-1}(z) - N\mu_2 P_{N0}(1-z) + \mu_1 z\sum_{m=0}^{N-1}(N-m)z^m P_{Nm} \tag{2.14}$$

The sets (2.12), (2.13) and (2.14) comprise a system of linear equations of the form

$$A(z)\vec{G}(z) = \vec{P}(z)\,,$$

where, the vectors $\vec{G}(z)$ and $\vec{P}(z)$ and the matrix $A(z)$ are defined as follows:

$$\vec{G}(z) = \left(G_0(z), G_1(z), \ldots, G_N(z)\right)^t$$

$$\vec{P}(z) = \left(P_0(z), P_1(z), \ldots, P_N(z)\right)^t$$

with

$$P_n(z)=\begin{cases}-\mu_1 P_{10}, & n=0\\ \mu_1 z\left(\sum_{m=0}^{n-1}(n-m)z^m P_{nm}-\sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}\right)-n\mu_2 P_{n0}(1-z), & 1\le n\le N-1\\ \mu_1 z\sum_{m=0}^{N-1}(N-m)z^m P_{Nm}-N\mu_2 P_{N0}(1-z), & n=N\end{cases}$$

$$A_{(N+1)\times(N+1)}(z)=\begin{pmatrix}\alpha_0^{(N)}(z) & -\mu_1 & 0 & \cdots & \cdots & 0\\ -\lambda_1 z & \alpha_1^{(N)}(z) & -2\mu_1 z & 0 & \cdots & 0\\ 0 & \ddots & \alpha_2^{(N)}(z) & \ddots & 0 & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & -N\mu_1 z\\ 0 & \cdots & \cdots & 0 & -\lambda_1 z & \alpha_N^{(N)}(z)\end{pmatrix}$$

where

$$\alpha_0^{(N)}(z)=\lambda_1+\lambda_2(1-z)$$

$$\underline{1\le n\le N-1:}\quad \alpha_n^{(N)}(\mathrm{z})=(\lambda_1+n\mu_1)\mathrm{z}+(\lambda_2\mathrm{z}-n\mu_2)(1-z)$$

$$\alpha_N^{(N)}(z)=N\mu_1 z+(\lambda_2 z-N\mu_2)(1-z)\ .$$

To obtain $G_n(z)$ we use Cramer's rule. I.e., for every $0\le n\le N$,

$$G_n(z)=\frac{|A_n(z)|}{|A(z)|},$$

where $|A|$ is the determinant of the matrix A and $A_n(z)$ is a matrix obtained from $A(z)$ by replacing its n-th column by $\vec{P}(z)$. This leads to an expression of $G_n(z)$ in terms of the $N(N+1)/2$ unknown probabilities, $P_{10};\ P_{20},\ P_{21};\ \ldots;\ P_{N0},\ P_{N1},\ \ldots,\ P_{N,N-1}$, appearing in $\vec{P}(z)$. In order to find $\vec{P}(z)$ we need to find $N(N+1)/2$ equations relating those $N(N+1)/2$ variables. We do that in the next section by characterizing and using the roots of $|A(z)|$. Since $G_n(z)$ is a probability generating function defined for all $0\le z\le 1$, each root of $|A(z)|$ in that interval is a root of $|A_n(z)|$, for every $0\le n\le N$.

2.3 Derivation of $P_{10};\ P_{20},\ P_{21};\ \ldots;\ P_{N0},\ P_{N1},\ \ldots,\ P_{N,N-1}$ and $E[L_2]$

Theorem 2.1 For any $\lambda_1>0$, $\mu_1,\lambda_2\ge 0$, $\mu_2>0$ and $N\ge 1$, $|A(z)|$ is a polynomial of degree $2N+1$ possessing $N-1$ distinct roots in the open interval $(0,1)$, a single root at $z=1$, and N roots in the open interval $(1,\infty)$. Another root exists in the open interval $(0,1)$ if the condition $\lambda_2>\mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$ holds.

Proof. Let $q_0(z)=1$. Define the minors of the diagonal of $A(z)$**,** starting from the higher left side corner, as follows

$$q_1^{(N)} = \alpha_0^{(N)}(z),\ q_2^{(N)}(z) = \begin{vmatrix} \alpha_0^{(N)}(z) & -\mu_1 \\ -\lambda_1 z & \alpha_1^{(N)}(z) \end{vmatrix},\ \ldots,\ q_{N+1}^{(N)}(z) = |A(z)| \tag{2.15}$$

The polynomials $q_n^{(N)}(z),\ 1 \le n \le N+1$ satisfy the following equations

$$\begin{aligned} q_1^{(N)}(z) &= \alpha_0^{(N)}(z) q_0^{(N)}(z) \\ q_2^{(N)}(z) &= \alpha_1^{(N)}(z) q_1^{(N)}(z) - \lambda_1\mu_1 z q_0^{(N)}(z) \\ q_n^{(N)}(z) &= \alpha_{n-1}^{(N)}(z) q_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 q_{n-2}^{(N)}(z) \quad \text{for} \quad 3 \le n \le N+1 \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we conclude that

1. $q_0^{(N)}(z) = 1$ and therefore has no roots.
2. $q_n^{(N)}(z)$ and $q_{n+1}^{(N)}(z)$ have no joint roots in $(0,\infty)$. Otherwise, suppose they have a joint root, then it would also be a root for $q_{n-1}^{(N)}(z),\ q_{n-2}^{(N)}(z),\ \ldots,\ q_0^{(N)}(z)$ which contradicts 1.
3. $\mathrm{Sign}\left(q_n^{(N)}(0)\right) = (-1)^{n+1}$ for $1 \le n \le N+1$.
4. $\mathrm{Sign}\left(q_n^{(N)}(\infty)\right) = (-1)^n$ for all *n*.
5. $q_n^{(N)}(1) = \lambda_1^{\ n}$ for $0 \le n \le N$ and $q_{N+1}(1) = 0$.
6. $\mathrm{Sign}\left(\alpha_n^{(N)}(0)\right) = -1$ for $1 \le n \le N$.
7. Given $\tilde{z} > 0$ a root of $q_n^{(N)}(z)$ then $\mathrm{sign}\left(q_{n-1}^{(N)}(\tilde{z}) q_{n+1}^{(N)}(\tilde{z})\right) = -1$.
8. $q_n^{(N)}(z)$ is a polynomial of degree $2n-1$ for $1 \le n \le N+1$.
9. For $n \le N$ the polynomial $q_n^{(N)}(z)$ has $2n-1$ distinct roots, where $n-1$ of them are in the open interval $(0,1)$ and the other *n* are in the open interval $(1,\infty)$.

From the above we conclude that $q_1^{(N)}(z)$ has only one root, $z_{1,1} = 1 + \lambda_1/\lambda_2 > 1$, $q_2^{(N)}(0) < 0$, $q_2^{(N)}(1) = \lambda_1^{\ 2} > 0,\ q_2^{(N)}(z_{1,1}) < 0,\ q_2^{(N)}(\infty) > 0$. Therefore, the 3 roots of $q_2^{(N)}(z)$ satisfy: $z_{2,1} \in (0,1)$, $z_{2,2} \in (1, z_{1,1})$, $z_{2,3} \in (z_{1,1}, \infty)$. Similarly, $q_3^{(N)}(z)$ is of degree 5 and therefore can have no more than 5 roots. Also $q_3^{(N)}(0) > 0$, $q_3^{(N)}(z_{2,1}) < 0$, $q_3^{(N)}(1) = \lambda_1^{\ 3} > 0$, $q_3^{(N)}(z_{2,2}) < 0$, $q_3^{(N)}(z_{2,3}) < 0$, $q_3^{(N)}(\infty) < 0$. This implies that $q_3^{(N)}(z)$ has exactly 5 distinct roots satisfying: $z_{3,1} \in (0, z_{2,1})$, $z_{3,2} \in (z_{2.1}, 1)$, $z_{3,3} \in (1, z_{2,2})$, $z_{3,4} \in (z_{2,2}, z_{2,3})$, $z_{3,5} \in (z_{2,3}, \infty)$.

