POLLING SYSTEMS WITH BREAKDOWNS AND REPAIRS

Oren Nakdimon and Uri Yechiali

Department of Statistics and Operations Research
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
[Email: uriy@post.tau.ac.il; phone +972-3-6409637]

Abstract

This work analyzes various polling systems with both random breakdowns and repairs. A few works in the literature investigated polling networks with failing nodes, but none has treated the associated repair process or the combined effect of breakdowns and repairs on such systems.

We consider three service mechanisms: Gated, Exhaustive and Globally-Gated. For each service regime we study several variations, differing from each other by (i) whether the arrival process to a queue being repaired continues or stops during the repair process, and (ii) whether the failure is observed immediately when it occurs or only at the end of a service duration.

For each of the twelve models studied we provide analyses regarding the system state at polling instants (law of motion, probability generating functions, first and second order moments) and derive expressions for several performance measures, such as (distribution and mean of) number of customers at the different queues, their waiting and sojourn times, server's cycle times, etc. We derive stability conditions for the various models and express all results in a unified generalized form.

Keywords: Polling; Gated; Exhaustive; Globally-Gated; Breakdowns; Repairs.

1 Introduction

Only a few works in the literature deal with the important phenomenon of nodes breakdowns in polling systems. Recently Kofman and Yechiali studied models with failing nodes, analyzing the Gated and Exhaustive [8], as well as the Globally Gated [9] service regimes. However, we know of no works studying the combined effect of breakdowns and the associated repair processes on such systems. This work addresses this issue.

Queueing systems consisting of N queues (stations, nodes, or channels) served by a single server who incurs switchover periods when moving from one queue to another have been studied widely in the literature and used as a central model for the analysis of a large variety of applications in the areas of telecommunication systems, computer networks, multiple access protocols, multiplexing schemes in ISDN, readerhead movements in a computer's hard disk, manufacturing systems, road traffic control, repair problems, etc. Very often, such applications are modeled as polling systems in which the server visits the queues in a cyclic or some other order.

In many of these applications, as well as in most polling models, it is customary to control the amount of service time allocated to each queue during the server's visit. Two common service policies are the Gated and the Exhaustive regimes. Under the Gated regime, in each cycle only customers who are present when the server polls the queue are served during its current visit, while customers arriving when the queue is attended will be served during the next visit. Under the Exhaustive regime, at each visit the server attends the queue until it becomes completely empty, and only then is the sever allowed to move on. There is extensive literature on the theory and applications of these models (see Takagi [10],[11], Yechiali [12] and references there).

Another service regime is the Globally-Gated, introduced by Boxma, Levy and Yechiali [1] and extended in Boxma, Weststrate and Yechiali [2]. Under this regime, the server uses the instant of cycle beginning as a reference point of time, serving in each queue, during each cycle, only those customers that were present there at the cycle-beginning.

In this work we consider a polling system with N infinite-capacity stations, where customers arrive to the various queues according to independent Poisson processes, requiring general independent service times. A single server visits the stations in a cyclic order, incurring random switchover times when moving from one station to another. A station being served may fail due to a breakdown process (described in the sequel), and a repair process is initiated immediately after such a failure is observed. We assume that during the repair process the server stays *dormant* in the station, and once the repair is completed, the server continues serving customers in that station, starting anew with the interrupted customer.

We study two models, differing from each other by the behavior of the arrival process to queues being repaired: In the first model, called Arrival Continues [AC], the arrival processes continue even when a station is being repaired, while in the second, called Arrival Stops [AS], the arrival process to the station being repaired stops for the entire repair period. We also distinguish between two versions for determining when the breakdowns (failures) are observed. In the first version a failure is observed *immediately* when it occurs, while in the second version it is observed only at the *end* of the service. We analyze these systems under each of the above mentioned regimes, namely the Gated, Exhaustive and Globally Gated service protocols.

The [AC] model may be viewed as a regular polling model, with N $M/G/1$-type queues and a single server, where each customer requires a generalized service time, composed of several unsuccessful and one successful service attempts. Therefore, once the required expressions for such a generalized service time are derived, we can use well known results and apply them in the analysis of this model.

The [AS] model is the more interesting one in this work. We introduce a new parameter of the system, η, which is the "loss" of potential customers to a queue during a service of a customer, due to arrival stoppage. Using η in the results, rather than using repair time expressions, makes the [AS] model a generalization of the [AC] model. Moreover, the [AS] model is a generalized polling model, which may be reduced to the standard one when the mean time to breakdown tends to infinity (implying $\eta \to 0$). An important generalization is of the queue work rate: we show that if one defines a generalized work-load rate, $\overline{\rho} = \frac{\text{(arrival rate)} \times \text{(mean service time)}}{1+\eta}$, then $\overline{\rho}$ preserves its characteristics as work rate in all relevant expressions.

The structure of the paper is as follows: In section 2 we present the general description of the models along with a set of assumptions, definitions and notations used throughout the work. In section 3 we derive some general results, independent of the service regime, for mean number of service attempts of a customer, Laplace-Stieltjes transforms, means and second moments of successful and unsuccessful service attempts (for both versions), etc. In sections 4 and 5 we analyze the Gated and Exhaustive service regimes, respectively. In section 6 we obtain expressions – common to all models, versions and service regimes discussed in the previous sections – for various important performance measures, such as mean number of customers at polling instants and mean cycle time. In addition, stability conditions are derived. Section 7 concludes the paper with the analysis of the Globally Gated service regime.

2 Model and Notation

We consider a polling system consisting of N infinite-capacity queues (stations, channels, nodes), labeled $1, 2, \dots, N$, and a single server. Customers arrive to queue i according to a Poisson process with rate λ_i. The server visits (polls) the queues in a cyclic order. Each customer in queue i requires a random service time, distributed as B_i, with Laplace-Stieltjes Transform (LST) $B_i^*(\cdot)$, mean b_i and second moment $b_i^{(2)}$. The random switchover time from queue i to queue $i+1$ is denoted by D_i, with LST $D_i^*(\cdot)$, mean d_i and second moment $d_i^{(2)}$.

If the server enters a non-empty station and starts serving customers present there, the station may fail due to a breakdown process. There are two versions for determining when the breakdowns are observed: (i) the breakdown is observed immediately when it occurs; (ii) the breakdown is observed only at the end of the current service (such as in packet transmission applications). In both versions, a repair process is initiated immediately after the breakdown is observed. The repair time for station i is V_i, with LST $V_i^*(\cdot)$, mean v_i and second moment $v_i^{(2)}$. During the repair process the server stays dormant in the station, and only when the repair is completed the server continues serving customers in the station, starting anew with the interrupted customer (whose service time is resampled) until it moves to the next station, following the Gated, Exhaustive or Globally-Gated service discipline, whichever applies.

The time to breakdown of station i is denoted by T_i and is distributed Exponentially with parameter γ_i. This process is regenerated at the beginning of every new visit of the server to the station and after the completion of every repair.

We consider two models: in the first, the arrival processes to the various stations never stop, while in the second, the arrival process to the station being repaired stops for the entire repair period, whereas the arrival streams to other queues continue uninterruptedly. We denote the first model by [AC] (Arrival Continues), and the second by [AS] (Arrival Stops).

We assume that the underlying arrival processes, the service times, the breakdown processes, the repair times and the switchover times are all mutually independent.

We use the following notation:

- $S^*(\omega) \equiv E[e^{-\omega S}]$ = LST of a non-negative continuous random variable S.
- X_i^j = number of jobs present in queue j at a polling instant of queue i.
- $\underline{X}_i \equiv (X_i^1, X_i^2, \dots, X_i^N)$ = state of the system at a polling instant of queue i.
- $F_i(\underline{z}) = F_i(z_1, \dots, z_N) \equiv E\Big[\prod_{j=1}^{N} z_j^{X_i^j}\Big]$ = Probability Generating Function (PGF) of $\underline{X}_i$.

- $A^i(t)$ = number of Poisson arrivals to queue i during a random time interval of length t, in which the arrival process doesn't stop.

3 Some General Results

3.1 Number of Service Attempts of a Customer

Let a_i be the probability of a successful service attempt in queue i, i.e., the probability that no breakdown occurs during a service time of a customer. Then,

$$a_i = P(B_i < T_i) = E[P(B_i < T_i|B_i)] = E[e^{-\gamma_i B_i}] = B_i^*(\gamma_i) \ . \tag{1}$$

Due to the memoryless property of the exponential distribution, we can assume that the 'timer' of the breakdown process is initiated each time a service starts.

Let K_i be the number of *unsuccessful* service attempts of a customer in queue i *before* a successful service completion. Then $P(K_i = k) = (1 - a_i)^k a_i \quad (k = 0, 1, 2, \dots)$. K_i is a shifted Geometric variable with parameter a_i. Thus,

$$E[K_i] = \frac{1 - a_i}{a_i} \ , \tag{2}$$

$$E[K_i^2] = \frac{(1 - a_i)(2 - a_i)}{a_i^2} \ , \tag{3}$$

and

$$E[K_i(K_i - 1)] = E[K_i^2] - E[K_i] = 2 \cdot \left(\frac{1 - a_i}{a_i}\right)^2 = 2 \cdot [E(K_i)]^2 \ . \tag{4}$$

3.2 A Successful Service Attempt

Let S_i^+ be the duration of a successful service attempt. Then (for both versions), $S_i^+ \sim B_i|_{B_i<T_i}$. Hence,

$$\begin{aligned} S_i^{+*}(\omega) \quad &= E[e^{-\omega S_i^+}] = E[e^{-\omega B_i}|B_i < T_i] = \frac{E[e^{-\omega B_i} P(B_i < T_i|B_i)]}{P(B_i < T_i)} = \\ &= \frac{1}{a_i} E[e^{-\omega B_i} e^{-\gamma_i B_i}] = \frac{1}{a_i} E[e^{-(\omega+\gamma_i)B_i}] = \frac{B_i^*(\omega + \gamma_i)}{a_i} \ . \end{aligned} \tag{5}$$

Therefore,

$$E[S_i^+] = -\frac{B_i^{*'}(\gamma_i)}{a_i} \ , \quad E[(S_i^+)^2] = \frac{B_i^{*''}(\gamma_i)}{a_i} \ . \tag{6}$$

3.3 An Unsuccessful Service Attempt

Let S_i^- be the duration of an unsuccessful service attempt. S_i^- is distributed *differently* for each version:
In **version (i)** the service is interrupted whenever a breakdown occurs, thus $S_i^- \sim T_i|_{B_i > T_i}$, and

$$S_i^{-*}(\omega) = E[e^{-\omega S_i^-}] = E[e^{-\omega T_i}|B_i > T_i] = \frac{E[e^{-\omega T_i} P(B_i > T_i|T_i)]}{P(B_i > T_i)} \; . \tag{7}$$

Now, with $f_{B_i}(\cdot)$ denoting the probability density function of B_i,

$$\begin{aligned} E[e^{-\omega T_i} P(B_i > T_i|T_i)] &= \int_{t=0}^{\infty} e^{-\omega t}\Big(\int_{x=t}^{\infty} f_{B_i}(x)dx\Big)\gamma_i e^{-\gamma_i t}dt = \gamma_i \int_{x=0}^{\infty}\Big(\int_{t=0}^{x} e^{-(\omega+\gamma_i)t}dt\Big) f_{B_i}(x)dx \\ &= \frac{\gamma_i}{\omega+\gamma_i}\int_0^{\infty}\Big(1-e^{-(\omega+\gamma_i)x}\Big)\cdot f_{B_i}(x)dx = \frac{\gamma_i}{\omega+\gamma_i}[1 - B_i^*(\omega+\gamma_i)] \end{aligned} \tag{8}$$

Substituting (8) in (7) we get, for **version (i)**,

$$S_i^{-*}(\omega) = \frac{\gamma_i}{\omega+\gamma_i} \cdot \frac{1 - B_i^*(\omega+\gamma_i)}{1-a_i} \; . \tag{9}$$

Hence,

$$E[S_i^-] = \frac{B_i^{*'}(\gamma_i)}{1-a_i} + \frac{1}{\gamma_i} \quad , E[(S_i^-)^2] = \frac{2}{\gamma_i^2} + \frac{2B_i^{*'}(\gamma_i)}{\gamma_i(1-a_i)} - \frac{B_i^{*''}(\gamma_i)}{1-a_i} \; . \tag{10}$$

In **version (ii)** the failure is observed only upon service completion. Thus, $S_i^- \sim B_i\Big|_{B_i > T_i}$, and

$$\begin{aligned} S_i^{-*}(\omega) &= E[e^{-\omega B_i}|B_i > T_i] = \frac{E[e^{-\omega B_i} P(B_i > T_i|B_i)}{P(B_i > T_i)} \\ &= \frac{E[e^{-\omega B_i}(1 - e^{-\gamma_i B_i})]}{1-a_i} = \frac{B_i^*(\omega) - B_i^*(\omega+\gamma_i)}{1-a_i} \; . \end{aligned} \tag{11}$$

