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ON RELATIVE EFFECTIVENESS IN DUELS

by

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ABSTRACT

We study the relative effectiveness of two multi-unit forces, A and B, fighting a successive series of duels between randomly selected pairs of opposing units. A duel consists of a series of games – each of at most s units of time – and the duel terminates as soon as one of the two duelists is hit. A new independent duel then starts, and so on.

Each force is characterized by a common probability distribution function representing the time required by a single unit of the force to hit a non-firing unit of the other force in a game of unbounded duration. The relative effectiveness of force A with respect to force B is expressed by the *exchange rate*, $R(s)$, defined as the expected number of units of force B hit by a single unit of force A until unit A itself is hit.

We derive a general characterization of $R(s)$ and develop analytic results for several choices of pairs of families of time-to-hit distributions. We show that the shape of the distributions can strongly affect the outcome of the duels and that this effect changes with the duration of the duel. We further show that duels are not necessarily transitive. Some Numerical results are presented.

INTRODUCTION

We study the relative effectiveness of two forces, A and B, whose units fight a series of sequential independent duels. A duel is a series of games, where in each game the two contestants fire at each other for at most s units of time. As soon as one of the duelists is hit, the game and the entire duel end. We consider as measure of effectiveness (MOE) the *exchange rate*, i.e., the expected number of B-targets hit by a single unit of A, until A itself is hit. This is an appropriate MOE when studying the relative effectiveness of two forces, independent of their size. Our model and assumptions are somewhat different from those leading to the so-called "fundamental duel" [2]. We assume that each force is characterized by a common time-to-hit distribution function representing an underlying firing process resulting from the type of weapon being used. Such distributions may be empirical or analytical. Specifically, we consider the Gamma and Beta families of distribution functions, as well as constant-time intervals between rounds. Nevertheless, our method of analysis is general and may be applied to the study of other distribution functions.

We show that the shape of the time-to-hit distributions can strongly affect the outcome of duels even when the duelists have the same expected time to hit and the same type of distribution function. We further show that the effect of the shape of the distribution function on the outcome of dules is not constant but changes with he duration of the duel. Moreover, we show that duels are not necessarily transitive, that is, if A is "better" than B and B is "better" than C, it does not necessarily follow that A is "better" than C.

In section 1 the model is presented and the basic formula for calculating the exchange rate, $R(s)$, is derived. In section 2 duels of Gamma versus Gamma are analyzed. We show that if the two distributions have the same mean, the duelist with fewer exponential stages is superior for every $s \geq 0$. Section 3 studies duels of Beta versus Beta. It is shown that a duelist with a uniform distribution, $\beta(1,1)$, is inferior to a duelist with distribution $\beta(1,b), b > 1$, and that $R(s) = 1/b$ for all $0 \leq s \leq 1$. Duels with fixed time intervals between rounds are studied in section 4. Some numerical results are presented for the Gamma and Beta cases.

1. THE MODEL

The model deals with a sequence of duels between units of two opposing forces. A duel is a firing process in which a single unit of force A fights a single unit of force B. Each duel consists of a random number of games where a game terminates as soon as one of the following two mutually exclusive conditions is met: (i) a fighting unit is hit, or (ii) s units of time ($0 < s \leq \infty$) have elapsed and none of the fighting units has been hit. The entire duel terminates only when one of the units is hit in the course of some game. Following the termination of a given duel, a new duel starts between two randomly selected units of A and B, and so on.

Let T be the time required for a unit of A to hit a non-firing unit of B in a game of unlimited duration (i.e., $s = \infty$), and let $G(t) = P(T \leq t)$ denote the probability distribution function (pdf) of T, where $G(t) = 0$ for $t < 0$.

Similarly, let Y be the time required for a unit of B to hit a non-firing unit of A in an unbounded-duration game, and let $F(t) = P(Y \leq t)$ denote its pdf ($F(t) = 0, t < 0$). The duration of a game, then, is $\min(s, T, Y)$ where T and Y are independent.