In general, for $2 \le n \le N$, given $2n-3$ distinct roots of $q_{n-1}^{(N)}(z)$, the roots of $q_n^{(N)}(z)$ satisfy: $z_{n,1} \in (0\ ,\ z_{n-1,1})$, $z_{n,2} \in (z_{n-1,1}\ ,\ z_{n-1,2})$, $\ldots$, $z_{n,n-1} \in (z_{n-1,n-2}\ ,\ 1)$, $z_{n,n} \in (1\ ,\ z_{n-1,n-1})$, $\ldots$, $z_{n,2n-1} \in (z_{n-1,2n-3}\ ,\infty)$.

$q_{N+1}^{(N)}(z)$ has $2N+1$ roots where the first $N-1$ are within the interval $(0,1)$ satisfying: $z_{N+1,1} \in (0, z_{N,1})$, $z_{N+1,2} \in (z_{N,1}, z_{N,2})$, $\ldots$, $z_{N+1,N-1} \in (z_{N,N-2}, z_{N,N-1})$.

As for the N-th root, $z_{N+1,N}$, we observe that, since $z_{N,N-1} \in (z_{N-1,N-2}\ , z_{N-1,N-1})$, where $z_{N-1,N-1} < 1$, we have that $q_{N-1}(z_{N,N-1}) > 0$ and therefore:

$$q_{N+1}^{(N)}(z_{N,N-1}) = -\lambda_1\mu_1(z_{N,N-1})^2 q_{N-1}^{(N)}(z_{N,N-1}) < 0 \,.$$

$q_{N+1}^{(N)}(1)=0$, and we need to check whether another root (besides the $N-1$ already mentioned) exists in $(z_{N,N-1},1)$.

We will show that under a stationary condition, such a root does not exist. In such a case, the $N-1$ distinct roots of $q_{N+1}(z)$ in $(0,1)$ will provide $N-1$ equations relating the $N(N+1)/2$ unknown probabilities.

By induction over n we obtain (see Appendix 1)

$$q_n^{(N)}(z) = \lambda_1{}^n z^{n-1} + (1-z)h_n^{(N)}(z) \quad , \quad 1 \le n \le N \tag{2.17}$$

$$q_{N+1}^{(N)}(z) = (1-z)h_{N+1}^{(N)}(z) \tag{2.18}$$

Another root exists in $(z_{N,N-1}\,,1)$ if and only if $h_{N+1}^{(N)}(1) > 0$.

Substituting (2.17) in the third part of (2.16) yields the following (see Appendix 1):

$$h_1^{(N)}(z) = \lambda_2$$

$$h_2^{(N)}(z) = (\lambda_2 z - \mu_2)(\lambda_1 + \lambda_2(1-z)) + \lambda_2 z(\lambda_1+\mu_1)$$

$$h_n^{(N)}(z) = \lambda_1{}^{n-1} z^{n-2}(\lambda_2 z - (n-1)\mu_2) + \alpha_{n-1}^{(N)}(z)h_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 h_{n-2}^{(N)}(z) \,, \qquad 3 \le n \le N$$

$$h_{N+1}^{(N)}(z) = \lambda_1{}^N z^{N-1}(\lambda_2 z - N\mu_2) + \alpha_N^{(N)}(z)h_N^{(N)}(z) - N\lambda_1\mu_1 z^2 h_{N-1}^{(N)}(z)$$

Substituting $z=1$ in the above gives

$$h_1^{(N)}(1) = \lambda_2$$

$$h_2^{(N)}(1) = (\lambda_2 - \mu_2)\lambda_1 + \lambda_2(\lambda_1+\mu_1)$$

$$h_n^{(N)}(1) = \lambda_1{}^{n-1}(\lambda_2 - (n-1)\mu_2) + (\lambda_1 + (n-1)\mu_1)h_{n-1}^{(N)}(1) - (n-1)\lambda_1\mu_1 h_{n-2}^{(N)}(1) \;,\; 3 \le n \le N \tag{2.19}$$

$$h_{N+1}^{(N)}(1) = \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1 h_N^{(N)}(1) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) \,. \tag{2.20}$$

Therefore (see Appendix 2), for every $N \ge 2$,

$$h_{N+1}^{(N)}(1) = \lambda_2 N!\mu_1{}^N \sum_{n=0}^{N}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2 N!\mu_1{}^N \sum_{n=1}^{N} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} \,.$$

Since another root for $|A(z)|$ exists if and only if $h_{N+1}^{(N)}(1) > 0$,

$$h_{N+1}^{(N)}(1) = \lambda_2 N!\mu_1{}^N \sum_{n=0}^{N}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2 N!\mu_1{}^N \sum_{n=1}^{N} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} > 0 \,.$$

This implies that another root exists if and only if

$$\frac{\lambda_2}{\mu_2} > \frac{\sum_{n=1}^{N} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}}{\sum_{n=0}^{N}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}} = E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$$

This completes the proof of Theorem 2.1. □

The condition described in Theorem 1 regarding the existence of the N-th root in the open interval (0,1), contradicts the condition for the system's stability, given by (2.21).

To find the $N(N+1)/2$ unknown probabilities appearing in $\vec{P}(z)$, when (2.21) holds, we use the N-1 distinct roots in the open interval (0,1), which provide us with N-1 equations for those probabilities. Another $N(N-1)/2$ equations are taken from the balance equations for states (n,m), $2 \le n \le N$, $0 \le m \le n-2$. Together with equation (2.11), and since $P_{N\bullet} = G_N(1)$, we have a linear set of $N(N+1)/2$ distinct equations in the $N(N+1)/2$ unknown probabilities.

The mean total number of customers in Q_2, $E[L_2]$, is obtained by summing the derivatives of $G_n(z)$ over n at $z=1$. That is,

$$E[L_2] = \sum_{n=0}^{N} G_n'(1) = \sum_{n=0}^{N} E[L_2 \mid L_1 = n] P\left(L_1 = n\right).$$

Also, by multiplying equation (2.12) by z, summing it with (2.13) over n, and adding (2.14) we get

$$\lambda_1 z \sum_{n=0}^{N-1} G_n(z) + \mu_1 z \sum_{n=1}^{N} n G_n(z) + (1-z) \sum_{n=0}^{N} (\lambda_2 z - n\mu_2) G_n(z)$$
$$= \lambda_1 z \sum_{n=1}^{N} G_{n-1}(z) + \mu_1 z \sum_{n=0}^{N-1} (n+1) G_{n+1}(z) - (1-z)\mu_2 \sum_{n=0}^{N} n P_{n0}$$
$$+ \mu_1 \left(\sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) z^m P_{nm} - \sum_{n=0}^{N-1} \sum_{m=0}^{n} (n+1-m) z^m P_{n+1,m} \right),$$

implying that

$$\sum_{n=0}^{N} (\lambda_2 z - n\mu_2) G_n(z) = -\mu_2 \sum_{n=0}^{N} n P_{n0}$$

