THE CONDITIONAL RESIDUAL SERVICE TIME IN THE M/G/1 QUEUE*

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Abstract

For the M/G/1 queue in steady state we derive an explicit formula for the expectation of the conditional residual service time given the queue size at epochs of arrival.

1 Introduction

In various control-studies of M/G/1–type queues, certain actions taken at instants of customers' arrival depend on the residual service time, R, of the customer being served. In a stationary queue it is well known that $E[R \mid j \geq 1] = E(V^2)/[2E(V)]$, where V denotes the service time and j stands for the number of customers present. However, very often decisions depend heavily on the exact number of customers in the system, and not only on the fact that the queue is non-empty.

We present an explicit formula for $E[R \mid j]$ in terms of the stationary distribution of the queue. Denoting by λ the arrival rate and setting $\rho = \lambda E(V)$, we show that

$$E[R \mid j] \equiv R_j = \frac{1-\rho}{\lambda \pi_j}[1 - \sum_{k=0}^{j} \pi_k] \quad j = 1, 2, 3, \ldots$$

where π_k is the probability that there are k customers in the system.

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2 Proof

Consider an arrival instant, and denote by η and R the number of customers present, and the residual service time of the customer being served, respectively. For $j \geq 1$ let

$$P_j(x) = P[R \leq x, \eta = j] = P[R \leq x \mid \eta = j]\pi_j \tag{1}$$

$$\Omega_j(s) = \int_0^\infty e^{-sx} dP_j(x) \ , \qquad \mathrm{Re}(s) \geq 0 \tag{2}$$

and

$$U(s,z) = \sum_{j=1}^\infty \Omega_j(s) z^j \ , \qquad |z| \leq 1 \tag{3}$$

In [1] it is shown that:

$$U(s,z) = \frac{(1-\rho)\lambda z(1-z)}{z - \psi(\lambda - \lambda z)} \left[\frac{\psi(s) - \psi(\lambda - \lambda z)}{s - \lambda(1-z)} \right] \tag{4}$$

where $\psi(s) = \int_0^\infty e^{-sv} dP(V \leq v)$.

It follows that

$$R_j = E[R \mid \eta = j] = \int_0^\infty x dP[R \leq x \mid \eta = j] = \int_0^\infty x dP_j(x)/\pi_j = -\Omega_j'(0)/\pi_j \ . \tag{5}$$

By differentiating $U(s,z)$ in (3) with respect to s and using (5) we get,

$$\frac{d}{ds} U(s,z) \Big|_{s=0} = \sum_{j=1}^\infty \Omega_j'(0) z^j = -\sum_{j=1}^\infty \pi_j R_j z^j \ . \tag{6}$$

On the other hand, differentiation of $U(s,z)$ in (4) yields

$$U'(0,z) = \frac{(1-\rho)z}{z - \psi(\lambda - \lambda z)} \left[\frac{\psi(\lambda - \lambda z) - z + (1-z)(\rho - 1)}{\lambda(1-z)} \right] \tag{7}$$

Equating (6) and (7) results in

$$\sum_{j=1}^\infty \pi_j R_j z^j = \frac{1-\rho}{\lambda} \left[\frac{z}{1-z} + \frac{(1-\rho)z}{z - \psi(\lambda - \lambda z)} \right] \ . \tag{8}$$

Now, observe that

$$\sum_{j=0}^\infty \left(\sum_{k=0}^j \pi_k \right) z^j = \sum_{k=0}^\infty \pi_k \sum_{j=k}^\infty z^j = \sum_{k=0}^\infty \pi_k z^k/(1-z) \equiv U(z)/(1-z) \tag{9}$$

where $U(z)$, the generating function of $\{\pi_k\}$, is given by

$$U(z) \equiv \sum_{k=0}^{\infty} \pi_k z^k = \frac{(1-\rho)(1-z)\psi(\lambda-\lambda z)}{\psi(\lambda-\lambda z)-z} \ . \tag{10}$$

From (9) and (10), and using $(1-\rho)=\pi_0$, we obtain

$$\sum_{j=1}^{\infty}\left(\sum_{k=0}^{j}\pi_k\right)z^j = \frac{\pi_0\psi(\lambda-\lambda z)}{\psi(\lambda-\lambda z)-z} - \pi_0 = \frac{(1-\rho)z}{\psi(\lambda-\lambda z)-z} \ . \tag{11}$$

Substituting (11) into (8) leads to

$$\sum_{j=1}^{\infty}\pi_j R_j z^j = \frac{1-\rho}{\lambda}\left[\sum_{j=1}^{\infty} z^j - \sum_{j=1}^{\infty}\left(\sum_{k=0}^{j}\pi_k\right)z^j\right] = \frac{1-\rho}{\lambda}\sum_{j=1}^{\infty}\left[1-\sum_{k=0}^{j}\pi_k\right]z^j \tag{12}$$

By equating coefficients of z^j we derive,

$$\pi_j R_j = \frac{1-\rho}{\lambda}\left[1-\sum_{k=0}^{j}\pi_k\right] \tag{13}$$

or, as we claimed,

$$R_j = \frac{1-\rho}{\lambda\pi_j}\left[1-\sum_{k=0}^{j}\pi_k\right] \qquad j=1,2,3,\ldots$$

For the M/M/1 queue, where $\pi_k=(1-\rho)\rho^k$ and $\sum\limits_{k=j+1}^{\infty}\pi_k=\rho^{j+1}$, we calculate:

$$R_j = \frac{1-\rho}{\lambda(1-\rho)\rho^j}\left[\rho^{j+1}\right] = E(V) \ ,$$

which clearly agrees with the Markovian property of the service time.

The expected residual service time, given that the system is not empty, may now be derived using (10) and (13). From (10),

$$U'(0) = \sum_{j=1}^{\infty} j\pi_j = \frac{\lambda^2 E(V^2)}{2(1-\rho)} + \rho \equiv L_q + \rho$$

where

$$L_q = \sum_{j=2}^{\infty}(j-1)\pi_j \ .$$

Employing (13), we get

$$\sum_{j=1}^{\infty}\pi_j R_j = \frac{1-\rho}{\lambda}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\pi_k = \frac{1-\rho}{\lambda}\sum_{k=2}^{\infty}\pi_k(k-1) = \frac{1-\rho}{\lambda}L_q \ .$$

Since $\sum\limits_{j=1}^{\infty}\pi_j R_j = \rho E[R]j \geq 1]$, it follows that

$$E[R \mid j \geq 1] = \frac{1-\rho}{\lambda\rho}L_q = \frac{E(V^2)}{2E(V)} \ .$$

References

[1] Wishart, D.M.G., "An Application of Ergodic Theorems in the Theory of Queues", Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability Vol. **2**, pp. 581–592, (1961).