Now, for $0 \leq s \leq \infty$ define by $P[\widetilde{A}(s)]$ the probability that a game terminates with only B being hit (i.e. A "wins"); by $P[\widetilde{B}(s)]$ the probability that a game terminates with only A being hit (B wins); and by $P[\widetilde{D}(s)]$ the probability that a game terminates with both A and B being hit simultaneously. Then,

$$
\begin{aligned}
P[\widetilde{A}(s)] &= P[T < Y | \text{a duration of a game is bounded by } s] \\
&= \int_0^s [1 - F(t)] dG(t) \qquad (1)
\end{aligned}
$$

$$
P[\widetilde{B}(s)] = P[Y < T | s] = \int_0^s [1 - G(t)] dF(t) \qquad (2)
$$

and

$$P[\widetilde{D}(s)] = \int_0^s [G(t) - G(t^-)]dF(t) = \int_0^s [F(t) - F(t^-)]dG(t) .$$

Clearly, P[a game terminates with at least one hit]

$$= 1 - [1 - G(s)][1 - F(s)] = G(s) + F(s) - G(s)F(s) .$$

Define

$$q(s) = P[\text{a game terminates with B being hit}] = P[\widetilde{A}(s)] + P[\widetilde{D}(s)]$$

and

$$\theta(s) = P[\text{a game terminates with A being hit}] = P[\widetilde{B}(s)] + P[\widetilde{D}(s)].$$

We now derive a characterization of the inherent effectiveness of force A relative to force B. As presented in the Introduction, the measure of effectiveness is the *exchange rate*, $R(s)$, defined as the expected number of hits made by an A dualist until he himself is hit. $R(s)$ may be interpreted as the ration between the number of "casualties" caused among units of force B, and those caused among units of force A after a large number of duels has taken place.

For simplification in presentation we assume for a while that the probability of simultaneous hits is null, i.e., $P[\widetilde{D}(s)] = 0$. This is clearly the case when $G(\cdot)$ and $F(\cdot)$ are continuous or when they are jump functions with jumps at non-coinciding instants of time. In such a case, $q(s) = P[\widetilde{A}(s)]$ and $\theta(s) = P[\widetilde{B}(s)]$. This assumption will later be relaxed when duels with fixed time intervals between rounds are analyzed in Section 4.

Our main result in this section is expressed in the following theorem.

Theorem 1: $R(s) = q(s)/\theta(s)$.

Proof: Let $\alpha(s)$ be the probability that a duel is terminated with A hitting B. Then,

$$\alpha(s) = \sum_{j=0}^{\infty}[1-\theta(s)-q(s)]^j q(s) = q(s)/[\theta(s)+q(s)] \ .$$

Let I_s denote the number of units of B hit by a unit of A before A itself is hit, i.e., I_s is the number of duels in which A wins before his first (and last) loss. Clearly, $P(I_s = i) = [\alpha(s)]^i[1-\alpha(s)]$. Hence,

$$R(s) = E[I_s] = \sum_{i=0}^{\infty} iP(I_s = i) = \sum_{i=0}^{\infty} i\alpha^i(s)[1-\alpha(s)]$$
$$= \alpha(s)/[1-\alpha(s)] = q(s)/\theta(s) \ . \qquad Q.E.D.$$

This result has an intuitive explanation: The expected number of games until unit A loses is $\theta(s)^{-1}$, while $q(s)$ is the probability that unit A wins a given game. Thus, the expected number of duels won by A prior to its loss is $q(s)\theta(s)^{-1}$.

Substituting (1) and (2) in Theorem 1, yields

$$R(s) = [G(s) - \int_0^s F(t)dG(t)]/[F(s) - \int_0^s G(t)dF(t)] \tag{3}$$

Since $q(s) = G(s) - G(s)F(s) + \int_0^s G(t)dF(t)$, it follows that $R(s) \geq 1$ if and only if $F(s) - G(s) + G(s)F(s) \leq 2\int_0^s G(t)dF(t)$. In particular, $R(\infty) \geq 1$ if and only if

$$\theta(\infty) = 1 - \int_0^{\infty} G(t)dF(t) \leq 1/2 \ . \tag{4}$$

However, whether or not $R(s) \geq 1$ depends on the *combined* properties of $G(\cdot)$ and $F(\cdot)$. Thus, we proceed by studying various families of distribution functions and focus our analysis on conditions under which $R(s) \geq 1$.