Differentiating both sides of the above and setting $z=1$ yields

$$\sum_{n=0}^{N} \lambda_2 G_n(1) + \sum_{n=0}^{N} (\lambda_2 - n\mu_2) G_n'(1) = \lambda_2 + \lambda_2 E[L_2] - \mu_2 E[L_1 \cdot L_2] = 0.$$

Therefore,

$$E[L_1 \cdot L_2] = \frac{\lambda_2 + \lambda_2 E[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2} + \frac{\lambda_2 E[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2}\left(1 + E[L_2]\right).$$

Thus,

$$Cov\left(L_1, L_2\right) = E[L_1 \cdot L_2] - E[L_1] \cdot E[L_2] = \frac{\lambda_2}{\mu_2} + \frac{\lambda_2}{\mu_2} E[L_2] - E[L_1] \cdot E[L_2]$$
$$= \frac{\lambda_2}{\mu_2} - E[L_2]\left(E[L_1] - \frac{\lambda_2}{\mu_2} \right).$$

We argue that $Cov\left(L_1, L_2\right) \le 0$, since increasing values of L_1 reduce the magnitude of L_2. Hence, we can derive a closed-form lower bound for $E[L_2]$:

$$E[L_2] \ge \frac{\frac{\lambda_2}{\mu_2}}{E[L_1] - \frac{\lambda_2}{\mu_2}} = \frac{\lambda_2}{\mu_2 E[L_1] - \lambda_2}$$

Clearly, by Little's Law, $E[W_2]=\dfrac{E[L_2]}{\lambda_2}\geq\dfrac{1}{\mu_2 E[L_1]-\lambda_2}$.

2.4 Matrix Geometric Method for Deriving $(P_{nm})_{0\leq n\leq N\,,\,0\leq m}$

We now use Matrix Geometric approach for deriving $(P_{nm})_{0\leq n\leq N\,,\,0\leq m}$ and for further analysis of the system. Our queueing system can be described as having $N+1$ 'phases', where phase n indicates that the service rate in Q_2 is $n\mu_2$. State (m,n) denotes that there are m jobs in Q_2, $0\leq m$, and the system is in phase n, $0\leq n\leq N$.

We construct a quasi birth-and-death process (Nuets [4], Latouche and Ramaswami [3]) with generator Q, given by

$$Q=\begin{pmatrix} A_1^0 & A_0^0 & 0 & \dots & & & & \dots \\ A_2^1 & A_1^1 & A_0 & 0 & \dots & & & \\ 0 & A_2 & A_1^2 & A_0 & 0 & \dots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \vdots & \ddots & A_2 & A_1^N & A_0 & 0 & \\ & & \vdots & \ddots & A_2 & A_1^N & A_0 & \ddots \\ & & & \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where, starting from the upper diagonal, $A_0^0, A_0; A_1^0, A_1^1,\dots,A_1^N; A_2^1, A_2^2,\dots,A_2^N$ are the following matrices: A_0^0 is of size $N\times(N+1)$; A_0 is of size $(N+1)\times(N+1)$; A_1^0 is of size $N\times N$; $A_1^1,\dots,A_1^N$ are each of size $(N+1)\times(N+1)$; A_2^1 is of size $(N+1)\times N$ and A_2 is of size $(N+1)\times(N+1)$. They are given by

$$A_0^0=\begin{pmatrix} 0 & \lambda_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \lambda_2 \end{pmatrix}$$

$$A_0=diag\left(\lambda_2\right)$$

$$A_1^0=\begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

For all $1 \le n \le N$

$$\left(A_1^n\right)_{ij} = \begin{cases} -(\lambda_1+\lambda_2) & j=i=0 \\ -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) & j=i,\ i=1,\ldots,n \\ -(\lambda_1+\lambda_2+n\mu_1+i\mu_2) & j=i,\ i=n+1,\ldots,N-1 \\ -(\lambda_2+n\mu_1+N\mu_2) & j=i=N \\ \lambda_1 & j=i+1,\ i=0,1,\ldots,N-1 \\ i\mu_1 & j=i-1,\ i=1,\ldots,n \\ n\mu_1 & j=i-1,\ i=n+1,\ldots,N \\ 0 & otherwise \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \ldots & \ldots & \vdots \\ 0 & 2\mu_2 & 0 & \ldots & \vdots \\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & N\mu_2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & \mu_2 & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & 2\mu_2 & \ddots & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & N\mu_2 \end{pmatrix}$$

Let $A_1 = A_1^N$, then the matrix $\tilde{Q} = A_0 + A_1 + A_2$ is the infinitesimal generator of Erlang's classical loss system $M(\lambda_1)/M(\mu_1)/N/N$ (see [2]). Let $\vec{\pi} = \left(\pi_0, \pi_1, \ldots, \pi_N\right)$ be the stationary probability vector of the matrix $\tilde{Q}$, i.e. $\vec{\pi}\tilde{Q} = \vec{0}$ and $\vec{\pi} \cdot \vec{e} = 1$. Then, $\pi_n = \dfrac{\frac{1}{n!}\left(\frac{\lambda_1}{\mu_1}\right)^n}{\sum_{k=0}^{N} \frac{1}{k!}\left(\frac{\lambda_1}{\mu_1}\right)^k}$ for all $0 \le n \le N$. Substituting $\vec{\pi}$ in the stability condition $\vec{\pi} A_2 \vec{e} > \vec{\pi} A_0 \vec{e}$ (see [4, p. 83]), we arrive at

$$\vec{\pi} A_0 \vec{e} = \lambda_2 \sum_{n=0}^{N} \pi_n = \lambda_2 < \vec{\pi} A_2 \vec{e} = \mu_2 \sum_{n=0}^{N} n\pi_n = \mu_2 \frac{\sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{(n-1)!}}{\sum_{k=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^k \frac{1}{k!}} = \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$$

That is, the stability condition is $\lambda_2 / \mu_2 < E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$. (2.21)

Define the steady-state probability vector $\vec{P}_m = \left(P_{0m}, P_{1m}, \ldots, P_{Nm}\right)$ for all $m \ge 0$. Then,

$$\vec{P}_m = \vec{P}_{N-1} R^{m-(N-1)}, \quad m \ge N-1,$$

where R is the minimal non negative solution of the matrix quadratic equation $A_0 + RA_1 + R^2 A_2 = 0$ (see [4, Section 1.9] and [3], where computational procedures for finding the matrix R are discussed).

The vectors $\vec{P}_0, \vec{P}_1, \ldots, \vec{P}_{N-1}$ can be found by solving the following linear system of equations:

$$\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0}$$
$$\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0}$$
$$\vec{P}_{m-1} A_0 + \vec{P}_m A_1^m + \vec{P}_{m+1} A_2 = \vec{0} \quad , \quad 2 \le m \le N-2$$
$$\vec{P}_{N-2} A_0 + \vec{P}_{N-1} \left(A_1^{N-1} + R A_2 \right) = \vec{0}$$
$$\sum_{m=0}^{N-2} \vec{P}_m \vec{e} + \vec{P}_{N-1} \left[I - R \right]^{-1} \vec{e} = 1$$

where I is the identity matrix.

The mean total number of customers in Q_2, $E[L_2]$, is given by

$$E[L_2] = \sum_{m=0}^{\infty} m \vec{P}_m \cdot \vec{e} = \sum_{m=0}^{N-2} m \vec{P}_m \cdot \vec{e} + \sum_{m=N-1}^{\infty} m \vec{P}_{N-1} R^{m-N+1} \cdot \vec{e}$$
$$= \sum_{m=0}^{N-2} m \vec{P}_m \cdot \vec{e} + (N-2) \vec{P}_{N-1} \left[I - R \right]^{-1} \cdot \vec{e} + \vec{P}_{N-1} \left[I - R \right]^{-2} \cdot \vec{e}$$

3 Model 2

In this model we consider two connected queueing systems, similar to Model 1. Q_1 is an $M(\lambda_1)/M(\mu_1)/Min(N, L_2)/N$ system, as in Model 1. Q_2, however, is an $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system. That is, each customer present in Q_1 individually acts as a server for the customers in Q_2.