Hence, for **version (ii)**,

$$E[S_i^-] = \frac{b_i + B_i^{*'}(\gamma_i)}{1-a_i} \; , \quad E[(S_i^-)^2] = \frac{b_i^{(2)} - B_i^{*''}(\gamma_i)}{1-a_i} \; . \tag{12}$$

3.4 A Generalized Service and Repair Time

For both versions, let $\overline{B}_i$ denote the *total* length of time starting from the moment a service of a type-i customer is initiated until he leaves the system (after a successful completion of service). Let $\overline{b}_i$ and $\overline{b}_i^{(2)}$ denote the mean and second moment of $\overline{B}_i$, respectively. To calculate the LST of $\overline{B}_i$ we use a similar approach to the one used in [3] when studying the residence time of a job in a queue under a preemptive repeat rule with resampling. Considered as a generalized service time, $\overline{B}_i$ can be expressed as

$$\overline{B}_i = \sum_{j=1}^{K_i} [S_i^{-(j)} + V_i^{(j)}] + S_i^+ \ , \tag{13}$$

where $S_i^{-(j)} \sim S_i^-$, $V_i^{(j)} \sim V_i$ for $j = 1, \ldots, K_i$ and $\{S_i^{-(j)}\}_{j=1}^{K_i}$, $\{V_i^{(j)}\}_{j=1}^{K_i}$, S_i^+ are all mutually independent. Therefore:

$$\begin{aligned} E\Big[e^{-\omega\overline{B}_i}|K_i = k\Big] &= E\Big[e^{-\omega\sum_{j=1}^{k}(S_i^{-(j)}+V_i^{(j)})} e^{-\omega S_i^+}\Big] = \prod_{j=1}^{k}\Big[E\Big(e^{-\omega S_i^{-(j)}}\Big)\cdot E\Big(e^{-\omega V_i^{(j)}}\Big)\Big]\cdot E\Big[e^{-\omega S_i^+}\Big] \\ &= \Big(S_i^{-*}(\omega)\cdot V_i^*(\omega)\Big)^k \cdot S_i^{+*}(\omega) \ . \end{aligned} \tag{14}$$

Hence,

$$\overline{B}_i^*(\omega) = \sum_{k=0}^{\infty}(1-a_i)^k a_i \Big(S_i^{-*}(\omega)V_i^*(\omega)\Big)^k S_i^{+*}(\omega) = \frac{a_i S_i^{+*}(\omega)}{1-(1-a_i)S_i^{-*}(\omega)V_i^*(\omega)} \ . \tag{15}$$

Substituting (5) in (15) we get

$$\overline{B}_i^*(\omega) = \frac{B_i^*(\omega+\gamma_i)}{1-(1-a_i)S_i^{-*}(\omega)V_i^*(\omega)} \ . \tag{16}$$

The first and second moments of $\overline{B}_i$ may be calculated by taking derivatives of (16), or directly from (13), using (2),(4) and (6):

$$\overline{b}_i = E[K_i]\cdot E[S_i^- + V_i] + E[S_i^+] = \frac{1-a_i}{a_i}\cdot(E[S_i^-]+v_i) \ - \ \frac{B_i^{*'}(\gamma_i)}{a_i} \tag{17}$$

$$\begin{aligned} \overline{b}_i^{(2)} &= E[(S_i^+)^2] + 2E[K_i]\cdot E[S_i^+(S_i^- + V_i)] + E[K_i]\cdot E[(S_i^- + V_i)^2] + E[K_i(K_i-1)]\cdot[E(S_i^-)+v_i]^2 \\ &= \frac{B_i^{*''}(\gamma_i)}{a_i} \ + \ 2\overline{b}_i\Big[\overline{b}_i + \frac{B_i^{*'}(\gamma_i)}{a_i}\Big] \ + \ \frac{1-a_i}{a_i} \ \cdot \ \Big\{E[(S_i^-)^2] + v_i^{(2)} + 2E[S_i^-]\cdot v_i\Big\} \ . \end{aligned} \tag{18}$$

Substituting (9) and (10) for version (i), and (11) and (12) for version (ii), respectively, in (16), (17) and (18) yields:

Version (i):

$$\overline{B}_i^*(\omega) \;=\; \frac{B_i^*(\omega+\gamma_i)}{1 \;-\; \frac{\gamma_i}{\omega+\gamma_i}[1-B_i^*(\omega+\gamma_i)]V_i^*(\omega)}\;; \tag{19}$$

$$\overline{b}_i \;=\; \frac{1-a_i}{a_i}\left(\frac{B_i^{*'}(\gamma_i)}{1-a_i} \;+\; \frac{1}{\gamma_i} \;+v_i\right) \;-\; \frac{B_i^{*'}(\gamma_i)}{a_i} = \frac{1-a_i}{a_i} \;\cdot\; \left(\frac{1}{\gamma_i}+v_i\right) \tag{20}$$

That is, $E[\overline{B}_i] \;=\; E[K_i]\cdot\left(\frac{1}{\gamma_i} \;+\; v_i\right)$, which is the mean number of unsuccessful service attempts ($E[K_i]$) multiplied by $(E[T_i]+v_i)$.

$$\overline{b}_i^{(2)} = 2\overline{b}_i^2 + 2\Big[\frac{B_i^{*'}(\gamma_i)}{a_i(1-a_i)} \;+\; \frac{1}{\gamma_i}\Big]\overline{b}_i + \;\frac{1-a_i}{a_i}v_i^{(2)} \tag{21}$$

(Clearly, when $\gamma_i\to 0, \overline{b}_i\to b_i$, since $\lim\limits_{\gamma_i\to 0}\;\frac{1-a_i}{\gamma_i} = -B_i^{*'}(0)$, while $\frac{B_i^{*'}(\gamma_i)}{a_i\cdot(1-a_i)} \;+\; \frac{1}{\gamma_i} \underset{\gamma_i\to 0}{\longrightarrow} \frac{b_i^{(2)}-2b_i^2}{2b_i}$, implying that $\overline{b}_i^{(2)}\to b_i^{(2)}$).

Version (ii):

$$\overline{B}_i^*(\omega) = \frac{B_i^*(\omega+\gamma_i)}{1-[B_i^*(\omega)-B_i^*(\omega+\gamma_i)]\cdot V_i^*(\omega)} \tag{22}$$

$$\overline{b}_i = \frac{1-a_i}{a_i}\cdot\Big(\frac{b_i+B_i^{*'}(\gamma_i)}{1-a_i}+v_i\Big) \;-\; \frac{B_i^{*'}(\gamma_i)}{a_i} \;=\; \frac{b_i}{a_i} \;+\; \frac{1-a_i}{a_i} \;\cdot\; v_i\;. \tag{23}$$

That is, the mean generalized service time is comprised of the total service time devoted to a customer, namely $b_i\times E[K_i+1]$, augmented by $E[K_i]$ times the mean length of a repair, v_i. The second moment of $\overline{B}_i$ is given by

$$\overline{b}_i^{(2)} = 2\overline{b}_i^2 + \frac{b_i^{(2)}+(1-a_i)v_i^{(2)}+2v_ib_i}{a_i} \;+\; 2\frac{\overline{b}_i+v_i}{a_i}B_i^{*'}(\gamma_i) \tag{24}$$

It is clear that in version (ii), as in version (i), when $\gamma_i\to 0$, $\overline{b}_i\to b_i$ and $\overline{b}_i^{(2)}\to b_i^{(2)}$.

Similarly to the definition of $\overline{B}_i$, we define $\overline{V}_i=\sum\limits_{j=1}^{K_i}V_i^{(j)}$ as the generalized repair time in queue i. That is, the period of time, out of $\overline{B}_i$, in which the station is being repaired. Now,

$$\begin{aligned}\overline{V}_i^*(\omega) \;&=\; \sum_{k=0}^{\infty}P(K_i=k)\cdot E[e^{-\omega\overline{V}_i}|K_i=k] \\ &=\; \sum_{k=0}^{\infty}(1-a_i)^k a_i\cdot[V_i^*(\omega)]^k \;=\; \frac{a_i}{1-(1-a_i)\cdot V_i^*(\omega)}\end{aligned} \tag{25}$$

$$\overline{v}_i \;\equiv\; E[\overline{V}_i]=E[K_i]\cdot E[V_i] \;=\; \frac{1-a_i}{a_i}\;\cdot\;v_i\;. \tag{26}$$

Define $\widehat{B}_i$ as the *effective time*, out of $\overline{B}_i$, in which customers arrive to queue i, and denote the mean and second moment of $\widehat{B}_i$ by $\widehat{b}_i$ and $\widehat{b}_i^{(2)}$, respectively. Clearly, in the **[AC] model**, $\widehat{B}_i = \overline{B}_i$. To find the distribution of $\widehat{B}_i$ in the **[AS] model**, we apply the general results for $\overline{B}_i$ to the special case where $V_i \equiv 0 (\Rightarrow V_i^*(\omega) \equiv 1;\ v_i = v_i^{(2)} \equiv 0)$:

In the **[AS] model, version (i)**, Eqs. (19),(20) and (21) are reduced, respectively, to

$$\widehat{B}_i^*(\omega) = \frac{B_i^*(\omega+\gamma_i)}{1 - \frac{\gamma_i}{\omega+\gamma_i}\left[1 - B_i^*(\omega+\gamma_i)\right]} \tag{27}$$

$$\widehat{b}_i = \frac{1-a_i}{a_i} \cdot \frac{1}{\gamma_i} \tag{28}$$

and

$$\widehat{b}_i^{(2)} = \frac{2}{a_i^2\gamma_i^2}\left[1 - a_i + \gamma_i B_i^{*'}(\gamma_i)\right] \tag{29}$$

In the **[AS] model, version (ii),** Eqs. (22),(23) and (24) are reduced, respectively, to

$$\widehat{B}_i^*(\omega) = \frac{B_i^*(\omega+\gamma_i)}{1 - \left[B_i^*(\omega) - B_i^*(\omega+\gamma_i)\right]} \tag{30}$$

$$\widehat{b}_i = \frac{b_i}{a_i} \tag{31}$$

and

$$\widehat{b}_i^{(2)} = \frac{2b_i}{a_i^2}\left[b_i + B_i^{*'}(\gamma_i)\right] + \frac{b_i^{(2)}}{a_i} \tag{32}$$

Again $\widehat{b}_i^{(2)} \to b_i^{(2)}$ as $\gamma_i \to 0$.

Note that for both versions, from the definitions of $\overline{B}_i$, $\overline{V}_i$ and $\widehat{B}_i$, the following relations hold:

$$\widehat{B}_i = \begin{cases} \overline{B}_i, & \text{[AC] model} \\ \overline{B}_i - \overline{V}_i, & \text{[AS] model} \end{cases}$$

In the sequel, we will need the value $E[\overline{B}_i\widehat{B}_i]$. (Clearly, since $\widehat{B}_i$ is stochastically smaller than $\overline{B}_i$, $\widehat{b}_i^{(2)} \le E[\overline{B}_i\widehat{B}_i] \le \overline{b}_i^{(2)}$.)

In the **[AC] model**, $E[\overline{B}_i\widehat{B}_i] = \overline{b}_i^{(2)}$, because $\widehat{B}_i = \overline{B}_i$.

In the **[AS] model**, using the definitions of $\overline{V}_i$, equations (13), (2) and (3):

$$\begin{aligned} E[\overline{B}_i\widehat{B}_i] &= E[\widehat{B}_i^2 + \overline{V}_i\widehat{B}_i] = \widehat{b}_i^{(2)} + E[E(\overline{V}_i\widehat{B}_i|K_i)] \\ &= \widehat{b}_i^{(2)} + E\Big[K_i v_i \cdot \Big(E(S_i^+) + K_i E(S_i^-)\Big)\Big] = \widehat{b}_i^{(2)} + \frac{1-a_i}{a_i} v_i[E(S_i^+) + \frac{2-a_i}{a_i}E(S_i^-)] \end{aligned} \tag{33}$$

Thus, in **version (i)**, by substituting (6),(10) and (29) in (33) we get

$$E[\overline{B}_i\widehat{B}_i] = \frac{1}{a_i^2}\Big(v_i + \frac{2}{\gamma_i}\Big)\Big(B_i^{*'}(\gamma_i) + \frac{1-a_i}{\gamma_i}\Big) + \Big(\frac{1-a_i}{a_i}\Big)^2\frac{v_i}{\gamma_i}\ , \tag{34}$$

while in **version (ii)**, by substituting (6),(12) and (32) in (33) we obtain

$$E[\overline{B}_i\widehat{B}_i] = \frac{b_i^{(2)}}{a_i} + \frac{2b_i^2 + [2B_i^{*'}(\gamma_i) + (2-a_i)v_i]\cdot\ b_i + v_iB_i^{*'}(\gamma_i)}{a_i^2} \tag{35}$$

Clearly, when $\gamma_i \to 0$, $E[\overline{B}_i\widehat{B}_i] \to b_i^{(2)}$ for both versions.