2. DUELS OF GAMMA VERSUS GAMMA

In this section we study the properties of $R(s)$ where the duels are between units of forces having Gamma distributions as their probability distribution functions. Specifically, let T (A's time-to-hit) possess the Gamma distribution function with integer-valued shape parameter n and scale parameter $1/\lambda$ (we write $T \sim (G(n,\lambda))$. In this case T possesses a density function $g(t) = e^{-\lambda t}\lambda^n t^{n-1}/(n-1)!, t \geq 0$, with mean $ET = n/\lambda$. Similarly, let $Y \sim G(k,\mu)$ with density function $f(t) = e^{-\mu t}\mu^k t^{k-1}/(k-1)!, t \geq 0$, and mean $EY = k/\mu$.

Using $F(t) = 1 - \sum_{i=0}^{k-1} e^{-\mu t}(\mu t)^i/i!$ and interchanging the order of integration and summation we derive

$$q(s) = \int_0^s \sum_{i=0}^{k-1} e^{-\mu t}[(\mu t)^i/i!]e^{-\lambda t}[\lambda^n t^{n-1}/(n-1)!]dt$$
$$= \left(\frac{\lambda}{\lambda+\mu}\right)^n \sum_{i=0}^{k-1} \binom{i+n-1}{i} \left(\frac{\mu}{\lambda+\mu}\right)^i \left[1 - \sum_{j=0}^{i+n-1} e^{-(\lambda+\mu)s}\frac{[(\lambda+\mu)s]^j}{j!}\right].$$

Calculating $\theta(s)$ in a similar manner we get

$$R(s) = \frac{\left(\frac{\lambda}{\lambda+\mu}\right)^n \sum_{i=0}^{k-1} \binom{i+n-1}{i} \left(\frac{\mu}{\lambda+\mu}\right)^i \left[1 - \sum_{j=0}^{i+n-1} e^{-(\lambda+\mu)s}\frac{[(\lambda+\mu)s]^j}{j!}\right]}{\left(\frac{\mu}{\lambda+\mu}\right)^k \sum_{i=0}^{n-1} \binom{i+k-1}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left[1 - \sum_{j=0}^{i+k-1} e^{-(\lambda+\mu)s}\frac{[(\lambda+\mu)s]^j}{j!}\right]} \tag{5}$$

In particular

$$R(\infty) = \frac{q(\infty)}{\theta(\infty)} = \frac{\sum_{i=0}^{k-1} \binom{i+n-1}{i} \left(\frac{\mu}{\lambda+\mu}\right)^i \left(\frac{\lambda}{\lambda+\mu}\right)^n}{\sum_{i=0}^{n-1} \binom{i+k-1}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^k} \tag{6}$$

The expression for $q(\infty)$ may be given the following probabilistic interpretation: For $T \sim G(n,\lambda)$, the time-to-hit is the sum of n independent identically distributed exponential stages with common parameter λ. Similary,

$Y \sim G(k,\mu)$ is the sum of k i.i.d. exponential stages with the same parameter μ. The probability that a stage of T terminates before a stage of Y is $\lambda/(\lambda+\mu)$. Duelist A wins if he finishes n stages before B finishes k stages. The probability of n successes before k failures, in a sequence of Bernoulli trials with probability of success $\lambda/(\lambda+\mu)$ in each trial, is given by the negative binomial distribution. That is,

$$q(\infty) = \sum_{i=0}^{k-1} \binom{n-1+i}{i} \left(\frac{\mu}{\lambda+\mu}\right)^i \left(\frac{\lambda}{\lambda+\mu}\right)^n .$$

It is interesting to note that the expression for $q(\infty)$ is identical with the expression $p(k,n)$ in the linear law of Brown [4, p. 420] which describes the probability that force A with k units hits all units of force B before all its own units are hit, where in each stage of fight the probability of hitting a unit of force B is $\lambda/(\lambda+\mu)$.