3.1 Balance Equations

The pair (L_1, L_2) defines a continuous-time Markov process with transition rate diagram as shown in figure 3.1. The set of balance equations is given as follows:

$\underline{n = 0}$:

$$\begin{cases} m = 1: \quad (\lambda_1 + \lambda_2) P_{01} = \mu_1 P_{11} \\ m \ge 2: \quad (\lambda_1 + \lambda_2) P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m} \end{cases} \tag{3.1}$$

$\underline{n = 1}$:

$$\begin{cases} m = 0: \quad (\lambda_1 + \lambda_2) P_{10} = \mu_2 P_{11} \\ m = 1: \quad (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) P_{11} = \lambda_1 P_{0,1} + \lambda_2 P_{1,0} + \mu_1 P_{2,1} + \mu_2 P_{1,2} \\ m \ge 2: \quad (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) P_{1m} = \lambda_1 P_{0,m} + \lambda_2 P_{1,m-1} + 2\mu_1 P_{2,m} + \mu_2 P_{1,m+1} \end{cases} \tag{3.2}$$

$\underline{2 \le n \le N-1}$:

$$\begin{cases} m = 0: \quad (\lambda_1 + \lambda_2) P_{n0} = \lambda_1 P_{n-1,0} + \mu_2 P_{n1} \\ 1 \le m < n: \quad (\lambda_1 + \lambda_2 + m\mu_1 + m\mu_2) P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + (m+1)\mu_2 P_{n,m+1} \\ m = n: \quad (\lambda_1 + \lambda_2 + m\mu_1 + m\mu_2) P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \\ n < m: (\lambda_1 + \lambda_2 + n\mu_1 + n\mu_2) P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \end{cases} \tag{3.3}$$

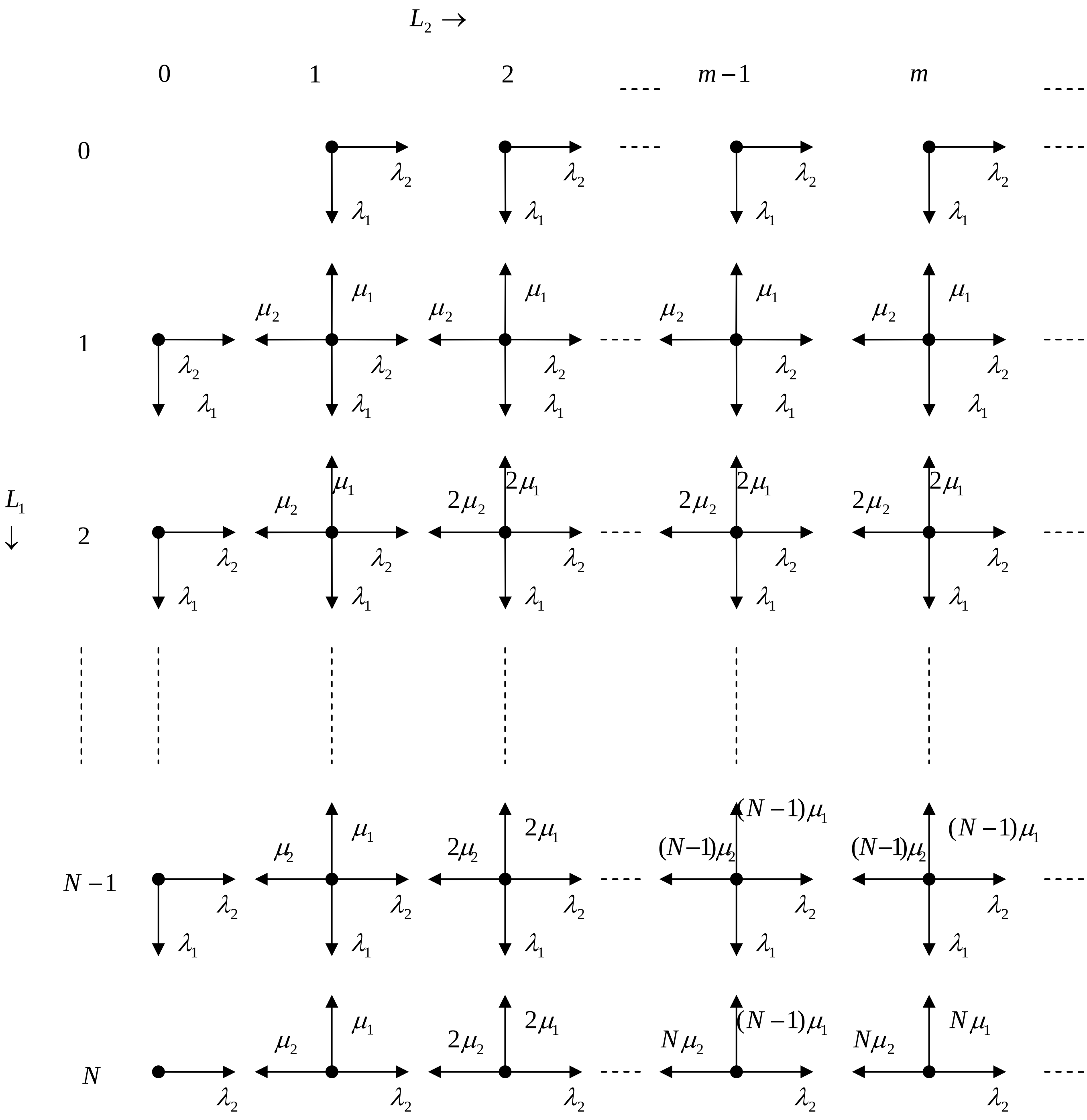

Figure 3.1: Transition-rate diagram of (L_1, L_2) for Model 2

$\underline{n = N}$: (3.4)

$$\begin{cases} m=0: \ \ \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + \mu_2 P_{N1} \\ 1 \le m < N: \ (\lambda_2 + m\mu_1 + m\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + (m+1)\mu_2 P_{N,m+1} \\ N \le m: (\lambda_2 + N\mu_1 + N\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \end{cases}$$

Define

$$P_{n\bullet} = \sum_{m=0}^{\infty} P_{nm} \quad \text{for} \ \ 0 \le n \le N$$

$$P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \quad \text{for} \ \ m = 0,1,2,\ldots$$

Then, for all $m = 0,\ 1,\ 2,\ldots$, summing equations (3.1) - (3.4) over n yields:

$$\underline{0 \le m \le N-1}: \quad \lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} E\left[L_1 \middle| L_2 = m+1\right] - \mu_2 \sum_{n=m+1}^{N} (n-m-1) P_{n,m+1} \tag{3.5}$$

$$\underline{N \le m}: \qquad \lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} E\left[L_1 \middle| L_2 = m+1\right]$$

By summing (3.5) over m we get

$$\lambda_2 = \mu_2 \left(E\left[L_1\right] - \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \right) \tag{3.6}$$

That is,

$$E[L_1] = \lambda_2 / \mu_2 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \tag{3.7}$$

Furthermore, by summing equations (3.1) - (3.4) over m we get, for all $0 \le n \le N-1$,

$$\lambda_1 P_{n\bullet} = (n+1)\mu_1 P_{n+1,\bullet} - \mu_1 \sum_{m=0}^{n} (n+1-m) P_{n+1,m} \tag{3.8}$$