4 The Gated Regime

4.1 System-State: Law of Motion, PGFs and First Moments

In the Gated service regime, in each visit the server serves only those customers that were present in the queue at the polling instant.

For the **[AC] model**, the evolution law of the system-state is given by:

$$X_{i+1}^j = \begin{cases} X_i^j + A^j\Big(\sum_{m=1}^{X_i^i}\overline{B}_i^{(m)}\Big) + A^j(D_i), & j \neq i \\ A^i\Big(\sum_{m=1}^{X_i^i}\overline{B}_i^{(m)}\Big) + A^i(D_i), & j = i \end{cases} \tag{36}$$

where $\overline{B}_i^{(m)} \sim \overline{B}_i$ for every m, and are mutually independent. Since the server moves in a cyclic order, all summations throughout the paper are cyclic ones. This model is actually the classical gated polling scheme, with N $M/G/1$-queues and a single server, where each customer in queue i requires a (generalized) service time of $\overline{B}_i$. Therefore, for $i = 1, 2, \ldots, N$ and for $j = 1, 2, \ldots, N$ the PGF of $\underline{X}_{i+1}$ is given by (see Takagi [10], Yechiali [12]):

$$F_{i+1}(\underline{z}) = F_i\Big(z_1, \ldots, z_{i-1}, \overline{B}_i^*\Big[\sum_{k=1}^N \lambda_k(1-z_k)\Big], z_{i+1}, \ldots, z_N\Big)\cdot D_i^*\Big(\sum_{k=1}^N\lambda_k(1-z_k)\Big) \tag{37}$$

Setting

$$f_i(j) \equiv E[X_i^j]\ ,\ \overline{\rho}_i \equiv \lambda_i\overline{b}_i\ ,\ \overline{\rho} \equiv \sum_{k=1}^N\overline{\rho}_k\ ,\ d \equiv \sum_{k=1}^N d_k$$

the first moments satisfy

$$f_{i+1}(j) = \begin{cases} f_i(j) + \lambda_j\overline{b}_if_i(i) + \lambda_jd_i & j \neq i \\ \lambda_i\overline{b}_if_i(i) + \lambda_id_i, & j = i \end{cases} \tag{38}$$

implying that

$$f_i(j) = \begin{cases} \lambda_j \cdot \sum_{k=j}^{i-1} \left(\overline{\rho}_k \dfrac{d}{1-\overline{\rho}} \; + \; d_k \right) , & j \neq i \qquad (39) \\ \lambda_i \cdot \dfrac{d}{1-\overline{\rho}} \; , & j = i \qquad (40) \end{cases}$$

In the **[AS] model**, the evolution of the state of the system is given by:

$$X_{i+1}^j = \begin{cases} X_i^j + A^j\Big(\sum_{m=1}^{X_i^i} \overline{B}_i^{(m)} \Big) + A^j(D_i), & j \neq i \\ A^i\Big(\sum_{m=1}^{X_i^i} \widehat{B}_i^{(m)} \Big) + A^i(D_i), & j = i \end{cases} \qquad (41)$$

where $\overline{B}_i^{(m)} \sim \overline{B}_i$ and $\widehat{B}_i^{(m)} \sim \widehat{B}_i$ for every m. Note that $\left\{ \overline{B}_i^{(m)} \right\}_{m=1}^{X_i^i}$ are independent of each other and so are $\left\{ \widehat{B}_i^{(m)} \right\}_{m=1}^{X_i^i}$. However, $\overline{B}_i^{(m)}$ and $\widehat{B}_i^{(m)}$ are not independent.

Then,

$$F_{i+1}(\underline{z}) = E\Big[\prod_{j=1}^{N} z_j^{X_{i+1}^j} \Big] = E\Bigg[\Bigg(\prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{X_i^j + A^j(\sum_{m=1}^{X_i^i} \overline{B}_i^{(m)})} \Bigg) \cdot z_i^{A^i(\sum_{m=1}^{X_i^i} \widehat{B}_i^{(m)})} \cdot \prod_{j=1}^{N} z_j^{A^j(D_i)} \Bigg]$$

$$= E\Bigg[\Bigg(\prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{X_i^j} \Bigg) \cdot E\Bigg(z_i^{A^i\left(\sum_{m=1}^{X_i^i} \widehat{B}_i^{(m)} \right)} \prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{A^j(\sum_{m=1}^{X_i^i} \overline{B}_i^{(m)})} \Bigg| \underline{X}_i \Bigg) \Bigg] \cdot E\Big[\prod_{j=1}^{N} z_j^{A^j(D_i)} \Big] \; . \qquad (42)$$

Let $\sigma(\underline{z}) \equiv \sum_{j=1}^{N} \lambda_j (1 - z_j)$ and $\sigma_i(\underline{z}) \equiv \sum_{\substack{j=1 \\ j\neq i}}^{N} \lambda_j (1 - z_j)$. Then, as the arrival is Poissonian,

$$E\left[\prod_{j=1}^{N} z_j^{A^j(D_i)} \right] = D_i^*(\sigma(\underline{z})) \; . \qquad (43)$$

We now use (13) and the fact that $\widehat{B}_i = \sum_{j=1}^{K_i} S_i^{-(j)} + S_i^+$ and write

$$E\Bigg[z_i^{A^i(\sum_{m=1}^{X_i^i} \widehat{B}_i^{(m)})} \Bigg(\prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{A^j(\sum_{m=1}^{X_i^i} \overline{B}_i(m))} \Bigg) \Bigg| \underline{X}_i \Bigg] = \Bigg\{ E\Bigg[z_i^{A^i(\widehat{B}_i)} \prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{A^j(\overline{B}_i)} \Bigg] \Bigg\}^{X_i^i}$$

$$= \Bigg\{ \sum_{k=0}^{\infty} (1-a_i)^k a_i \cdot E\Bigg[z_i^{A^i\left(S_i^+ + \sum_{m=1}^{k} S_i^{-(m)} \right)} \prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{A^j\left(S_i^+ + \sum_{m=1}^{k} [S_i^{-(m)} + V_i^{(m)}] \right)} \Bigg] \Bigg\}^{X_i^i}$$

$$= \left\{ \sum_{k=0}^{\infty} (1-a_i)^k a_i \cdot S_i^{+*}(\sigma(\underline{z})) \cdot \Big[S_i^{-*}(\sigma(\underline{z})) \cdot V_i^*(\sigma_i(\underline{z})) \Big]^k \right\}^{X_i^i} \tag{44}$$

$$= \left\{ \frac{a_i S_i^{+*}(\sigma(\underline{z}))}{1-(1-a_i)S_i^{-*}(\sigma(\underline{z})) \cdot V_i^*(\sigma_i(\underline{z}))} \right\}^{X_i^i} .$$

Combining (42), (43) and (44), we get:

$$F_{i+1}(\underline{z}) = D_i^*(\sigma(\underline{z})) \cdot E\left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{N} z_j^{X_i^j} \right) \cdot \left\{ \frac{a_i S_i^{+*}(\sigma(\underline{z}))}{1-(1-a_i)S_i^{-*}(\sigma(\underline{z})) \cdot V_i^*(\sigma_i(\underline{z}))} \right\}^{X_i^i} \right] \tag{45}$$

$$= D_i^*(\sigma(\underline{z})) \cdot F_i\Big(z_1, \dots, z_{i-1} , \frac{a_i S_i^{+*}(\sigma(\underline{z}))}{1-(1-a_i)S_i^{-*}(\sigma(\underline{z})) \cdot V_i^*(\sigma_i(\underline{z}))} , z_{i+1}, \dots, z_N \Big) .$$

Since $f_i(j) \equiv E[X_i^j] = \frac{\partial F_i(\underline{z})}{\partial z_j}\Big|_{\underline{z}=\underline{1}}$, we get, by taking derivatives of (45) or directly from (41), the following relations between the first-order moments of $\{X_i^j\}$:

$$f_{i+1}(j) = \begin{cases} f_i(j) + \lambda_j \overline{b}_i f_i(i) + \lambda_j d_i, & j \neq i \\ \lambda_i \widehat{b}_i f_i(i) + \lambda_i d_i, & j = i \end{cases} \tag{46}$$

Let η_i denote the "loss" of potential customers to queue i, during a generalized service time of a customer, due to arrival stoppage. That is $\underline{\eta_i \equiv \lambda_i(\overline{b}_i - \widehat{b}_i)}$. Thus, for $j = i$, we can write (46) as:

$$f_{i+1}(i) = \lambda_i \overline{b}_i f_i(i) - \eta_i f_i(i) + \lambda_i d_i . \tag{47}$$

We can use (46) and (47) to express $\{f_i(j)\}_{i\neq j}$ in terms of $\{f_i(i)\}$:

$$f_i(j) = \lambda_j \sum_{k=j}^{i-1} (\overline{b}_k f_k(k) + d_k) - \eta_j f_j(j) . \tag{48}$$

Thus, finding $\{f_i(i)\}_{i=1}^N$ readily gives all values of $\{f_i(j)\}$.

Summing (46) over all i, and using (47), we get:

$$\sum_{i=1}^N f_i(j) = \sum_{i=1}^N f_{i+1}(j) = \sum_{\substack{i=1 \\ i\neq j}}^N f_i(j) + \lambda_j \sum_{i=1}^N \overline{b}_i f_i(i) - \eta_j f_j(j) + \lambda_j d ,$$

implying that $f_j(j) + \eta_j f_j(j) = \lambda_j \Big(d + \sum_{i=1}^N \overline{b}_i f_i(i) \Big)$. That is,

$$f_j(j) = \frac{\lambda_j}{1+\eta_j} \Big(d + \sum_{i=1}^N \overline{b}_i f_i(i) \Big) . \tag{49}$$

Multiplying (49) by $\overline{b}_j$ and summing over all j lead to

$$\sum_{j=1}^{N} \overline{b}_j f_j(j) = \Big(d + \sum_{i=1}^{N} \overline{b}_i f_i(i)\Big) \sum_{j=1}^{N} \frac{\lambda_j \overline{b}_j}{1+\eta_j} \ . \tag{50}$$

Let $\overline{\rho}_j \equiv \frac{\lambda_j \overline{b}_j}{1+\eta_j}$. Note that for the [AC] model we have already defined $\overline{\rho}_j$ as $\lambda_j \overline{b}_j$. Nevertheless, the definition here for the [AS] model holds for the [AC] model as well, where, by its definition, $\eta_j \equiv 0$. We again use $\overline{\rho}$ to denote $\sum_{j=1}^{N} \overline{\rho}_j$. (In section 6 we'll show that $\overline{\rho}$ represents the total traffic load of the system). Thus, Eq. (50) is expressed as

$$\sum_{j=1}^{N} \overline{b}_j f_j(j) = \Big(d + \sum_{i=1}^{N} \overline{b}_i f_i(i)\Big)\overline{\rho} \ , \quad \text{leading to}$$

$$\sum_{i=1}^{N} \overline{b}_i f_i(i) = \frac{d\overline{\rho}}{1-\overline{\rho}} \ . \tag{51}$$

Substituting (51) in (49), we get

$$f_j(j) = \frac{\lambda_j}{1+\eta_j}\Big(d + \frac{d\overline{\rho}}{1-\overline{\rho}}\Big) = \frac{\lambda_j}{1+\eta_j} \cdot \frac{d}{1-\overline{\rho}} \ . \tag{52}$$

It will be shown that for all models, the mean cycle time is given by $E[C] = \frac{d}{1-\overline{\rho}}$. Hence

$$f_j(j) = \frac{\lambda_j}{1+\eta_j} E[C] \quad \text{for} \quad j = 1, 2, \ldots, N \ .$$

Substituting the values of $\{f_i(i)\}$ from (52) and (48) yields

$$f_i(j) = \lambda_j \left[\sum_{k=j}^{i-1} \Big(\overline{\rho}_k \frac{d}{1-\overline{\rho}} \ + \ d_k \Big) \ - \ \frac{\eta_j}{1+\eta_j} \frac{d}{1-\overline{\rho}} \right] \ . \tag{53}$$

4.2 Second-Order Moments

The second-order moments of $\{X_i^j\}$ are derived from the PGFs (37) for the [AC] model and (45) for the [AS] model. Let

$$f_i(j,k) \ = \ E[X_i^j X_i^k] = \left.\frac{\partial^2 F_i(\underline{z})}{\partial z_j \partial z_k}\right|_{\underline{z}=\underline{1}} \quad (i,j,k = 1,\ldots,N; j \neq k) \ ,$$

$$f_i(j,j) \ = \ E[X_i^j (X_i^j - 1)] = \left.\frac{\partial^2 F_i(\underline{z})}{\partial z_j^2}\right|_{\underline{z}=\underline{1}} \quad (i,j = 1,\ldots,N) \ . \tag{54}$$

Eqs. (54) define a set of N^3 linear equations, that its solution gives the values of the second-order moments $\{f_i(j,k)\}$.