Using the relation (Brown [4])

$$\sum_{i=0}^{k-1} \binom{n+i-1}{i} p^i (1-p)^n = \sum_{i=0}^{k-1} \binom{n+k-1}{i} p^i (1-p)^{n+k-1-i}$$

equation (6) is rewritten as

$$R(\infty) = \frac{\left[\sum_{i=0}^{k-1} \binom{k+n-1}{i} \left(\frac{\mu}{\lambda+\mu}\right)^i \left(\frac{\lambda}{\lambda+\mu}\right)^{k+n-1-i}\right]}{\left[\sum_{i=0}^{n-1} \binom{k+n-1}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{k+n-1-i}\right]} . \tag{7}$$

The calculation of $R(\infty)$ in (7) may become easier by using the Normal approximation of the Binomial distribution. When $(k+n-1)\left(\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right) > 9$ we have (see [1]),

$$R(\infty) \approx \frac{\Phi[(k-1+\frac{1}{2}-M_1)/\sigma]}{\Phi[n-1+\frac{1}{2}-M_2)/\sigma]}$$

where, $\sigma = \sqrt{(k+n-1)\left(\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right)}$, $M_1 = (k+n-1)\frac{\mu}{\lambda+\mu}$, $M_2 = (k+n-1)\frac{\lambda}{\lambda+\mu}$, and $\Phi(\cdot)$ is the Standard Normal distribution function.

Setting $z = [k\lambda - n\mu + \frac{1}{2}(\mu - \lambda)]/\sqrt{(k+n-1)\lambda\mu}$ the expression for $R(\infty)$ takes the form

$$R(\infty) \cong \Phi(z)/\Phi(-z) .$$

Going back to (7) we wish to study the properties of $R(\infty)$ as a function of the parameters n, λ, k and μ. We first show that whenever the expected times to hit a passive target are equal, the force that possesses the Gamma distribution with fewer number of stages is superior. We state

Theorem 2: Let ET = EY (i.e., $n/\lambda = k/\mu$). Then, $R(\infty) > 1$ whenever $k > n$.

Proof: As $R(\infty) = q(\infty)/\theta(\infty)$ and $q(\infty) + \theta(\infty) = 1$, it suffices to show that $\theta(\infty) < 1/2$. Since $\lambda/(\lambda+\mu) = n/(n+k)$ we must show that $\theta(\infty) = \sum_{i=0}^{n-1} \binom{n+k-1}{i} \left(\frac{n}{n+k}\right)^i \left(\frac{k}{n+k}\right)^{n+k-1-i} < 1/2$ whenever $k > n$.

Write $a_i = \binom{n+k-1}{i} \left(\frac{n}{k}\right)^i \left(\frac{k}{n+k}\right)^{n+k-1}$. Then, as $\sum_{i=0}^{n+k-1} a_i = 1$, it is enough to show that

$$\sum_{i=0}^{n-1} a_i < \sum_{i=n}^{n+k-1} a_i \tag{8}$$

Letting $b_i = \binom{n+k-1}{i} \left(\frac{n}{k}\right)^i$, relation (8) is equivalent to

$$\sum_{i=0}^{n-1} b_i < \sum_{i=n}^{n+k-1} b_i \tag{9}$$

We will now show that for all $i = 0, 1, \ldots, n-1$, $b_{n+i}/b_{n-1-i} \geq 1$.
For $i = 0$, $b_n/b_{n-1} = 1$. For $i \geq 1$, $\frac{b_{n+i}}{b_{n-1-i}} = \frac{(k+i)\cdots k \cdots (k-i)}{(n+i)\cdots n \cdots (n-i)} \left(\frac{n}{k}\right)^{2i+1}$
$= \frac{k}{n} \prod_{j=1}^{i} \frac{(k^2-j^2)}{(n^2-j^2)} \cdot \left(\frac{n}{k}\right)^{2i+1} > \frac{k}{n}\left(\frac{k^2}{n^2}\right)^i \left(\frac{n}{k}\right)^{2i+1} = 1$, where the inequality follows from the fact that for $k > n \geq 1$, $\frac{k^2-j^2}{n^2-j^2} > \frac{k^2}{n^2}$ for $j = 1, \ldots, n$.

The proof is concluded by noting that in relation (9) the number of terms in the right hand side is greater than the number of terms in the left hand side, since $k > n$. Q.E.D.