Summing equation (3.8) over n yields

$$\sum_{n=0}^{N-1} \lambda_1 P_{n\bullet} = \mu_1 \sum_{n=0}^{N-1} (n+1) P_{n+1,\bullet} - \mu_1 \sum_{n=}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m}$$

$$\lambda_1 (1 - P_{N\bullet}) = \mu_1 E[L_1] - \mu_1 \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m}$$

That is,

$$E[L_1] = (1 - P_{N\bullet})\, \lambda_1 / \mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \tag{3.9}$$

By substituting equation (3.7) in equation (3.9) we get

$$E[L_1]=(1-P_{N\bullet})\,\lambda_1/\mu_1+\sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m}=\lambda_2/\mu_2+\sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m}\ , \tag{3.10}$$

implying that

$$P_{N\bullet}=1-\frac{\lambda_2/\mu_2}{\lambda_1/\mu_1} \tag{3.11}$$

3.2 Generating Functions

Define, for each $0\le n\le N$, the probability generating functions, $G_n(z)=\sum_{m=0}^{\infty}P_{nm}z^m$.

By multiplying each equation for m in (3.1) - (3.4) by z^m, summing over m and rearranging terms we get

(3.12)

$\underline{n=0}$:

$$\left(\lambda_1+\lambda_2(1-z)\right)G_0(z)=\mu_1G_1(z)-\mu_1P_{10}$$

$\underline{1\le n\le N-1}$:

$$\begin{aligned}\left(\left(\lambda_1+n\mu_1\right)z+\left(\lambda_2z-n\mu_2\right)(1-z)\right)G_n(z)&=\lambda_1zG_{n-1}(z)+(n+1)\mu_1zG_{n+1}(z)\\&+(\mu_1z-\mu_2(1-z))\sum_{m=0}^{n-1}(n-m)P_{nm}z^m\\&-\mu_1z\sum_{m=0}^{n}(n+1-m)z^mP_{n+1,m}\end{aligned} \tag{3.13}$$

$\underline{n=N}$:

$$\left(N\mu_1z+\left(\lambda_2z-N\mu_2\right)(1-z)\right)G_N(z)=\lambda_1zG_{N-1}(z)+(\mu_1z-\mu_2(1-z))\sum_{m=0}^{N-1}(N-m)z^mP_{Nm} \tag{3.14}$$

The sets (3.12) – (3.14) define a system of linear equations having a matrix form

$$A(z)\vec{G}(z)=\vec{P}(z)\ ,$$

where , $A(z)$ and $\vec{G}(z)$ are defined similarly to model 1, but $\vec{P}(z)$ is given by

$$\vec{P}_n(z)=\begin{cases}-\mu_1P_{10}\ , & n=0\\ \left(\mu_1z-\mu_2(1-z)\right)\sum_{m=0}^{n-1}(n-m)z^mP_{nm}-\mu_1z\sum_{m=0}^{n}(n+1-m)z^mP_{n+1,m}\ , & 1\le n\le N-1\\ \left(\mu_1z-\mu_2(1-z)\right)\sum_{m=0}^{N-1}(N-m)z^mP_{Nm}\ , & n=N\end{cases}$$

As before, to obtain $G_n(z)$ we use Cramer's rule. This leads to an expression of $G_n(z)$ in terms of $N(N+1)/2$ unknown probabilities appearing in $\vec{P}(z)$.

3.3 Derivation of $P_{10}, P_{20}, P_{21}, \ldots, P_{N0}, P_{N1}, \ldots, P_{N,N-1}$ and $E[L_2]$

In order to find $P_{10}, P_{20}, P_{21}, \ldots, P_{N0}, P_{N1}, \ldots, P_{N,N-1}$ we need to find $N(N+1)/2$ equations relating those $N(N+1)/2$ variables. We do that by using the roots of $|A(z)|$. From Theorem 2.1, $|A(z)|$ has $N-1$ distinct roots in the open interval (0,1) (for $\lambda_1 > 0$, μ_1, $\lambda_2 \geq 0$, $\mu_2 > 0$ and provided that $\lambda_2 < \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$). Another $N(N-1)/2$ equations are taken from the balance equations for states (n,m) $2 \leq n \leq N$, $0 \leq m \leq n-2$. The last equation we use is (3.11). Thus, we have $N(N+1)/2$ equations relating those $N(N+1)/2$ variables as requested.

The mean total number of customers in Q_2, $E[L_2]$, is obtained by summing the derivatives of $G_n(z)$ over n at $z=1$. That is,

$$E[L_2] = \sum_{n=0}^{N} G_n{}'(1) = \sum_{n=0}^{N} E[L_2 \mid L_1 = n] P(L_1 = n)$$

Also, by multiplying equation (3.12) by z and summing it with (3.13) over n, adding (3.14), and differentiating at $z=1$, yields

$$\sum_{n=0}^{N} \lambda_2 G_n(1) + \sum_{n=0}^{N} (\lambda_2 - n\mu_2) G_n{}'(1) = \lambda_2 + \lambda_2 E[L_2] - \mu_2 E[L_1 \cdot L_2] = -\mu_2 \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm}$$

Therefore,

$$E[L_1 \cdot L_2] = \frac{\lambda_2 + \lambda_2 E[L_2] + \mu_2 \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm}}{\mu_2} = \frac{\lambda_2}{\mu_2} + \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm} + \frac{\lambda_2 E[L_2]}{\mu_2}$$

$$= \frac{\lambda_2}{\mu_2}\left(1 + E[L_2]\right) + \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm}$$

Thus,

$$Cov(L_1, L_2) = E[L_1 \cdot L_2] - E[L_1] \cdot E[L_2]$$

$$= \frac{\lambda_2}{\mu_2} + \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm} + \frac{\lambda_2}{\mu_2} E[L_2] - E[L_1] \cdot E[L_2]$$

$$= \frac{\lambda_2}{\mu_2} + \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm} - E[L_2] \left(E[L_1] - \frac{\lambda_2}{\mu_2} \right)$$

We argue that $Cov(L_1, L_2) \leq 0$, since increasing values of L_1 reduce the magnitude of L_2. Hence, we can derive a lower bound for $E[L_2]$,

$$E[L_2] \geq \frac{\frac{\lambda_2}{\mu_2} + \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm}}{E[L_1] - \frac{\lambda_2}{\mu_2}} = \frac{\lambda_2 + \mu_2 \sum_{n=2}^{N} \sum_{m=1}^{n-1} m(n-m) P_{nm}}{\mu_2 E[L_1] - \lambda_2}$$

Clearly, by Little's Law, $E[W_2] = \dfrac{E[L_2]}{\lambda_2}$.