In the **[AC] model**, Eqs. (54) are given by (see also Takagi [10])

$$\left.\begin{array}{l}
f_{i+1}(i,i) = \lambda_i^2 d_i^{(2)} + \lambda_i^2 f_i(i)\cdot[2d_i\overline{b}_i + \overline{b}_i^{(2)}] + \lambda_i^2 \overline{b}_i^2 f_i(i,i) \\
\left.\begin{array}{l} f_{i+1}(i,j) = \lambda_i\lambda_j d_i^{(2)} + \lambda_i\lambda_j f_i(i)\cdot[2d_i\overline{b}_i + \overline{b}_i^{(2)}] + \lambda_i\lambda_j \overline{b}_i^2 f_i(i,i) \\ \qquad + \lambda_i d_i f_i(j) + \lambda_i \overline{b}_i f_i(i,j) \end{array}\right\} j \neq i \\
\left.\begin{array}{l} f_{i+1}(j,k) = \lambda_j\lambda_k d_i^{(2)} + \lambda_j\lambda_k f_i(i)\cdot[2d_i\overline{b}_i + \overline{b}_i^{(2)}] + \lambda_j\lambda_k \overline{b}_i^2 f_i(i,i) \\ \qquad + \lambda_j d_i f_i(k) + \lambda_k d_i f_i(j) + \lambda_k \overline{b}_i f_i(i,j) + \lambda_j \overline{b}_i f_i(i,k) \\ \qquad + f_i(j,k) \end{array}\right\} \begin{array}{l} j\neq i \\ k \neq i \end{array}
\end{array}\right\} \qquad (55)$$

In the **[AS] model**, Eqs. (54) become (after a lengthy calculation)

$$\left.\begin{array}{l}
f_{i+1}(i,i) = \lambda_i^2 d_i^{(2)} + \lambda_i^2 f_i(i)\cdot[2d_i\widehat{b}_i + \widehat{b}_i^{(2)}] + \lambda_i^2 \widehat{b}_i^2 f_i(i,i) \\
\left.\begin{array}{l} f_{i+1}(i,j) = \lambda_i\lambda_j d_i^{(2)} + \lambda_i\lambda_j f_i(i)\cdot[d_i(\overline{b}_i + \widehat{b}_i) + E(\overline{B}_i\widehat{B}_i)] + \lambda_i\lambda_j \overline{b}_i\widehat{b}_i f_i(i,i) \\ \qquad + \lambda_i d_i f_i(j) + \lambda_i \widehat{b}_i f_i(i,j) \end{array}\right\} j \neq i \\
\left.\begin{array}{l} f_{i+1}(j,k) = \lambda_j\lambda_k d_i^{(2)} + \lambda_j\lambda_k f_i(i)\cdot[2d_i\overline{b}_i + \overline{b}_i^{(2)}] + \lambda_j\lambda_k \overline{b}_i^2 f_i(i,i) \\ \qquad + \lambda_j d_i f_i(k) + \lambda_k d_i f_i(j) + \lambda_k \overline{b}_i f_i(i,j) + \lambda_j \overline{b}_i f_i(i,k) \\ \qquad + f_i(j,k) \end{array}\right\} \begin{array}{l} j\neq i \\ k \neq i \end{array}
\end{array}\right\} \qquad (56)$$

Note the following:

(i) The expressions for $f_{i+1}(i,i)$ are similar in the [AC] and [AS] models, except that in the latter the moments of $\widehat{B}_i$ replace those of $\overline{B}_i$ in the former.

(ii) The expressions for $f_{i+1}(j,k)$ when $j \neq i$ and $k \neq i$ are the same for both models.

The symmetric case.

When all stations are (stochastically) identical, we set, for all i.

$$\begin{array}{l}
\lambda_i = \lambda_0,\ d_i = d_0, \quad d_i^{(2)} = d_0^{(2)}\ , \\
\overline{b}_i = \overline{b}_0, \quad \overline{b}_i^{(2)} = \overline{b}_0^{(2)}, \quad \widehat{b}_i = \widehat{b}_0, \quad \widehat{b}_i^{(2)} = \widehat{b}_0^{(2)}\ , \\
\eta_i = \eta_0, \quad \overline{\rho}_i = \overline{\rho}_0, \quad E[\overline{B}_i\widehat{B}_i] = E[\overline{B}_0\widehat{B}_0]\ .
\end{array}$$

Now, in the **[AC] model**, $f_i(i,i)$ is given by (see Hashida [6], Takagi [10]:

$$f_i(i,i) = \frac{N\lambda_0^2}{(1+\overline{\rho}_0)\cdot(1-\overline{\rho})}\left\{ d_0^{(2)} + \frac{(N-1)\cdot d_0^2}{1-\overline{\rho}} + \frac{2Nd_0^2\overline{\rho}_0}{1-\overline{\rho}} + \frac{N\lambda_0 d_0 \overline{b}_0^{(2)}}{1-\overline{\rho}} \right\}, \qquad (57)$$

and in the **[AS] model** we get:

$$f_i(i,i) = \tag{58}$$

$$\frac{N\lambda_0^2}{(1+\lambda_0\widehat{b}_0)(1-\overline{\rho})(1+\eta_0)}\left\{d_0^{(2)} + \frac{(N-1)\cdot d_0^2}{1-\overline{\rho}} + \frac{2Nd_0^2}{1-\overline{\rho}}\,\frac{\lambda_0\widehat{b}_0}{1+\eta_0} \;+\; \frac{N\lambda_0 d_0}{1-\overline{\rho}}\,\frac{\beta}{1+\eta_0}\right\}$$

where β is the following convex combination of $\overline{b}_0^{(2)}$, $\widehat{b}_0^{(2)}$ and $E[\overline{B}_0\widehat{B}_0]$:

$$\beta = \frac{(N-1)\cdot(1-\overline{\rho}_0)}{N}\,\overline{b}_0^{(2)} \;+\; \frac{1-(N-1)\cdot\overline{\rho}_0}{N}\,\widehat{b}_0^{(2)} \;+\; \frac{2(N-1)\cdot\overline{\rho}_0}{N}\,E[\overline{B}_0\widehat{B}_0]\ . \tag{59}$$

4.3 Number of Customers at Departing Instants

Let L_i be the number of customers left behind by an arbitrary departing customer from queue i, and let $Q_i(z)$ be its PGF. Let M_i be the total number of customers served in queue i during a visit of the server to that queue, and let $L_i(n)$ be the sequence of random variables denoting the number of customers that the n-th departing customer from queue i (in the current visit of the server) leaves behind him ($n = 1, 2, \ldots, M_i$). Then it is well known (cf. Takagi [10])

$$Q_i(z) = \frac{E\Big[\sum_{n=1}^{M_i} z^{Li(n)}\Big]}{E[M_i]}\ . \tag{60}$$

Let $G_i(z) \equiv E[z^{X_i^i}] = F_i(1, \ldots, 1, z, 1, \ldots, 1)$ (where z is in the i-th coordinate).

In the **[AC] model** the PGF of L_i and its expected value are given by (see Takagi [10], Yechiali [12]):

$$Q_i(z) = \frac{1-\overline{\rho}}{\lambda_i d}\ \cdot\ \frac{\overline{B}_i^*(\lambda_i-\lambda_i z)}{z-\overline{B}_i^*(\lambda_i-\lambda_i z)}\ \cdot\ [G_i(z) - G_i(\overline{B}_i^*(\lambda_i-\lambda_i z))] \tag{61}$$

and

$$E[L_i] = \overline{\rho}_i \;+\; \frac{(1+\overline{\rho}_i)f_i(i,i)(1-\overline{\rho})}{2\lambda_i d} \;=\; \overline{\rho}_i \;+\; \frac{(1+\overline{\rho}_i)f_i(i,i)}{2f_i(i)}\ . \tag{62}$$

In the **[AS] model**:

$$L_i(n) = X_i^i - n + A^i\Big(\sum_{m=1}^{n}\widehat{B}_i^{(m)}\Big) \quad\text{and}\quad M_i = X_i^i, \quad\text{where}\quad \widehat{B}_i^{(m)} \sim \widehat{B}_i\ .$$

Hence,

$$E\Big[\sum_{n=1}^{M_i} z^{L_i(n)}\Big] \;=\; E\left[E\Big(\sum_{n=1}^{X_i^i} z^{X_i^i-n+A^i(\sum_{m=1}^{n}\widehat{B}_i^{(m)})}\Big|X_i^i\Big)\right] = E\left[z^{X_i^i}\sum_{n=1}^{X_i^i}\Big(\frac{E[z^{A^i(\widehat{B}_i)}]}{z}\Big)^n\right]$$

$$= E\left[z^{X_i^i} \cdot \frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z} \cdot \frac{1 - \left(\frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z}\right)^{X_i^i}}{1 - \frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z}} \right] \tag{63}$$

$$= \frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z - \widehat{B}_i^*(\lambda_i - \lambda_i z)} \cdot E\left[z^{X_i^i} - (\widehat{B}_i^*(\lambda_i - \lambda_z))^{X_i^i} \right]$$

$$= \frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z - \widehat{B}_i^*(\lambda_i - \lambda_i z)} \cdot \left[G_i(z) - G_i(\widehat{B}_i^*(\lambda_i - \lambda_i z)) \right]$$

and

$$E[M_i] = E[X_i^i] = f_i(i) \; . \tag{64}$$

Combining (52), (60), (63) and (64), we get

$$Q_i(z) = \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \frac{\widehat{B}_i^*(\lambda_i - \lambda_i z)}{z - \widehat{B}_i^*(\lambda_i - \lambda_i z)} \cdot \left[G_i(z) - G_i(\widehat{B}_i^*(\lambda_i - \lambda_i z)) \right] . \tag{65}$$

Differentiating (65) and performing the required calculations lead to

$$\begin{aligned} E[L_i] &= Q_i'(1) = \lambda_i \widehat{b}_i + \frac{(1+\lambda_i \widehat{b}_i) f_i(i,i)}{2 f_i(i)} \\ &= \lambda_i \widehat{b}_i + \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \frac{(1+\lambda_i \widehat{b}_i) f_i(i,i)}{2} . \end{aligned} \tag{66}$$

Note the similarity between (66) and (62) where $\widehat{b}_i$ replaces $\overline{b}_i$.

In the **symmetric case**, the mean number of customers is given for the **[AC] model** by

$$E[L_i] = \overline{\rho}_0 + \frac{\lambda_0 d_0^{(2)}}{2 d_0} + \frac{(N-1)\cdot \lambda_0 d_0}{2(1-\overline{\rho})} + \frac{N \lambda_0 d_0 \overline{\rho}_0}{1-\overline{\rho}} + \frac{N \lambda_0^2 \overline{b}_0^{(2)}}{2(1-\overline{\rho})} , \tag{67}$$

and in the **[AS] model** it becomes

$$E[L_i] = \lambda_0 \widehat{b}_0 + \frac{\lambda_0 d_0^{(2)}}{2 d_0} + \frac{(N-1)\cdot \lambda_0 d_0}{2(1-\overline{\rho})} + \frac{N \lambda_0^2 d_0 \widehat{b}_0}{(1-\overline{\rho})(1+\eta_0)} + \frac{N \lambda_0^2 \beta}{2(1-\overline{\rho})(1+\eta_0)} . \tag{68}$$

4.4 Waiting and Sojourn Times

Let W_{q_i} denote the waiting time (excluding service time) of an arbitrary customer at queue i, and let $\widehat{W}_{q_i}$ denote the period of time, out of W_{q_i}, in which the arrival process to queue i is active.