Note that the conclusion of Theorem 2 follows readily if one uses the approximation $R(\infty) = \Phi(z)/\Phi(-z)$ above. When $n/\lambda = k/\mu$ then $z > 0$ if and only if $\mu > \lambda$. In such a case $R(\infty) > 1$ as $\Phi(z) > \Phi(-z)$. But $\mu > \lambda$ if and only if $k > n$.

Corollaries:

(i) If ET = EY, then there is an advantage to the duelist having greater chances of hitting at the beginning of the duel.

(ii) For the Gamma family there exists a transitivity property: if three forces have the same mean of time-to-hit such that A is better than B (i.e., $R(\infty) > 1$) and B is better than C, then A is also better than C. However, this transitivity property is *not true in general*, as will be evident from an example presented in the sequel.

An interesting implication of Theorem 2 and the transitivity property within the Gamma family with same mean, is that effectiveness increases with the variance of the hitting-time distribution. This follows since $VAR(T)/VAR(Y) = \mu/\lambda > 1$ if and only if $k > n$.

So far we have dealt mainly with $R(\infty)$. For the Gamma vs. Gamma duel with $n/\lambda = k/\mu$ we claim the following:

Theorem 3: Let $n/\lambda = k/\mu$. If $k > n$, then $R(s) > 1$ for all $0 \le s \le \infty$.

Proof: Define $\overline{g}(t) = g(t)/[1 - G(t)] = \lambda^n t^{n-1}/[(n-1)! \sum_{j=0}^{n-1} [(\lambda t)^j/j!]]$ and $\overline{f}(t) = f(t)/[1 - F(t)] = \mu^k t^{k-1}/[(k-1)! \sum_{j=0}^{k-1} [\mu t)^j/j!]]$. For $n, k \ge 1$, the distributions $G(\cdot)$ and $F(\cdot)$ are IFR, i.e., the failure rates $\overline{g}(t)$ and $\overline{f}(t)$ are monotonically non-decreasing continuous functions (see [3],p. 75). If $n < k$ and $n/\lambda = k/\mu$ then (i) $\overline{f}(0) = 0$ for $n \ge 1$, and $\overline{g}(0) = \lambda$ for $n = 1$, $\overline{g}(0) = 0$ for $n > 1$. (ii) $\lim_{t\to\infty} \overline{g}(t) = \lambda$, and $\lim_{t\to\infty} \overline{f}(t) = \mu$. (iii) $\lim_{t\to 0}[\overline{g}(t)/\overline{f}(t)] = \infty$.

From (iii) it follows that $\overline{g}(t) > \overline{f}(t)$ for $0 < t < \varepsilon$, for small enough $\varepsilon > 0$. On the other hand, $\overline{g}(t) < \overline{f}(t)$ for t large enough, since $\lambda < \mu$ and (ii) holds. As $\overline{g}(t)$ and $\overline{f}(t)$ are continuous, it follows that there exists a point t_0 such that $\overline{g}(t_0) = \overline{f}(t_0)$ and $\overline{g}(t) > \overline{f}(t)$ for $0 < t < t_0$. Further, using the fact that $k > n$, it can be shown that $\overline{g}(t)$ and $\overline{f}(t)$ intersect exactly once in $(0, \infty)$ so that $\overline{g}(t) < \overline{f}(t)$ for all $t > t_0$.

Now, for all $s \leq t_0$, $R(s) > 1$ since

$$R(s) = \frac{\int_0^s [1-F(t)][1-G(t)]\overline{g}(t)dt}{\int_0^s [1-G(t)]f(t)dt} > \frac{\int_0^s [1-F(t)][1-G(t)]\overline{f}(t)dt}{\int_0^s [1-G(t)]f(t)dt} = 1$$

The proof will be completed by showing that $R(s) > 1$ for all $s > t_0$. Suppose, contrarily, that there exists $s^* > t_0$ such that $R(s^*) < 1$. Then, since $R(\infty) > 1$ and $R(s)$ is continuous, there exists z such that $R(z) = 1$ and $R'(z) \geq 0$. However, $q'(z) = [1-F(z)]g(z)$, $\theta'(z) = [1-G(z)]f(z)$ and hence, $q'(z)/\theta'(z) = \overline{g}(z)/\overline{f}(z)$. Therefore, $R'(z) = [\theta'(z)/\theta(z)][\overline{g}(z)/\overline{f}(z) - R(z)]$. But, $\overline{g}(z)/\overline{f}(z) < 1$ for $z > t_0$, while $R(z) = 1$ and $\theta'(z)/\theta(z) > 0$ for all $z > 0$. Thus, $R'(z) < 0$, which contradicts the above implication that $R'(z) \geq 0$. Hence, $R(s^*) \not< 1$. Q.E.D.