3.4 Matrix Geometric Method for Deriving $(P_{nm})_{0\le n\le N,\,0\le m}$

Applying the Matrix Geometric approach, the system of balance equations can be described as a queueing system with N+1 phases, where phase n indicates that there are n servers available at Q_1. We construct a quasi birth-and-death process with generator Q, given by

$$Q=\begin{pmatrix} A_1^0 & A_0^0 & 0 & \cdots & & & & \cdots \\ A_2^1 & A_1^1 & A_0 & 0 & \cdots & & & \\ 0 & A_2^2 & A_1^2 & A_0 & 0 & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \vdots & \ddots & A_2^N & A_1^N & A_0 & 0 & \\ & & \vdots & \ddots & A_2^N & A_1^N & A_0 & \ddots \\ & & & \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & & & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Where, $A_0^0, A_0, A_1^0, A_1^1, \ldots, A_1^N, A_2^1, A_2^2, \ldots, A_2^N$ are the following matrices: A_0^0 is of size $N\times(N+1)$; A_0 is of size $(N+1)\times(N+1)$; A_1^0 is of size $N\times N$; $A_1^1,\ldots, A_1^N$ are each of size $(N+1)\times(N+1)$; A_2^1 is of size $(N+1)\times N$, and $A_2^2,\ldots, A_2^N$ are each of size $(N+1)\times(N+1)$. They are given by

$$A_0^0=\begin{pmatrix} 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_2 \end{pmatrix}$$

$$A_0 = diag(\lambda_2)$$

$$A_1^0=\begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

For all $1 \le n \le N$

$$\left(A_1^n\right)_{ij} = \begin{cases} -(\lambda_1+\lambda_2) & j=i=0 \\ -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) & j=i,\ i=1,\dots,n \\ -(\lambda_1+\lambda_2+n\mu_1+n\mu_2) & j=i,\ i=n+1,\dots,N-1 \\ -(\lambda_2+n\mu_1+n\mu_2) & j=i=N \\ \lambda_1 & j=i+1,\ i=0,1,\dots,N-1 \\ i\mu_1 & j=i-1,\ i=1,\dots,n \\ n\mu_1 & j=i-1,\ i=n+1,\dots,N \\ 0 & otherwise \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \dots & \dots & \vdots \\ 0 & \mu_2 & 0 & \dots & \vdots \\ \vdots & \ddots & \mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mu_2 \end{pmatrix}$$

For all $2 \le n \le N$

$$\left(A_2^n\right)_{ij} = \begin{cases} i\mu_2 & j=i=0,1,\dots,n-1 \\ n\mu_2 & j=i=n,\dots,N \\ 0 & otherwise \end{cases}$$

Again, letting $A_1 = A_1^N$, $A_2 = A_2^N$, then the matrix $\tilde{Q} = A_0 + A_1 + A_2$ is the infinitesimal generator of the classical Erlang's loss system $M(\lambda_1)/M(\mu_1)/N/N$. Let $\vec{\pi} = \left(\pi_0, \pi_1, \dots, \pi_N\right)$ be the stationary vector of the matrix $\tilde{Q}$, i.e. $\vec{\pi}\tilde{Q} = \vec{0}$ and $\vec{\pi}\cdot\vec{e} = 1$. Then, $\pi_n = \dfrac{\frac{1}{n!}\left(\frac{\lambda_1}{\mu_1}\right)^n}{\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{\lambda_1}{\mu_1}\right)^k}$ for all $0 \le n \le N$. Substituting $\vec{\pi}$ in the stability condition $\vec{\pi}A_2\vec{e} > \vec{\pi}A_0\vec{e}$, we arrive at

$$\vec{\pi}A_0\vec{e} = \lambda_2\sum_{n=0}^{N}\pi_n = \lambda_2 < \vec{\pi}A_2\vec{e} = \mu_2\sum_{n=0}^{N}n\pi_n = \mu_2\frac{\sum_{n=0}^{N}\left(\frac{\lambda_1}{\mu_1}\right)^n\frac{1}{(n-1)!}}{\sum_{n=0}^{N}\left(\frac{\lambda_1}{\mu_1}\right)^n\frac{1}{n!}} = \mu_2 E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$$

That is, the stability condition is $\lambda_2/\mu_2 < E[L_{M(\lambda_1)/M(\mu_1)/N/N}]$, exactly as in Model 1.

Define the steady-state probability vector $\vec{P}_m = \left(P_{0m}, P_{1m}, \dots, P_{Nm}\right)$ for all $m \ge 0$. Then,

$$\vec{P}_m = \vec{P}_1 R^{m-1}, \quad m \ge 1$$

where R is the minimal solution of the matrix quadratic equation $A_0 + RA_1 + R^2A_2 = 0$.

The vectors $\vec{P}_0, \vec{P}_1, \ldots, \vec{P}_{N-1}$ can be calculated by solving the following linear system of equations:

$$\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0}$$

$$\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2^2 = \vec{0}$$

$$\vec{P}_{m-1} A_0 + \vec{P}_m A_1^m + \vec{P}_{m+1} A_2^{m+1} = \vec{0} \quad , \quad 2 \le m \le N-2$$

$$\vec{P}_{N-2} A_0 + \vec{P}_{N-1} \left(A_1^{N-1} + R A_2^{N-1} \right) = \vec{0}$$

$$\sum_{m=0}^{N-2} \vec{P}_m \vec{e} + \vec{P}_{N-1} \left[I - R \right]^{-1} \vec{e} = 1$$

where I is the identity matrix.
The mean total number of customers in Q_2, $E[L_2]$, is given by

$$E[L_2] = \sum_{m=0}^{\infty} m\vec{P}_m \cdot \vec{e} = \sum_{m=0}^{N-2} m\vec{P}_m \cdot \vec{e} + \sum_{m=N-1}^{\infty} m\vec{P}_{N-1} R^{m-N+1} \cdot \vec{e}$$

$$= \sum_{m=0}^{N-2} m\vec{P}_m \cdot \vec{e} + (N-2)\vec{P}_{N-1} \left[I - R \right]^{-1} \cdot \vec{e} + \vec{P}_{N-1} \left[I - R \right]^{-2} \cdot \vec{e}$$

4 Numerical results

We present some numerical results for both models. In Table 4.1 we show results for Model 1, using the set of parameters $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$ and $\mu_2 = 2$ for $N = 2$ and $N = 3$.

	P_{10}	P_{20}	P_{21}	P_{30}	P_{31}	P_{32}	$E[L_1]$	$E[L_2]$	$Cov(L_1, L_2)$
$N = 2$	0.0282	0.6209	0.1411	—	—	—	1.7702	0.6642	-0.6436
$N = 3$	0.0056	0.0379	0.0256	0.751	0.1125	0.0164	2.8342	0.2999	-0.12

Table 4.1: Numerical results for Model 1 with $N = 2$ and $N = 3$

In Table 4.2 we show results for Model 2, using the same set of parameters $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$ and $\mu_2 = 2$ for $N = 2$ and $N = 3$.

	P_{10}	P_{20}	P_{21}	P_{30}	P_{31}	P_{32}	$E[L_1]$	$E[L_2]$	$Cov(L_1, L_2)$
$N = 2$	0.0394	0.4719	0.1966	—	—	—	1.6798	0.9254	-0.3951
$N = 3$	0.0141	0.0524	0.0645	0.5017	0.1984	0.0407	2.48	0.5631	-0.0721

Table 4.2: Numerical results for Model 2 with $N = 2$ and $N = 3$

It is seen that, for same parameter values, $E[L_1]$ in Model 1 is larger than in Model 2. This follows since in Model 1 all customers of Q_1 join hands together in serving Q_2

reducing its size. This affects the size of Q_1 since less customers are present in Q_2 to serve Q_1. The opposite holds for $E[L_2]$.

5 Conclusions

In this paper we extend the scope of analytic investigation of 2-queue models where customers of only one queue act as servers, to the case where both customers in both queues act as servers, each group serving the opposite queue. We derived the stability conditions of such queues, revealing their connection to the roots of a given matrix related to the PGFs, obtain the stationary probabilities, calculate the mean queue size of each queue, as well as the correlation between them. Numerical results further exhibit the Inter relationship between the two queues.

6 References

[1] A. Arazi, E. Ben-Jacob and U. Yechiali, "Controlling an Oscillating Jackson-Type Network Having State-Dependent Service Rates", Mathematical Methods of Operations Research, 62 (2005) 453-466.