In the **[AC] model** $\widehat{W}_{q_i} = W_{q_i}$, and their LST and expected value are given by (see Takagi [10], Yechiali [12]):

$$W^*_{q_i}(\omega) = \frac{1-\overline{\rho}}{\lambda_i d} \cdot \frac{G_i(1-\omega/\lambda_i) - G_i(\overline{B}^*_i(\omega))}{1-\omega/\lambda_i - \overline{B}^*_i(\omega)} \tag{69}$$

and

$$E[W_{q_i}] = \frac{1-\overline{\rho}}{d} \cdot \frac{(1+\overline{\rho}_i) f_i(i,i)}{2\lambda_i^2} \tag{70}$$

In the **[AS] model**:

The number of customers left behind by a departing customer from queue i is the number of arrivals to that queue during the sojourn time of this customer in the system. Therefore

$$Q_i(z) = \widehat{W}^*_{q_i}(\lambda_i - \lambda_i z) \cdot \widehat{B}^*_i(\lambda_i - \lambda_i z)\ . \tag{71}$$

Hence, from (65),

$$\widehat{W}^*_{q_i}(\omega) = \frac{Q_i(1-\omega/\lambda_i)}{\widehat{B}^*_i(\omega)} = \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \cdot \frac{G_i(1-\omega/\lambda_i) - G_i(\widehat{B}^*_i(\omega))}{1-\omega/\lambda_i - \widehat{B}^*_i(\omega)} \tag{72}$$

To find the expected value of $\widehat{W}_{q_i}$, we can differentiate (72) or use Little's Law:

$$E[\widehat{W}_{q_i}] = -\widehat{W}^{*'}_{q_i}(0) = \frac{E[L_i]}{\lambda_i} - \widehat{b}_i = \frac{1+\lambda_i \widehat{b}_i}{2\lambda_i} \cdot \frac{f_i(i,i)}{f_i(i)}\ . \tag{73}$$

L_i is also the number of customers found in the queue by an arriving customer. This follows since the system-state changes by unit jumps (see Kleinrock [7]). Then the waiting time of a customer in queue i is $\widehat{W}_{q_i}$ with the addition of the arrival stoppage periods that took place during the service periods of all the customers who where present in the system when he arrived:

$$W_{q_i} = \widehat{W}_{q_i} + \sum_{j=1}^{L_i} \left[\overline{B}^{(j)}_i - \widehat{B}^{(j)}_i\right]\ . \tag{74}$$

Thus,

$$\begin{aligned} E[W_{q_i}] &= E\Big[\widehat{W}_{q_i}\Big] + E[L_i] \cdot E\Big[\overline{B}_i - \widehat{B}_i\Big] = \frac{E[L_i]}{\lambda_i} - \widehat{b}_i + E[L_i] \cdot (\overline{b}_i - \widehat{b}_i) \\ &= E[L_i] \cdot \frac{1+\eta_i}{\lambda_i} - \widehat{b}_i\ . \end{aligned} \tag{75}$$

Substituting $E[L_i]$ from (66) in (75) we get:

$$E[W_{q_i}] = \left(\lambda_i\widehat{b}_i + \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \cdot \frac{(1+\lambda_i\widehat{b}_i)f_i(i,i)}{2}\right) \cdot \frac{1+\eta_i}{\lambda_i} - \widehat{b}_i$$

$$= \left(\frac{1+\eta_i}{\lambda_i}\right)^2 \cdot \frac{1-\overline{\rho}}{d} \cdot \frac{(1+\lambda_i\widehat{b}_i)f_i(i,i)}{2} + \eta_i\widehat{b}_i \ . \tag{76}$$

In the **symmetric case**, the mean waiting time is given for the **[AC] model** by

$$E[W_{q_i}] = \frac{d_0^{(2)}}{2d_0} + \frac{(N-1)\cdot d_0}{2(1-\overline{\rho})} + \frac{Nd_0\overline{\rho}_0}{1-\overline{\rho}} + \frac{N\lambda_0\overline{b}_0^{(2)}}{2(1-\overline{\rho})} \ , \tag{77}$$

and in the **[AS] model**, using (75) and (68), it becomes

$$E[W_{q_i}] = \frac{(1+\eta_0)\cdot d_0^{(2)}}{2d_0} + \frac{(N-1)\cdot(1+\eta_0)\cdot d_0}{2(1-\overline{\rho})} + \frac{N\lambda_0 d_0\widehat{b}_0}{1-\overline{\rho}} + \frac{N\lambda_0\beta}{2(1-\overline{\rho})} + \eta_0\widehat{b}_0 \ . \tag{78}$$

Finally, the mean sojourn time of an arbitrary customer in queue i is given by

$$E[W_i] = E[W_{q_i}] + \overline{b}_i = E[L_i] \cdot \frac{1+\eta_i}{\lambda_i} - \widehat{b}_i + \overline{b}_i$$

$$= \frac{E[L_i]\cdot(1+\eta_i)+\eta_i}{\lambda_i} \ . \tag{79}$$

5 The Exhaustive Regime

5.1 System-State: Law of Motion, PGFs and First Moments

In the exhaustive regime, in each visit, the server leaves a station only when it becomes empty.

Let Θ_i denote the length of a 'busy period' generated by a single customer in queue i. Let $\Theta_i^*(\cdot)$, θ_i and $\theta_i^{(2)}$ denote the LST of Θ_i, its mean and its second moment, respectively.

The evolution laws for the system-state are:

$$X_{i+1}^j = \begin{cases} X_i^j + A^j\Big(\sum_{m=1}^{X_i^i}\Theta_i^{(m)}\Big) + A^j(D_i), & j \neq i \\ A^i(D_i), & j = i \end{cases} \tag{80}$$

where $\Theta_i^{(m)} \sim \Theta_i$ for every m, and they are mutually independent. Then (see Takagi [10], Yechiali [12])

$$F_{i+1}(\underline{z}) = F_i[z_1,\ldots,z_{i-1},\Theta_i^*(\sigma_i(\underline{z})),z_{i+1},\ldots,z_N]\cdot D_i^*(\sigma(\underline{z})) \ , \tag{81}$$

and by differentiating (81) or directly from (80),

$$f_{i+1}(j) = \begin{cases} f_i(j) + \lambda_j \theta_i f_i(i) + \lambda_j d_i, & j \neq i \\ \lambda_i d_i, & j = i \end{cases} \tag{82}$$

In the **[AC] model**, Θ_i is a *regular* busy period of an $M/G/1$ type, but with service times $\overline{B}_i$ to customers in queue i. It is well known (see [7]) that

$$\Theta_i^*(\omega) = \overline{B}_i^*[\omega + \lambda_i \cdot (1 - \Theta_i^*(\omega))] \ , \tag{83}$$

$$\theta_i = E[\Theta_i] = \frac{\overline{b}_i}{1 - \overline{\rho}_i} \ , \tag{84}$$

$$\theta_i^{(2)} = E[\Theta_i^2] = \frac{\overline{b}_i^{(2)}}{(1 - \overline{\rho}_i)^3} \ . \tag{85}$$

Therefore, the corresponding polling model may be viewed as a 'standard' one for which (see [10], [12])

$$f_i(j) = \begin{cases} \lambda_j \cdot \left[\frac{d}{1-\overline{\rho}} \cdot \sum_{k=j+1}^{i-1} \overline{\rho}_k + \sum_{k=j}^{i-1} d_k \right], & j \neq i \quad \text{(86a)} \\ \lambda_i (1 - \overline{\rho}_i) \ \frac{d}{1-\overline{\rho}} \ , & j = i \quad \text{(86b)} \end{cases}$$

In the **[AS] model**:

$$\Theta_i = \overline{B}_i + \sum_{m=1}^{A^i(\widehat{B}_i)} \Theta_i^{(m)} \ . \tag{87}$$

Hence,

$$\begin{aligned} \Theta_i^*(\omega) &= E\Big\{ \sum_{n=0}^{\infty} P[A^i(\widehat{B}_i) = n | \widehat{B}_i] \cdot E[e^{-\omega \Theta_i} | \overline{B}_i; \widehat{B}_i; A^i(\widehat{B}_i) = n] \Big\} \\ &= E\left\{ \sum_{n=0}^{\infty} e^{-\lambda_i \widehat{B}_i} \frac{(\lambda_i \widehat{B}_i)^n}{n!} \cdot e^{-\omega \overline{B}_i} \cdot E\left[e^{-\omega \sum_{m=1}^{n} \Theta_i^{(m)}} \right] \right\} \\ &= E\left\{ e^{-\lambda_i \widehat{B}_i - \omega \overline{B}_i} \sum_{n=0}^{\infty} \frac{(\lambda_i \widehat{B}_i \Theta_i^*(\omega))^n}{n!} \right\} = E\left[e^{-\omega \overline{B}_i - \lambda_i (1 - \Theta_i^*(\omega)) \widehat{B}_i} \right] \ . \end{aligned} \tag{88}$$

Using (13), the definition of $\widehat{B}_i$, and by conditioning on K_i we get

$$\Theta_i^*(\omega) = \sum_{k=0}^{\infty} (1 - a_i)^k a_i E\left[e^{-\omega \left[S_i^+ + \sum_{m=1}^{k} (S_i^{-(m)} + V_i^{(m)}) \right] - \lambda_i (1 - \Theta_i^*(\omega)) \left[S_i^+ + \sum_{m=1}^{k} S_i^{-(m)} \right]} \right]$$

$$= a_i S_i^{+*}[\omega + \lambda_i(1-\Theta_i^*(\omega))] \cdot \sum_{k=0}^{\infty} (1-a_i)^k \cdot \{S_i^{-*}[\omega + \lambda_i(1-\Theta_i^*(\omega))]\}^k \cdot [V_i^*(\omega)]^k$$

$$= \frac{a_i S_i^{+*}[\omega + \lambda_i(1-\Theta_i^*(\omega))]}{1-(1-a_i)\cdot S_i^{-*}[\omega+\lambda_i(1-\theta_i^*(\omega))]\cdot V_i^*(\omega)} \ . \qquad (89)$$

The first and second moments of the busy period in the [AS] model can be calculated by differentiating (89), or directly from (87), as follows:

$$\theta_i = \overline{b}_i + \lambda_i \widehat{b}_i \theta_i \ , \quad \text{implying}$$

$$\theta_i = \frac{\overline{b}_i}{1-\lambda_i \widehat{b}_i} \ . \qquad (90)$$

$$\theta_i^{(2)} = E[\Theta_i^2] = \overline{b}_i^{(2)} + 2\lambda_i E[\overline{B}_i \widehat{B}_i]\cdot\theta_i + \lambda_i \widehat{b}_i \theta_i^{(2)} + E\Big[A^i(\widehat{B}_i)\cdot\Big(A^i(\widehat{B}_i)-1\Big)\Big]\cdot\theta_i^2 \ . \qquad (91)$$

Using (90) and the definition of $\overline{\rho}_i$, we get:

$$\lambda_i\theta_i = \frac{\lambda_i\overline{b}_i}{1-\lambda_i\widehat{b}_i} = \frac{\lambda_i\overline{b}_i}{1+\eta_i}\Bigg/\left(\frac{1-\lambda_i\widehat{b}_i}{1+\eta_i}\right) = \overline{\rho}_i\Bigg/\left(\frac{1+\eta_i-\lambda_i\overline{b}_i}{1+\eta_i}\right) = \frac{\overline{\rho}_i}{1-\overline{\rho}_i} \ . \qquad (92)$$

Now,

$$E\Big[A^i(\widehat{B}_i)\cdot\Big(A^i(\widehat{B}_i)-1\Big)\Big] = E_{\widehat{B}_i}\left[\sum_{k=0}^{\infty} k(k-1)\cdot e^{-\lambda_i\widehat{B}_i}\frac{(\lambda_i\widehat{B}_i)^k}{k!}\right]$$

$$= E_{\widehat{B}_i}\left[(\lambda_i\widehat{B}_i)^2 e^{-\lambda_i\widehat{B}_i}\sum_{k=2}^{\infty}\frac{(\lambda_i\widehat{B}_i)^{k-2}}{(k-2)!}\right] = \lambda_i^2\cdot\widehat{b}_i^{(2)} \ . \qquad (93)$$

By combining (91), (92) and (93), we get:

$$\theta_i^{(2)} = \overline{b}_i^{(2)} + \frac{2\overline{\rho}_i}{1-\overline{\rho}_i}E[\overline{B}_i\widehat{B}_i] + \lambda_i\widehat{b}_i\theta_i^{(2)} + \left(\frac{\overline{\rho}_i}{1-\overline{\rho}_i}\right)^2\widehat{b}_i^{(2)}$$

leading to

$$\theta_i^{(2)} = \frac{(1-\overline{\rho}_i)^2\overline{b}_i^{(2)} + 2\overline{\rho}_i(1-\overline{\rho}_i)E[\overline{B}_i\widehat{B}_i] + \overline{\rho}_i^2\widehat{b}_i^{(2)}}{(1-\lambda_i\widehat{b}_i)\cdot(1-\overline{\rho}_i)^2}$$

$$= \frac{(1-\overline{\rho}_i)^2\overline{b}_i^{(2)} + 2\overline{\rho}_i(1-\overline{\rho}_i)E[\overline{B}_i\widehat{B}_i] + \overline{\rho}_i^2\widehat{b}_i^{(2)}}{(1+\eta_i)\cdot(1-\overline{\rho}_i)^3} \qquad (94)$$