<u>Remark:</u> For the exponential distribution ($n = k = 1$) it is easy to see that $R(s) = \lambda/\mu$ for all $s \geq 0$. Thus, A is preferrable to B if and only if $\lambda > \mu$, i.e., if and only if ET<EY.

We conclude this section by demonstrating that properties which are true within the Gamma family (see Corollaries above) do not necessarily hold in general.

<u>Example 1:</u> Duels are not transitive.

Consider three distinct forces characterized by the following distribution functions.

$$A: \qquad G(t) = \begin{cases} 0, & t < 0 \\ a + (1-a)t^2, & 0 \le t \le 1, \qquad 0 < a < \frac{1}{28} \\ 1, & t > 1 \end{cases}$$

$$B: \qquad F(t) = \begin{cases} 0, & t < 0 \\ b + \frac{3}{4}(1-b)t, & 0 \le t \le \frac{4}{3}, \qquad 0 < b < \frac{a}{1+a} \\ 1, & t > \frac{4}{3} \end{cases}$$

$$C: \qquad H(t) = \begin{cases} 0, & t < \frac{1}{3} \\ \frac{3}{2}(t - \frac{1}{3}), & \frac{1}{3} \le t \le 1 \\ 1, & t > 1 \end{cases}$$

Suppose $s = \infty$. Then, recalling (4), it readily follows that A is better than B since

$$\int_0^\infty G(t)dF(t) = \int_0^1 [a + (1-a)t^2](1-b) \cdot \frac{3}{4}dt + \int_1^{4/3} \frac{3}{4}(1-b)dt$$
$$= \frac{1}{2} + \frac{a - b - ab}{2} > \frac{1}{2}\,.$$

Similarly, B is better than C since

$$\int_0^\infty F(t)dH(t) = \int_{1/3}^1 [b + \frac{3}{4}(1-b)t]\frac{3}{2}dt = \frac{1}{2} + \frac{b}{2} > \frac{1}{2}.$$

However, C is better than A since

$$\int_0^\infty G(t)dH(t) = \int_{1/3}^1 [a + (1-a)t^2]\frac{3}{2}dt = a + \frac{13}{27}(1-a) < \frac{1}{2}\,.$$

<u>**Example 2:**</u> ET = EY and greater chances of A hitting B at the beginning do not imply $R(\infty) > 1$.

Let

$$G(t) = \begin{cases} 0, & t < 0 \\ t^2, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases} \quad \text{and} \quad F(t) = \begin{cases} 0, & t < \frac{1}{3} \\ \frac{3}{2}(t - \frac{1}{3}), & \frac{1}{3} \le t \le 1 \, . \\ 1, & t > 1 \end{cases}$$

Clearly, ET = EY $= \frac{2}{3}$. Now,

$$R(\infty) = R(1) = \frac{q(1)}{\theta(1)} = \frac{1 - \theta(1)}{\theta(1)} = \left(\frac{1}{2} - \frac{1}{54}\right) / \left(\frac{1}{2} + \frac{1}{54}\right) < 1 \, .$$

Numerical Results for Gamma vs. Gamma

In Table 1 values of $R(s)$ are presented for a duel between $G(n, \lambda)$ and $G(4,4)$. The table demonstrates (as was shown above) that whenever ET$=$ $n/\lambda = 1 =$ EY and $n < k$, $R(s) > 1$ for all $s \geq 0$. On the other hand, for arbitrary values of n and λ, the advantage may change as s increases. For example, for $n = 3$, $\lambda = 2.7$, A is better than B for $s \leq 0.6$, whereas B is better than A for higher values of s.