[2] R. B. Cooper, "Introduction to Queueing Theory" (second edition), North Holland, 1981.

[3] G. Latouche and V. Ramaswami, "Introduction to Matrix Analytic Methods in Stochastic Modeling", ASA, Alexandria, Virginia, 1999.

[4] M. F. Neuts, "Matrix-Geometric Solutions in Stochastic Models – An algorithmic Approach", Johns Hopkins, Baltimore, 1981.

[5] E. Perel and U. Yechiali, "2-Queue Systems where Customers of One Queue Serve the Customers of the Other Queue", Qeueueing Systems, 60 (2008) 271-288.

[6] W. P. Sendfeld "Two-Dimensional Overflow Queueing Systems", Doctoral Thesis Institute for Mathematics, University of Osnabrück, July 2009.

Appendix:

Proposition a.1 $q_n^{(N)}(z)$ is of the form $q_n^{(N)}(z) = \lambda_1{}^n z^{n-1} + (1-z)h_n^{(N)}(z)$ for $1 \le n \le N$ and $q_{N+1}^{(N)}(z)$ is of the form $q_{N+1}^{(N)}(z) = (1-z)h_{N+1}^{(N)}(z)$, where

$h_n^{(N)}(z) = \alpha_{n-1}^{(N)}(z)h_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 h_{n-2}^{(N)}(z) + \lambda_1{}^{n-1}z^{n-2}(\lambda_2 z - (n-1)\mu_2)$ for all $3 \le n \le N+1$.

Proof.

$n = 1$: $\quad q_1^{(N)}(z) = \lambda_1 + \lambda_2(1-z)$

$n = 2$: $\quad q_2^{(N)}(z) = \lambda_1{}^2 z + (1-z)\Big(\lambda_2\big((\lambda_1+\mu_1)z + (\lambda_2 z - \mu_2)(1-z)\big) + \lambda_1(\lambda_2 z - \mu_2)\Big)$

Suppose the proposition is valid for some n $(2 \le n \le N-1)$. We will show that it is valid for $n+1$:

$$
\begin{aligned}
q_{n+1}^{(N)}(z) &= \alpha_n^{(N)}(z)q_n^{(N)}(z) - n\lambda_1\mu_1 z^2 q_{n-1}^{(N)}(z) \\
&= \left((\lambda_1 + n\mu_1)z + (\lambda_2 z - n\mu_2)(1-z)\right)\left(\lambda_1{}^n z^{n-1} + (1-z)h_n^{(N)}(z)\right) \\
&\quad -n\lambda_1\mu_1 z^2\left(\lambda_1{}^{n-1}z^{n-2} + (1-z)h_{n-1}^{(N)}(z)\right) \\
&= \lambda_1{}^{n+1}z^n + n\lambda_1{}^n\mu_1 z^n + \lambda_1{}^n z^{n-1}(\lambda_2 z - n\mu_2)(1-z) + (\lambda_1 + \mu_1)z(1-z)h_n^{(N)}(z) \\
&\quad +(\lambda_2 z - n\mu_2)(1-z)^2 h_n^{(N)}(z) - n\lambda_1{}^n\mu_1 z^n - n\lambda_1\mu_1 z^2(1-z)h_{n-1}^{(N)}(z) \\
&= \lambda_1{}^{n+1}z^n + (1-z)\left(\left((\lambda_1 + n\mu_1)z + (\lambda_2 z - n\mu_2)(1-z)\right)h_n^{(N)}(z)\right) \\
&\quad +(1-z)\left(\lambda_1{}^n z^{n-1}(\lambda_2 z - n\mu_2) - n\lambda_1\mu_1 z^2 h_{n-1}^{(N)}(z)\right) \\
&= \lambda_1{}^{n+1}z^n + (1-z)\left(\alpha_n^{(N)}(z)h_n^{(N)}(z) - n\lambda_1\mu_1 z^2 h_{n-1}^{(N)}(z) + \lambda_1{}^n z^{n-1}(\lambda_2 z - n\mu_2)\right)
\end{aligned}
$$

Therefore

$$h_{n+1}^{(N)}(z) = \alpha_n^{(N)}(z)h_n^{(N)}(z) - n\lambda_1\mu_1 z^2 h_{n-1}^{(N)}(z) + \lambda_1{}^n z^{n-1}(\lambda_2 z - n\mu_2)$$

$$
\begin{aligned}
q_{N+1}^{(N)}(1) &= \alpha_N^{(N)}(z)q_N^{(N)}(z) - N\lambda_1\mu_1 z^2 q_{N-1}^{(N)}(z) \\
&= \left(N\mu_1 z + (\lambda_2 z - N\mu_2)(1-z)\right)\left(\lambda_1{}^N z^{N-1} + (1-z)h_N^{(N)}(z)\right) \\
&\quad -N\lambda_1\mu_1 z^2\left(\lambda_1{}^{N-1}z^{N-2} + (1-z)h_{N-1}^{(N)}(z)\right) \\
&= N\lambda_1{}^N\mu_1 z^N + N\mu_1 z(1-z)h_N^{(N)}(z) + (\lambda_2 z - N\mu_2)(1-z)\lambda_1{}^N z^{N-1} \\
&\quad +(\lambda_2 z - N\mu_2)(1-z)^2 h_N^{(N)}(z) - N\lambda_1{}^N\mu_1 z^N - N\lambda_1\mu_1 z^2(1-z)h_{N-1}^{(N)}(z) \\
&= (1-z)\left((\mu_1 z + (\lambda_2 z - N\mu_2)(1-z))h_N^{(N)}(z) - \lambda_1\mu_1 h_{N-1}^{(N)}(z) + \lambda_1{}^N z^{N-1}(\lambda_2 z - N\mu_2)\right) \\
&= (1-z)\left(\alpha_N^{(N)}(z)h_N^{(N)}(z) - \lambda_1\mu_1 h_{N-1}^{(N)}(z) + \lambda_1{}^N z^{N-1}(\lambda_2 z - N\mu_2)\right)
\end{aligned}
$$

Therefore

$$h_{N+1}^{(N)}(z) = \lambda_1{}^N z^{N-1}(\lambda_2 z - N\mu_2) + (\mu_1 z + (\lambda_2 z - N\mu_2)(1-z))h_N^{(N)}(z) - \lambda_1\mu_1 h_{N-1}^{(N)}(z)$$

This completes the proof of proposition a.1.

Proposition a.2 For all $N \geq 2$ $h_{N+1}^{(N)}(1) = \lambda_2 N!\mu_1{}^N \sum_{n=0}^{N}\left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 N!\mu_1{}^N \sum_{n=1}^{N} n\left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}$.