(Note, while comparing to (85), that the numerator of (94) is a convex combination of $\overline{b}_i^{(2)}, \widehat{b}_i^{(2)}$ and $E[\overline{B}_i\widehat{B}_i]$). Summing (82) over all i gives

$$\sum_{i=1}^{N} f_{i+1}(j) = \sum_{\substack{i=1\\ i\neq j}}^{N} f_i(j) + \lambda_j\sum_{\substack{i=1\\ i\neq j}}^{N}\theta_i f_i(i) + \lambda_j d \Rightarrow f_j(j) = \lambda_j\sum_{i=1}^{N}\theta_i f_i(i) - \lambda_j\theta_j f_j(j) + \lambda_j d \ .$$

Thus,

$$f_j(j) = \frac{\lambda_j}{1+\lambda_j\theta_j} \cdot \Big(d + \sum_{i=1}^{N} \theta_i f_i(i)\Big) \ . \tag{95}$$

Multiplying (95) by θ_j and summing over all j yield

$$\sum_{j=1}^{N} \theta_j f_j(j) = \Big(d + \sum_{i=1}^{N} \theta_i f_i(i)\Big) \cdot \sum_{j=1}^{N} \frac{\lambda_j\theta_j}{1+\lambda_j\theta_j} \ . \tag{96}$$

Using (92) for the expression of $\lambda_j\theta_j$, we get

$$\frac{\lambda_j\theta_j}{1+\lambda_j\theta_j} = \frac{\overline{\rho}_j}{1-\overline{\rho}_j} \quad \frac{1}{1-\overline{\rho}_j} = \overline{\rho}_j \ . \tag{97}$$

Substituting (97) in (96) leads to

$$\sum_{j=1}^{N} \theta_j f_j(j) = \Big(d + \sum_{i=1}^{N} \theta_i f_i(i)\Big) \cdot \overline{\rho}$$

from which

$$\sum_{i=1}^{N} \theta_i f_i(i) = \frac{d\overline{\rho}}{1-\overline{\rho}} \ . \tag{98}$$

Now, using (92) again,

$$\frac{\lambda_j}{1+\lambda_j\theta_j} = \lambda_j \quad \frac{1}{1-\overline{\rho}_j} = \lambda_j(1-\overline{\rho}_j) \ . \tag{99}$$

Substituting results (98) and (99) in (95) we get

$$f_j(j) = \lambda_j(1-\overline{\rho}_j) \cdot \Big(d + \frac{d\overline{\rho}}{1-\overline{\rho}}\Big) = \lambda_j(1-\overline{\rho}_j) \cdot \frac{d}{1-\overline{\rho}} \ . \tag{100}$$

Using (82) and by proper summation we have

$$f_i(j) = \lambda_j \cdot \Big[\sum_{k=j+1}^{i-1} \theta_k f_k(k) + \sum_{k=j}^{i-1} d_k \Big] \ . \tag{101}$$

Combining (92) with (100) leads to:

$$\theta_k f_k(k) = \overline{\rho}_k \frac{d}{1-\overline{\rho}} \ , \tag{102}$$

and therefore, using (101),

$$f_i(j) = \lambda_j \Big[\frac{d}{1-\overline{\rho}} \cdot \sum_{k=j+1}^{i-1} \overline{\rho}_k + \sum_{k=j}^{i-1} d_k\Big] \tag{103}$$

Note that expressions (100) and (103) for the first-order moments in the [AS] model now look 'the same' as in the [AC] model. (See (86b) and (86a).) However, the values of the $\{\overline{\rho}_i\}$ in each model are *different*.

5.2 Second-Order Moments

In both [**AC**] **model** and [**AS**] **model**, Eqs. (54) have the 'same' expressions (of course, θ_i and $\theta_i^{(2)}$ have different values in each model). After lengthy calculations we derive:

$$\left.\begin{array}{l} f_{i+1}(i,i) = \lambda_i^2 d_i^{(2)} \\ \left. f_{i+1}(i,j) = \lambda_i\lambda_j d_i^{(2)} + \lambda_i d_i[f_i(j) + \lambda_j\theta_i f_i(i)] \right\} j \neq i \\ \left. \begin{array}{l} f_{i+1}(j,k) = \lambda_j\lambda_k d_i^{(2)} + \lambda_i\lambda_k f_i(i)\cdot[2d_i\theta_i + \theta_i^{(2)}] + \lambda_j\lambda_k\theta_i^2 f_i(i,i) \\ \qquad + \lambda_j d_i f_i(k) + \lambda_k d_i f_i(j) + \lambda_k\theta_i f_i(i,j) + \lambda_j\theta_i f_i(i,k) \\ \qquad + f_i(j,k) \end{array} \right\} \begin{array}{l} j\neq i \\ k \neq i \end{array} \end{array}\right\} \tag{104}$$

In the **symmetric case**, in both [**AC**] **model** and [**AS**] **model**, using similar definitions as for the Gated regime, we obtain:

$$f_i(i,i) = \frac{N\lambda_0^2(1-\overline{\rho}_0)}{(1-\overline{\rho})} \cdot \left\{ d_0^{(2)} + \frac{(N-1)\cdot d_0^2}{1-\overline{\rho}} + \frac{(N-1)\cdot\lambda_0 d_o(1-\overline{\rho}_0)^2}{1-\overline{\rho}}\,\theta_0^{(2)} \right\}, \tag{105}$$

where we set $\theta_i = \theta_0$ and $\theta_i^{(2)} = \theta_0^{(2)}$ for $i = 1, 2, \ldots, N$.

5.3 PGF and Mean of Number of Customers

We use (60) again to find the PGF of L_i and its expected value. In the [**AC**] **model**, the expressions are given by (see [10], [12])

$$Q_i(z) = \frac{1-\overline{\rho}}{\lambda_i d} \cdot \frac{\overline{B}_i^*(\lambda_i - \lambda_i z)}{z - \overline{B}_i^*(\lambda_i - \lambda_i z)} \cdot [G_i(z) - 1], \tag{106}$$

$$E[L_i] = \overline{\rho}_i + \frac{\lambda_i^2 \cdot \overline{b}_i^{(2)}}{2(1-\overline{\rho}_i)} + \frac{1-\overline{\rho}}{d} \cdot \frac{f_i(i,i)}{2\lambda_i(1-\overline{\rho}_i)} = \overline{\rho}_i + \frac{\lambda_i^2 \cdot \overline{b}_i^{(2)}}{2(1-\overline{\rho}_i)} + \frac{f_i(i,i)}{2f_i(i)}. \tag{107}$$

The [**AS**] **model** requires additional calculations: Let $\widehat{\Theta}_i$ denote the period of time, out of Θ_i, in which customers arrive to queue i, and let $\widehat{\Theta}_i^*(\cdot)$ and $\widehat{\theta}_i$ denote its LST and its expected value, respectively. Then, as in (83) and (84),

$$\widehat{\Theta}_i^*(\omega) = \widehat{B}_i^*[\omega + \lambda_i\cdot(1 - \widehat{\Theta}_i^*(\omega))] \tag{108}$$

and

$$\widehat{\theta}_i = E[\widehat{\Theta}_i] = \frac{\widehat{b}_i}{1-\lambda_i\widehat{b}_i}. \tag{109}$$

Now, using (100), and (109),

$$E[M_i] = f_i(i)\cdot[1+\lambda_i\widehat{\theta}_i] = \lambda_i(1-\overline{\rho}_i)\frac{d}{1-\overline{\rho}}\Big[1 + \frac{\lambda_i\widehat{b}_i}{1-\lambda_i\widehat{b}_i}\Big]$$

$$= \lambda_i \frac{1-\lambda_i \widehat{b}_i}{1+\eta_i} \cdot \frac{d}{1-\overline{\rho}} \cdot \frac{1}{1-\lambda_i \widehat{b}_i} = \frac{\lambda_i}{1+\eta_i} \cdot \frac{d}{1-\overline{\rho}} \,. \tag{110}$$

$$E\Big[\sum_{n=1}^{M_i} z^{L_i(n)}\Big] = \frac{P_i(z)}{z-P_i(z)}[G_i(z)-1] \quad \text{(see Takagi [10])} \,, \tag{111}$$

where $P_i(z)$ is the PGF of the number of customers arrived to queue i during a (generalized) service time of a single customer. Then, for the [AS] model,

$$P_i(z) = \widehat{B}_i^*(\lambda_i(1-z)) \,. \tag{112}$$

Combining (60), (110), (111) and (112) gives the PGF of L_i:

$$Q_i(z) = \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \cdot \frac{\widehat{B}_i^*(\lambda_i-\lambda_i z)}{z-\widehat{B}_i^*(\lambda_i-\lambda_i z)} \, [G_i(z)-1] \,. \tag{113}$$

Differentiating (113) and performing some calculations lead to

$$\begin{aligned} E[L_i] &= Q_i'(1) = \lambda_i \widehat{b}_i + \frac{\lambda_i^2 \cdot \widehat{b}_i^{(2)}}{2(1-\lambda_i \widehat{b}_i)} + \frac{f_i(i,i)}{2f_i(i)} \\ &= \lambda_i \widehat{b}_i + \frac{\lambda_i^2 \cdot \widehat{b}_i^{(2)}}{2(1-\lambda_i \widehat{b}_i)} + \frac{1-\overline{\rho}}{d} \cdot \frac{f_i(i,i)}{2\lambda_i(1-\overline{\rho}_i)} \,. \end{aligned} \tag{114}$$

In the **symmetric case**, the mean number of customers is given for the **[AC] model** by (see Takagi [10]):

$$E[L_i] = \overline{\rho}_0 + \frac{\lambda_0 d_0^{(2)}}{2d_0} + \frac{(N-1)\lambda_0 d_0}{2(1-\overline{\rho})} + \frac{N\lambda_0^2 \overline{b}_0^{(2)}}{2(1-\overline{\rho})} \,, \tag{115}$$

and in the **[AS] model** it becomes

$$E[L_i] = \lambda_0 \widehat{b}_0 + \frac{\lambda_0 d_0^{(2)}}{2d_0} + \frac{(N-1)\lambda_0 d_0}{2(1-\overline{\rho})} + \frac{N\lambda_0^2 \beta}{2(1-\overline{\rho})(1+\eta_0)} \,. \tag{116}$$

5.4 Waiting and Sojourn Times

In the **[AC] model** the LST and the expected value of the waiting times are given by (see [10], [12]):

$$W_{q_i}^*(\omega) = \frac{1-\overline{\rho}}{\lambda_i d} \cdot \frac{G_i(1-\omega/\lambda_i)-1}{1-\omega/\lambda_i - \overline{B}_i^*(\omega)} \,; \tag{117}$$

$$E[W_{q_i}] = \frac{\lambda_i \overline{b}_i^{(2)}}{2(1-\overline{\rho}_i)} + \frac{1-\overline{\rho}}{d} \cdot \frac{f_i(i,i)}{2\lambda_i^2(1-\overline{\rho}_i)} \,. \tag{118}$$

In **[AS] model**, define $\widehat{W}_{q_i}$ (as in the Gated regime) to be the period of time out of W_{q_i}, in which the arrival process to queue i is active.