Table 1: $R(s)$ for $G(n, \lambda)$ vs. $G(4,4)$

n	λ	ET	s=0.2	s=0.6	s=1	s=1.4	s=1.8	s=2.2	s=∞
1	1.0	1	23.26	2.96	1.79	1.53	1.47	1.45	1.45
1	0.9	1.11	20.82	2.63	1.57	1.34	1.28	1.26	1.26
1	0.8	1.25	18.40	2.30	1.37	1.16	1.10	1.08	1.08
1	0.7	1.43	16.01	1.98	1.17	0.98	0.93	0.91	0.91
2	2.0	1	7.05	1.80	1.32	1.20	1.18	1.18	1.18
2	1.8	1.11	5.81	1.5	1.10	1.00	0.98	0.98	0.98
2	1.6	1.25	11.68	1.23	0.90	0.82	0.80	0.80	0.80
3	3.0	1	2.57	1.59	1.12	1.07	1.07	1.07	1.07
3	2.7	1.11	1.95	1.03	0.90	0.87	0.87	0.87	0.87
3	2.4	1.25	1.43	0.79	0.71	0.68	0.68	0.68	0.68
4	4.0	1	1	1	1	1	1	1	1
4	3.6	1.11	0.70	0.76	0.78	0.79	0.79	0.79	0.79
4	3.2	1.25	0.46	0.55	0.59	0.61	0.61	0.61	0.61

3. BETA VERSUS BETA

In this section we consider duels between units of forces characterized by the Beta distribution functions. Specifically, let $T \sim \beta(m,n)$ with density $g(t) = B^{-1}(m,n)t^{m-1}(1-t)^{n-1}$, $0 \le t \le 1$, and mean ET $= m/(m+n)$, where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$. Similarly, let $Y \sim \beta(a,b)$ with distribution function $F(x) = \int_0^x B^{-1}(a,b)t^{a-1}(1-t)^{b-1}dt \equiv I_x(a,b), 0 \le x \le 1$. It is well known ([1], p. 945) that for N and a *integers* $I_x(a, N-a+1) = \sum_{i=a}^{N} \binom{N}{i} x^i(1-x)^{N-i}$. Letting $b = N-a+1$ we get

$$F(x) = I_x(a,b) = \sum_{i=a}^{a+b-1} \binom{a+b-1}{i} x^i(1-x)^{a+b-1-i} . \tag{10}$$

Applying (1), we derive

$$\begin{aligned} q(s) &= \int_0^s \sum_{i=0}^{a-1} \binom{a+b-1}{i} x^i(1-x)^{a+b-1-i} B^{-1}(m,n)x^{m-1}(1-x)^{n-1}dx \\ &= \frac{1}{B(m,n)} \sum_{i=0}^{a-1} \binom{a+b-1}{i} B(m+i,\tau_i)I_s(m+i,\tau_i) . \end{aligned}$$

where $\tau_i = a+b-1-i+n$. Repeated use of (10) leads to

$$q(s) = \frac{1}{B(m,n)} \sum_{i=0}^{a-1} \binom{a+b-1}{i} [B(m+i,\tau_i) \sum_{j=m+i}^{\gamma} \binom{\gamma}{j} s^j(1-s)^{\gamma-j}] \tag{11}$$

where $\gamma = a+b+m+n-2$. Hence,

$$q(\infty) = q(1) = \frac{1}{B(m,n)} \sum_{i=0}^{a-1} \binom{a+b-1}{i} B(m+i, a+b-1-i+n) .$$

Similarly,

$$\theta(s) = \frac{1}{B(a,b)} \sum_{i=0}^{m-1} \binom{m+n-1}{i} [B(a+i,\ell_i) \sum_{j=a+i}^{\gamma} \binom{\gamma}{j} s^j(1-s)^{\gamma-j}] \tag{12}$$

where $\ell_i = m+n-1-i+b$. Thus,

$$\theta(\infty) = \theta(1) = \frac{1}{B(a,b)} \sum_{i=0}^{m-1} \binom{m+n-1}{i} B(a+i, m+n-1-i+b) \ .$$