Proof. We will show that the proposition holds by induction over *N*.
For $N = 2$, we have

$$h_1^{(2)}(1) = \lambda_2$$

$$h_2^{(2)}(1) = \lambda_2(2\lambda_1 + \mu_2) - \mu_2\lambda_1$$

$$h_3^{(2)}(1) = \lambda_1{}^2(\lambda_2 - 2\mu_2) + 2\mu_1 h_2^{(2)}(1) - 2\lambda_1\mu_1 h_1^{(2)}(1)$$

Substituting $h_1^{(2)}(1)$ and $h_2^{(2)}(1)$ in the above formula for $h_3^{(2)}(1)$ we get

$$h_3^{(2)}(1) = \lambda_1{}^2(\lambda_2 - 2\mu_2) + 2\mu_1 h_2^{(2)}(1) - 2\lambda_1\mu_1 h_1^{(2)}(1)$$
$$= \lambda_1{}^2(\lambda_2 - 2\mu_2) + 2\mu_1\left(\lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1)\right) - 2\lambda_1\mu_1\lambda_2$$
$$= \lambda_2\left(2\mu_1{}^2 + 2\lambda_1\mu_1 + \lambda_1{}^2\right) - \mu_2\left(2\lambda_1\mu_1 + 2\lambda_1{}^2\right)$$
$$= \lambda_2 2\mu_1{}^2\left(1 + \tfrac{\lambda_1}{\mu_1} + \left(\tfrac{\lambda_1}{\mu_1}\right)^2 \tfrac{1}{2!}\right) - \mu_2 2\mu_1{}^2\left(\tfrac{\lambda_1}{\mu_1} + \left(\tfrac{\lambda_1}{\mu_1}\right)^2\right) = \lambda_2 2\mu_1{}^2 \sum_{n=0}^{2}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2 2\mu_1{}^2 \sum_{n=1}^{2} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}$$

For $N = 3$, we see that

$$h_1^{(3)}(1) = \lambda_2$$
$$h_2^{(3)}(1) = \lambda_2(2\lambda_1 + \mu_2) - \mu_2\lambda_1$$
$$h_3^{(3)}(1) = \lambda_1{}^2(\lambda_2 - 2\mu_2) + (\lambda_1 + 2\mu_1)h_2^{(3)}(1) - 2\lambda_1\mu_1 h_1^{(3)}(1)$$
$$h_4^{(3)}(1) = \lambda_1{}^3(\lambda_2 - 3\mu_2) + 3\mu_1 h_3^{(3)}(1) - 3\lambda_1\mu_1 h_2^{(3)}(1)$$

Substituting $h_1^{(2)}(1)$ and $h_2^{(2)}(1)$ in the above formula for $h_3^{(2)}(1)$ we get

$$h_3^{(3)}(1) = \lambda_1{}^2(\lambda_2 - 2\mu_2) + (\lambda_1 + 2\mu_1)h_2^{(3)}(1) - 2\lambda_1\mu_1 h_1^{(3)}(1)$$
$$= \lambda_1{}^2(\lambda_2 - 2\mu_2) + (\lambda_1 + 2\mu_1)\left(\lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1)\right) - 2\lambda_1\mu_1\lambda_2$$
$$= \lambda_2\left((\lambda_1 + 2\mu_1)(2\lambda_1 + \mu_1) - 2\lambda_1\mu_1 + \lambda_1{}^2\right) - \mu_2\left(\lambda_1(\lambda_1 + 2\mu_1) + 2\lambda_1{}^2\right)$$
$$= \lambda_2\left(2\mu_1{}^2 + 3\lambda_1\mu_1 + 3\lambda_1{}^2\right) - \mu_2\left(3\lambda_1{}^2 + 2\lambda_1\mu_1\right)$$

Substituting $h_2^{(3)}(1)$ and $h_3^{(3)}(1)$ in the above formula for $h_4^{(3)}(1)$ we get

$$h_4^{(3)}(1) = \lambda_1{}^3(\lambda_2 - 3\mu_2) + 3\mu_1 h_3^{(3)}(1) - 3\lambda_1\mu_1 h_2^{(3)}(1)$$
$$= \lambda_1{}^3(\lambda_2 - 3\mu_2) + 3\mu_1\left(\lambda_2\left(2\mu_1{}^2 + 3\lambda_1\mu_1 + 3\lambda_1{}^2\right) - \mu_2\left(3\lambda_1{}^2 + 2\lambda_1\mu_1\right)\right)$$
$$-3\lambda_1\mu_1\left(\lambda_2(2\lambda_1 + \mu_2) - \mu_2\lambda_1\right)$$
$$= \lambda_2\left(3\mu_1\left(2\mu_1{}^2 + 3\lambda_1\mu_1 + 3\lambda_1{}^2\right) - 6\lambda_1{}^2\mu_1 + \lambda_1{}^3\right) - \mu_2\left(3\mu_1\left(3\lambda_1{}^2 + 2\lambda_1\mu_1\right) + 3\lambda_1{}^3 - 3\lambda_1{}^2\mu_1\right)$$
$$= \lambda_2\left(6\mu_1{}^3 + 6\lambda_1\mu_1{}^2 + 3\lambda_1{}^2\mu_1 + \lambda_1{}^3\right) - \mu_2\left(6\lambda_1\mu_1{}^2 + 6\lambda_1{}^2\mu_1 + 3\lambda_1{}^3\right)$$
$$= \lambda_2 3!\mu_1{}^3\left(1 + \tfrac{\lambda_1}{\mu_1} + \left(\tfrac{\lambda_1}{\mu_1}\right)^2 \tfrac{1}{2!} + \left(\tfrac{\lambda_1}{\mu_1}\right)^3 \tfrac{1}{3!}\right) - \mu_2 3!\mu_1{}^3\left(\tfrac{\lambda_1}{\mu_1} + \left(\tfrac{\lambda_1}{\mu_1}\right)^2 + \left(\tfrac{\lambda_1}{\mu_1}\right)^3 \tfrac{1}{2!}\right)$$
$$= \lambda_2 3!\mu_1{}^3 \sum_{n=0}^{3}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2 3!\mu_1{}^3 \sum_{n=1}^{3} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}$$

Assume that the proposition holds for every $k \le N-1$, we now show that it holds for *N*. Notice that for all $k \le N$, $h_n^{(k)}(1) = h_n^{(N)}(1)$ for every $n \le k$. In particular, for $k = N-1$, $h_{N-1}^{(N-1)}(1) = h_{N-1}^{(N)}(1)$ and $h_N^{(N-1)}(1) = h_N^{(N)}(1) - \lambda_1 h_{N-1}^{(N-1)}(1) = h_N^{(N)}(1) - \lambda_1 h_{N-1}^{(N)}(1)$, meaning that $h_N^{(N)}(1) = h_N^{(N-1)}(1) + \lambda_1 h_{N-1}^{(N)}(1)$

Therefore, by the definition of $h_{N+1}^{(N)}(1)$ we have

$$
\begin{aligned}
h_{N+1}^{(N)}(1) &= \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1 h_N^{(N)}(1) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) \\
&= \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1\left(h_N^{(N-1)}(1) + \lambda_1 h_{N-1}^{(N)}(1)\right) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) \\
&= \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1 h_N^{(N-1)}(1) + N\mu_1\lambda_1\mu_1 h_{N-1}^{(N)}(1) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) \\
&= \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1 h_N^{(N-1)}(1) \\
&= \lambda_1{}^N(\lambda_2 - N\mu_2) + N\mu_1\left(\lambda_2(N-1)!\,\mu_1{}^{N-1}\sum_{n=0}^{N-1}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2(N-1)!\,\mu_1{}^{N-1}\sum_{n=1}^{N-1} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}\right) \\
&= \lambda_2\left(\lambda_1{}^N + N(N-1)!\,\mu_1{}^{N-1}\sum_{n=0}^{N-1}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}\right) - \mu_2\left(N\lambda_1{}^N + N(N-1)!\,\mu_1{}^{N-1}\sum_{n=1}^{N-1} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}\right) \\
&= \lambda_2 N!\,\mu_1{}^N\sum_{n=0}^{N}\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!} - \mu_2 N!\,\mu_1{}^N\sum_{n=1}^{N} n\left(\tfrac{\lambda_1}{\mu_1}\right)^n \tfrac{1}{n!}
\end{aligned}
$$

This completes the proof of proposition a.2