Then, using the general relation (71) and (113),

$$\widehat{W}^*_{q_i}(\omega) = \frac{Q_i(1-\omega/\lambda_i)}{\widehat{B}^*_i(\omega)} \;=\; \frac{1+\eta_i}{\lambda_i} \cdot \frac{1-\overline{\rho}}{d} \cdot \frac{G_i(1-\omega/\lambda_i)-1}{1-\omega/\lambda_i - \widehat{B}^*_i(\omega)} \tag{119}$$

To find the expected value of $\widehat{W}_{q_i}$, we can differentiate (119) or use Little's Law:

$$E[\widehat{W}_{q_i}] = -\widehat{W}^{*'}_{q_i}(0) = \frac{E[L_i]}{\lambda_i} \;-\; \widehat{b}_i = \frac{\lambda_i \widehat{b}_i^{(2)}}{2(1-\lambda_i\widehat{b}_i)} \;+\; \frac{f_i(i,i)}{2\lambda_i f_i(i)} \;. \tag{120}$$

By substituting $E[L_i]$ from (114) in (75), we finally obtain:

$$\begin{aligned} E[W_{q_i}] &= \left(\lambda_i\widehat{b}_i + \frac{\lambda_i^2\cdot\widehat{b}_i^{(2)}}{2(1-\lambda_i\widehat{b}_i)} \;+\; \frac{f_i(i,i)}{2f_i(i)}\right) \cdot \frac{1+\eta_i}{\lambda_i} \;-\widehat{b}_i \\ &= \left(\frac{\lambda_i^2\cdot\widehat{b}_i^{(2)}}{2(1-\lambda_i\widehat{b}_i)} \;+\; \frac{1-\overline{\rho}}{d} \cdot \frac{f_i(i,i)}{2\lambda_i(1-\overline{\rho}_i)}\right) \cdot \frac{1+\eta_i}{\lambda_i} \;+\; \eta_i\widehat{b}_i \;. \end{aligned} \tag{121}$$

In the **symmetric case**, the mean waiting time is given for the **[AC] model** by (see Hashida [5], Takagi [10]):

$$E[W_{q_i}] = \frac{d_0^{(2)}}{2d_0} \;+\; \frac{(N-1)\cdot d_0}{2(1-\overline{\rho})} \;+\; \frac{N\lambda_0\overline{b}_0^{(2)}}{2(1-\overline{\rho})} \;, \tag{122}$$

and in the **[AS] model**, using (75) and (116), it becomes:

$$E[W_{q_i}] = \frac{(1+\eta_0)\cdot d_0^{(2)}}{2d_0} \;+\; \frac{(N-1)\cdot(1+\eta_0)\cdot d_0}{2(1-\overline{\rho})} \;+\; \frac{N\lambda_0\beta}{2(1-\overline{\rho})} \;+\eta_0\widehat{b}_0 \;. \tag{123}$$

The above results have the following important consequence:

It follows from (77) and (122), as well as from (78) and (123), that for the symmetric cases in *both* models

$$E[W_{q_i}\text{ (Gated)}] = E[W_{q_i}\text{ (Exhaustive)}] \;+\; \frac{N\lambda_0 d_0\widehat{b}_0}{1-\overline{\rho}} \tag{124}$$

(remember that in the [AC] model $\overline{\rho}_0 = \lambda_0\overline{b}_0 = \lambda_0\widehat{b}_0$). That is, for *both* models, as in the regular symmetric polling schemes

$$E[W_{q_i}\text{ (Exhaustive)}] < E[W_{q_i}\text{ (Gated)}] \;. \tag{125}$$

6 Common Results

Combining the above results, it follows that for *both regimes* (Gated and Exhaustive), for *both models* ([AC] and [AS]) and for *both versions* (breakdown observation upon occurrence or at end of service), we can derive a common (generalized) expression for $f_i(i)$, the mean number of customers in a queue at a polling instant of that queue (see Eq. (126) below). As a result of that, we obtain some generalized expressions for other important parameters. With $\overline{b}_i$ having the corresponding values for each version, and with θ_i expressing the mean busy period generated by a customer (which for the Gated regime equals the mean service time of a single customer) we construct the following table:

Regime	Model	$\overline{\rho}_i$	θ_i	$f_i(i)$
Gated	AC	$\lambda_i \overline{b}_i$	$\overline{b}_i$	$\lambda_i \cdot \frac{d}{1-\overline{\rho}}$
	AS	$\frac{\lambda_i \overline{b}_i}{1+\eta_i}$	$\overline{b}_i$	$\frac{\lambda_i}{1+\eta_i} \cdot \frac{d}{1-\overline{\rho}}$
Exhaustive	AC	$\lambda_i \overline{b}_i$	$\frac{\overline{b}_i}{1-\overline{\rho}_i}$	$\lambda_i(1-\overline{\rho}_i) \cdot \frac{d}{1-\overline{\rho}}$
	AS	$\frac{\lambda_i \overline{b}_i}{1+\eta_i}$	$\frac{\overline{b}_i}{1-\lambda_i \hat{b}_i}$	$\lambda_i(1-\overline{\rho}_i) \cdot \frac{d}{1-\overline{\rho}}$

It follows that in all cases, for $i = 1, 2, \ldots, N$,

$$f_i(i) = \frac{\overline{\rho}_i}{\theta_i} \cdot \frac{d}{1-\overline{\rho}} \,. \tag{126}$$

By using (126) the mean cycle time (for all models, all versions and all regimes) is given by a common expression:

$$E[C] = d + \sum_{i=1}^{N} f_i(i) \cdot \theta_i = d + \frac{d}{1-\overline{\rho}} \sum_{i=1}^{N} \frac{\overline{\rho}_i}{\theta_i} \cdot \theta_i = d + \frac{d}{1-\overline{\rho}} \cdot \overline{\rho} = \frac{d}{1-\overline{\rho}} \,. \tag{127}$$

It follows from (127) that a necessary condition for stability is $\overline{\rho} < 1$. Now, from (126),

$$f_i(i) = \frac{\overline{\rho}_i}{\theta_i} \cdot E[C], \tag{128}$$

and therefore, the mean sojourn time of the *server* at queue i is

$$f_i(i) \cdot \theta_i = \overline{\rho}_i \cdot E[C] \,. \tag{129}$$

Thus, $\overline{\rho}_i$ is the fraction of time the server resides in queue i. We use this result to calculate the mean arrival rate to queue i, Λ_i, which is composed of weighted effective arrival rates, with weights $\overline{\rho}_i$ and $(1-\overline{\rho}_i)$, respectively:

$$\Lambda_i \;=\; \overline{\rho}_i \cdot \Big[\frac{\widehat{b}_i}{\overline{\overline{b}}_i} \;\cdot\; \lambda_i + \Big(1 \;-\; \frac{\widehat{b}_i}{\overline{\overline{b}}_i}\Big) \cdot 0\Big] + (1-\overline{\rho}_i)\cdot\lambda_i = \lambda_i\Big[1-\overline{\rho}_i\Big(1-\frac{\widehat{b}_i}{\overline{\overline{b}}_i}\Big)\Big]$$

$$= \lambda_i\Big[1 \;-\; \frac{\lambda_i\overline{b}_i}{1+\eta_i} \;\cdot\; \frac{\overline{\overline{b}}_i-\widehat{b}_i}{\overline{\overline{b}}_i}\Big] = \lambda_i\Big[1 \;-\; \frac{\eta_i}{1+\eta_i}\Big] \;=\; \frac{\lambda_i}{1+\eta_i} \;=\; \frac{\overline{\rho}_i}{\overline{\overline{b}}_i}\;. \tag{130}$$

Accordingly, the work rate (traffic load) of queue is

$$\Lambda_i\overline{\overline{b}}_i = \overline{\rho}_i\;, \tag{131}$$

and the total traffic load of the system is indeed the generalized $\overline{\rho}$ (see remark after equation (50)). It follows (see Fricker and Jaïbi [4]) that $\overline{\rho} < 1$ is not only a necessary condition for stability, but also a sufficient one. ¿From (129), the mean number of customers served in queue i during a cycle is

$$E[M_i] \;=\; \frac{\overline{\rho}_i \cdot E[C]}{\overline{\overline{b}}_i} \;=\; \frac{\lambda_i}{1+\eta_i} \;\cdot\; E[C]\;, \tag{132}$$

which coincides with the results obtained separately for each of the regimes (Eqs. (52) and (64) for the Gated, and Eq. (110) for the Exhaustive).

7 The Globally-Gated Regime

In the (cyclic) Globally-Gated (GG) regime ([1], [2]), as in the Gated and Exhaustive regimes, the server visits the queues in a cyclic order. However, at the *initiation* of every new cycle *all* gates are simultaneously closed, so that only those customers present in the system at that instant are served during that cycle.

We assume, without loss of generality, that a cycle starts from queue 1.

Let $X_j \equiv X_1^j$ = the number of customers at queue j at a cycle-beginning. Let $f_j \equiv E(X_j) \equiv f_1(j)$.

In the **[AC] model** (see Yechiali [12]):

$$C^*(\omega) \;=\; D^*(\omega)\cdot C^*\Big[\sum_{j=1}^{N}\lambda_j(1-\overline{B}_j^*(\omega))\Big] \tag{133}$$

$$E[C] \;=\; \frac{d}{1-\overline{\rho}}\;, \tag{134}$$

$$E[C^2] \;=\; \frac{1}{1-\overline{\rho}^2} \;\cdot\; \Big[d^{(2)} + \Big(2d\overline{\rho} + \sum_{j=1}^{N}\lambda_j\overline{b}_j^{(2)}\Big)\cdot E[C]\Big]\;, \tag{135}$$

where $D \equiv \sum_{j=1}^{N} D_j$ and $d^{(2)}$ is its second order moment.

In both models, the number of customers present at queue j at a cycle-beginning is the number of customers that arrived at queue j during a (previous) cycle. However, in the **[AS] model**, the arrival process to queue j stops during repair times at that queue, and therefore, in steady state,

$$X_j = A^j \Big(C - \sum_{k=1}^{X_j} \big(\overline{B}_j^{(k)} - \widehat{B}_j^{(k)} \big) \Big) \tag{136}$$

where $\overline{B}_j^{(k)} \sim \overline{B}_j$ and $\widehat{B}_j^{(k)} \sim \widehat{B}_j$ for every k. It follows that

$$\begin{aligned} f_j &= \lambda_j \cdot (E[C] - f_j(\overline{b}_j - \widehat{b}_j)) = \lambda_j E[C] - f_j \eta_j, \quad \text{leading to} \\ f_j &= \frac{\lambda_j E[C]}{1+\eta_j} \end{aligned} \tag{137}$$

Now,

$$C = D + \sum_{j=1}^{N} \sum_{k=1}^{X_j} \overline{B}_j^{(k)} \ . \tag{138}$$

Hence, with $F_1(\underline{z}) \equiv E\Big[\prod_{j=1}^{N} z_j^{X_j} \Big]$, we get

$$C^*(\omega) = D^*(\omega) \cdot F_1(\overline{B}_i^*(\omega), \overline{B}_2^*(\omega), \dots, \overline{B}_N^*(\omega)) \ , \tag{139}$$

and,

$$E[C] = d + \sum_{j=1}^{N} f_j \cdot \overline{b}_j \ . \tag{140}$$

Substituting f_j from (137), we have
$E[C] = d + \sum_{j=1}^{N} \frac{\lambda_j \overline{b}_j}{1+\eta_j} E[C] = d + \overline{\rho} \cdot E[C]$, leading to, as in the other regimes,

$$E[C] = \frac{d}{1-\overline{\rho}} \tag{141}$$

Thus,

$$f_j = \frac{\lambda_j}{1+\eta_j} \ \cdot \ \frac{d}{1-\overline{\rho}} \ . \tag{142}$$

That is, $f_1(j)_{GG} = f_j(j)_{\text{Gated}}$.

Waiting Times

To be able to obtain expressions for the mean waiting time of a customer in queue i in the two models we need an expression for the second-order moment of a cycle. ¿From (138) we get:

$$E[C^2] = E\Big\{D^2 + 2D\sum_{j=1}^{N}\sum_{k=1}^{X_j} \overline{B}_j^{(k)} + \Big(\sum_{j=1}^{N}\sum_{k=1}^{X_j} \overline{B}_j^{(k)}\Big)^2\Big\}\ . \tag{143}$$

After some algebraic manipulations (see Appendix) we obtain:

$$E[C^2] = \frac{d^{(2)} + \Big[2d\overline{\rho} + \sum_{j=1}^{N}\Big(\frac{\lambda_j \overline{b}_j^{(2)}}{1+\eta_j} + \frac{\lambda_j \overline{\rho}_j^2}{1-\eta_j}\,(\overline{b}_j^{(2)} - 2E[\overline{B}_j\widehat{B}_j] + \widehat{b}_j^{(2)})\Big)\Big]\cdot E[C]}{1-\overline{\rho}^2} \tag{144}$$

Now, for both models, let C_P and C_R denote, respectively, the past and residual duration of a cycle. Then (see Boxma, Levy and Yechiali [1]):

$$C_P^*(\omega) = C_R^*(\omega) = \frac{1 - C^*(\omega)}{\omega E[C]} \tag{145}$$

and

$$E[C_P] = E[C_R] = \frac{E[C^2]}{2E[C]}\ . \tag{146}$$

Consider an arbitrary customer J at queue j. His waiting time is composed of

(i) a residual cycle time C_R,

(ii) the service times of all customers who arrive at queues $1, 2, \ldots, j-1$ during the cycle in which J arrives,

(iii) the switchover times of the server between queues $1, 2, \ldots, j-1$ and j, and

(iv) the service times of all customers who arrived at queue j before J, i.e. during the past part C_P of the cycle in which J arrives.

Then,

$$E[W_{q_j}] = E[C_R] + \sum_{k=1}^{j-1} E[A^k(C_P + C_R)]\cdot \overline{b}_k + \sum_{k=1}^{j-1} d_k + E[A^j(C_P)]\cdot \overline{b}_j\ . \tag{147}$$

In the **[AC] model** Eq. (147) becomes (see [1])

$$E[W_{q_j}] = \left(1 + 2\sum_{k=1}^{j-1} \overline{\rho}_k + \overline{\rho}_j\right) \cdot E[C_R] + \sum_{k=1}^{j-1} d_k \ . \tag{148}$$

In the **[AS] model** the calculation of $E[A^k(C_P)]$ is much more complicated. In order to find $E[A^k(C_P)]$ we consider the three possible cases for the position of the server at the instant of arrival of customer J.

(1) the server is before queue k;

(2) the server is in queue k; and

(3) the server has passed queue k.

The probabilities for these cases are, respectively, $\frac{s_k}{E[C]}$, $\overline{\rho}_k$ and $\left(1 - \overline{\rho}_k - \frac{s_k}{E[C]}\right)$, where s_k is the mean time from the start of a cycle until the server enters queue k. From (129), $s_k = \sum_{m=1}^{k-1} (\overline{\rho}_m E[C] + d_m)$. The arrival rate to queue k when the server is in that queue is $\lambda_k \frac{\widehat{b}_k}{\overline{b}_k}$, and it equals λ_k when the server is *not* in queue k. Therefore, an approximation to $E[A^k(C_P)]$, based on an assumption of independence between the various elements, is given by:

$$\begin{aligned} E[A^k(C_P)] &= \frac{s_k}{E[C]} \cdot \lambda_k E[C_P] + \overline{\rho}_k \cdot \left[\lambda_k s_k + \lambda_k \frac{\widehat{b}_k}{\overline{b}_k} \cdot (E[C_P] - s_k)\right] \\ &+ \left(1 - \overline{\rho}_k - \frac{s_k}{E[C]}\right) \cdot \left[\lambda_k \frac{\widehat{b}_k}{\overline{b}_k} \cdot \overline{\rho}_k E[C] + \lambda_k \cdot (E[C_P] - \overline{\rho}_k E[C])\right] \\ &= \lambda_k \cdot \left\{\left(\overline{\rho}_k \frac{\widehat{b}_k}{\overline{b}_k} + 1 - \overline{\rho}_k\right) \cdot E[C_P] + \overline{\rho}_k \left(1 - \frac{\widehat{b}_k}{\overline{b}_k}\right)[2s_k - E[C] \cdot (1 - \overline{\rho}_k)]\right\} \\ &= \lambda_k \cdot \left\{E[C_P] - \frac{\eta_k}{1+\eta_k} \cdot E[C_P] + \frac{\eta_k}{1+\eta_k}[2s_k - E[C] \cdot (1 - \overline{\rho}_k)]\right\} \\ &= \frac{\lambda_k}{1+\eta_k} \cdot [E(C_P) + 2\eta_k s_k - \eta_k (1 - \overline{\rho}_k) \cdot E(C)] \ . \end{aligned} \tag{149}$$

In a similar way,

$$E[A^k(C_R)] = \frac{s_k}{E[C]} \cdot \left[\lambda_k \frac{\widehat{b}_k}{\overline{b}_k} \cdot \overline{\rho}_k E[C] + \lambda_k \cdot (E[C_R] - \overline{\rho}_k E[C])\right]$$

$$
\begin{aligned}
&+ \quad \overline{\rho}_k \cdot \left[\lambda_k \cdot [(1-\overline{\rho}_k)E[C] - s_k] + \lambda_k \frac{\widehat{b}_k}{\overline{b}_k} \cdot [E[C_R] - (1-\overline{\rho}_k)E[C] + s_k]\right] \\
&+ \quad \left(1 - \overline{\rho}_k - \frac{s_k}{E[C]}\right) \cdot \lambda_k E[C_R] \qquad (150)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_k \cdot \left\{ \left(\overline{\rho}_k \frac{\widehat{b}_k}{\overline{b}_k} + 1 - \overline{\rho}_k\right) \cdot E[C_R] - \overline{\rho}_k \left(1 - \frac{\widehat{b}_k}{\overline{b}_k}\right)[2s_k - E[C] \cdot (1-\overline{\rho}_k)] \right\} \\
&= \lambda_k \cdot \left\{ E[C_R] - \frac{\eta_k}{1+\eta_k} \cdot E[C_R] - \frac{\eta_k}{1+\eta_k}[2s_k - E[C] \cdot (1-\overline{\rho}_k)] \right\} \\
&= \frac{\lambda_k}{1+\eta_k} \cdot [E(C_R) - 2\eta_k s_k + \eta_k(1-\overline{\rho}_k) \cdot E[C)] \ .
\end{aligned}
$$

Substituting (149) and (150) in (147) while using (146) and (148), we get, for the **[AS] model**:

$$
\begin{aligned}
E[W_{q_j}(AS)] &= E[C_R] + \sum_{k=1}^{j-1} 2 \frac{\lambda_k \overline{b}_k}{1+\eta_k} E[C_R] + \sum_{k=1}^{j-1} d_k \\
&+ \frac{\lambda_j \overline{b}_j}{1+\eta_j}[E(C_R) + 2\eta_j s_j - \eta_j(1-\overline{\rho}_j) \cdot E(C)] \qquad (151) \\
&= \left(1 + 2\sum_{k=1}^{j-1} \overline{\rho}_k + \overline{\rho}_j\right) \cdot E[C_R] + \sum_{k=1}^{j-1} d_k + \overline{\rho}_j \eta_j \cdot [2s_j - (1-\overline{\rho}_j) \cdot E(C)] \\
&= E\Big[W_{q_j}(AC)\Big] + \overline{\rho}_j \eta_j \cdot [2s_j - (1-\overline{\rho}_j) \cdot E[(C)]
\end{aligned}
$$

which generalizes (148) since $\eta_j = 0$ in the [AC] model. Note, however, that $E[C|_{AC}] \neq E[C|_{AS}]$.

8 Conclusions

We have studied the combined effects of breakdowns and repairs on the performance measures of polling systems operating under the Gated, Exhaustive or Globally-Gated regimes. Twelve models were analyzed in a generalized and unified manner. The results can be applied to various manufacturing and communication systems and used as stepping stones for further analysis of complex polling models.

Appendix: Calculation of $E[C^2]$ for the GG Regime

Observing Eq. (143) we first calculate:

$$E\left[\Big(\sum_{j=1}^{N}\sum_{k=1}^{X_j}\overline{B}_j^{(k)}\Big)^2\right]$$

$$= E\left\{\sum_{j=1}^{N}\left[\sum_{k=1}^{X_j}(\overline{B}_j^{(k)})^2+\sum_{k=1}^{X_j}\sum_{\substack{\ell=1\\ \ell\neq k}}^{X_j}\overline{B}_j^{(k)}\overline{B}_j^{(\ell)}+\sum_{k=1}^{X_j}\overline{B}_j^{(k)}\sum_{\substack{m=1\\ m\neq j}}^{N}\sum_{\ell=1}^{X_m}\overline{B}_m^{(\ell)}\right]\right\}$$

$$= \sum_{j=1}^{N}\left\{\frac{\lambda_j}{1+\eta_j}\,E[C]\cdot\overline{b}_j^{(2)}+E[X_j(X_j-1)]\cdot\overline{b}_j^{\,2}+\frac{\lambda_j\overline{b}_j}{1+\eta_j}\sum_{\substack{m=1\\ m\neq j}}^{N}\frac{\lambda_m\overline{b}_m}{1+\eta_m}\,E[C^2]\right\}.$$

Similarly to the derivation of Eq. (93), we get,

$$E[X_j(X_j-1)] = \lambda_j^2\cdot E\left[\left(C-\sum_{k=1}^{X_j}(\overline{B}_j^{(k)}-\widehat{B}_j^{(k)})\right)^2\right]$$

$$= \lambda_j^2\cdot E\left[C^2-2C\sum_{k=1}^{X_j}(\overline{B}_j^{(k)}-\widehat{B}_j^{(k)})+\sum_{k=1}^{X_j}(\overline{B}_j^{(k)}-\widehat{B}_j^{(k)})^2\right]$$

$$+\lambda_j^2\cdot E\left[\sum_{k=1}^{X_j}\sum_{\substack{m=1\\ m\neq k}}^{X_j}(\overline{B}_j^{(k)}-\widehat{B}_j^{(k)})\cdot(\overline{B}_j^{(m)}-\widehat{B}_j^{(m)})\right]$$

$$= \lambda_j^2\cdot\left[E[C^2]-\ 2\frac{\lambda_j}{1+\eta_j}(\overline{b}_j-\widehat{b}_j)\cdot E[C^2]+\ \frac{\lambda_j}{1+\eta_j}E[C]\cdot[\overline{b}_j^{(2)}-2E[\overline{B}_j\widehat{B}_j]+\widehat{b}_j^{(2)}]\right]$$

$$+\lambda_j^2\cdot\left[E[X_j(X_j-1)]\cdot(\overline{b}_j-\widehat{b}_j)^2\right]$$

$$\Rightarrow E[X_j(X_j-1)] = \frac{\lambda_j^2}{1-\eta_j^2}\left\{\frac{1-\eta_j}{1+\eta_j}E[C^2]+\frac{\lambda_j}{1+\eta_j}E[C]\cdot[\overline{b}_j^{(2)}-2E[\overline{B}_j\widehat{B}_j]+\widehat{b}_j^{(2)}]\right\}$$

$$\Rightarrow E[X_j(X_j-1)]\cdot\overline{b}_j^{\,2} = \overline{\rho}_j^2\cdot E[C^2]+\frac{\lambda_j}{1-\eta_j}\overline{\rho}_j^2\cdot E[C]\left[\overline{b}_j^{(2)}-2E[\overline{B}_j\widehat{B}_j]+\widehat{b}_j^{(2)}\right]$$

$$\Rightarrow E\left[\left(\sum_{j=1}^{N}\sum_{k=1}^{X_j}\overline{B}_j^{(k)}\right)^2\right]$$

$$= \sum_{j=1}^{N}\left\{\frac{\lambda_j\overline{b}_j^{(2)}}{1+\eta_j}E[C]+\overline{\rho}_j^2E[C^2]+\frac{\lambda_j\overline{\rho}_j^2}{1-\eta_j}\left(\overline{b}_j^{(2)}-2E[\overline{B}_j\widehat{B}_j]+\widehat{b}_j^{(2)}\right)\cdot E[C]\right\} \qquad \text{(A1)}$$

$$+\left(\overline{\rho}^{\,2}-\sum_{j=1}^{N}\overline{\rho}_j^2\right)\cdot E[C^2]$$

By substituting (A1) in (143) we get

$$E[C^2]=d^{(2)}+2d\overline{\rho}E[C]$$
$$+\sum_{j=1}^{N}\left\{\frac{\lambda_j\overline{b}_j^{(2)}}{1+\eta_j}\;+\;\frac{\lambda_j\overline{\rho}_j^2}{1-\eta_j}(\overline{b}_j^{(2)}-2E[\overline{B}_j\widehat{B}_j]+\widehat{b}_j^{(2)})\right\}\cdot E[C]+\overline{\rho}^{\,2}E[C^2] \qquad \text{(A2)}$$

which leads to Equation (144).

References

[1] Boxma, O.J., Levy, H. and Yechiali,U., Cyclic Reservation Schemes for Efficient Operation of Multiple-Queue Single-Server Systems. *Annals of Operations Research*, **35** (1992), 187-208.

[2] Boxma, O.J., Weststrate, J.A., and Yechiali, U., A Globally Gated Polling System with Server Interruptions and Applications to the Repairman Problem. *Probability in the Engineering and Informational Sciences*, **7** (1993), 187-208.

[3] Conway, R.W., Maxwell, W.L. and Miller, L.W., *Theory of Scheduling*. Addison-Wesley (1967).

[4] Fricker, C. and Jaïbi, M.R., Monotonicity and Stability of Periodic Polling Models. *Queueing Systems*, **15**, (1994), 211-238.

[5] Hashida, O., Analysis of Multiqueue. *Review of the Electrical Communication Laboratories* **20**, (1972), 189-199.

[6] Hashida, O., Gating Multiqueues Served in Cyclic Order. *Systems-Computers-Controls* **1** (1970), 1-8.

[7] Kleinrock, L., *Queueing Systems, Volume 1: Theory*. John Wiley and Sons (1975).

[8] Kofman, D., and Yechiali, U., Polling Systems with Station Breakdowns. *Performance Evaluation* **27 & 28**, (1996), 647-672.

[9] Kofman, D., and Yechiali, U., Queueing Networks with Station Breakdowns and Globally Gated Regime. In: *Teletraffic Contributions for the Information Age* (V. Ramaswamy and P.E. Wirth, Eds). North Holland, (1997), pp. 285-296.

[10] Takagi, H., *Analysis of Polling Systems*. MIT Press (1986).

[11] Takagi, H., Queueing Analysis of Polling Models: an Update. In: *Stochastic Analysis of Computer and Communication Systems* (H. Takagi Ed.), North-Holland (1990), pp. 267-318.

[12] Yechiali, U., Analysis and Control of Polling Systems. In: *Performance Evaluation of Computer and Communications Systems* (L. Donatiello and R. Nelson, Eds.), Springer Verlag (1993), pp. 630-650.