One can get more tractable expressions for $q(1)$ and $\theta(1)$, i.e.,

$$q(1) = \frac{(m+n-1)!}{(m-1)!(n-1)!} \sum_{i=0}^{a-1} \frac{(a+b-1)!}{i!(a+b-1-i)!} \cdot \frac{(m+i-1)!(\tau_i-1)!}{(a+b+m+n-2)!}$$
$$= \sum_{i=0}^{a-1} \binom{m+i-1}{i} \binom{a+b-i+n-2}{n-1} \Big/ \binom{\gamma}{a+b-1},$$

and

$$\theta(1) = \sum_{j=0}^{m-1} \binom{a+j-1}{j} \binom{m+n-j+b-2}{b-1} \Big/ \binom{\gamma}{m+n-1} \ .$$

Thus, $R(\infty) = R(1) = q(1)/\theta(1)$. In particular, for $\beta(1,1)$ vs. $\beta(a,b)$, $R(1) = a/b$.

For $0 \le s \le \infty, R(s) = q(s)/\theta(s)$ may be readily calculated by using (11) and (12) above. Nevertheless, for a certain special case, we derive

Theorem 4: For $\beta(1,1)$ vs. $\beta(1,b)$ $(b > 1)$, $R(s) = \frac{1}{b}$ for all $0 \le s \le 1$.

Proof: For $\beta(1,1), g(x) = 1$, for all $0 \le x \le 1$, while for $\beta(1,b)$, $f(x) = b(1-x)^{b-1}$, $0 \le x \le 1$. Hence,

$$q(s) = \int_0^s [1-F(t)]g(t)dt = \int_0^s (1-t)^b dt = \frac{1}{b+1}[1-(1-s)^{b+1}],$$
$$\theta(s) = \int_0^s (1-x)b(1-x)^{b-1}dx = \frac{b}{b+1}[1-(1-s)^{b+1}],$$

and so $R(s) = b^{-1}$ for all $0 \le s \le 1$. Q.E.D.

In a similar manner, for $\beta(1,1)$ vs. $\beta(a,1)$ $(a > 1)$, one gets $R(s) = [(a+1)s - s^{a+1}]/[(a+1)s^a - as^{a+1}]$ for every $0 \le s \le 1$, so that $R(1) = a$.

Table 2: $R(s)$ for $\beta(m,n)$ vs. $\beta(a,b)$
for various cases where the advantage changes as s increases

m	n	ET	a	b	EY	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	1	0.50	2	3	0.40	1.19	0.80	0.69	0.67	0.67
1	1	0.50	3	4	0.43	2.30	1.03	0.80	0.75	0.75
2	1	0.67	3	2	0.60	1.49	0.91	0.74	0.68	0.67
2	1	0.67	4	2	0.67	6.10	1.99	1.20	0.96	0.91
2	1	0.67	4	3	0.57	2.41	1.92	0.64	0.56	0.56
2	1	0.67	4	4	0.50	1.22	0.54	0.41	0.39	0.38
2	2	0.50	3	4	0.43	1.07	0.76	0.69	0.68	0.68
3	2	0.60	4	3	0.57	1.62	1.02	0.87	0.83	0.83

To emphasize the significance of Theorem 4, note that ET/EY $= (b+1)/2$. Thus, for $b = 3$, ET/EY$= 2$, which means that, on the average, a unit of force B is twice as fast as a unit of force A. However, $R(s) = 1/3$, which implies that, on the average, each unit of B wins three duels before being hit.

It is also interesting to observe that the advantage of a duelist with respect to its opponent may change as s increases. This is demonstrated in Table 2 where $R(s)$ is calculated for various pairs of (m,n) and (a,b). $R(s)$ is greater than 1 for small s and drops below 1 for larger values of s.

4. DUELS WITH FIXED TIME INTERVALS BETWEEN ROUNDS

In this section we consider duels where each game is comprised of several rounds. Each round is a firing event where both units fire simultaneously at each other. A game terminates as soon as one of the following two conditions is satisfied: (i) one of the two fighting units is hit, or (ii) m rounds passed with none of the units being hit. A duel terminates as soon as at least one of the units is hit.

Let $p_i(q_i)$ be the conditional probability that a unit of force A (B) hits a unit of force B (A) at the i^{th} round given that the game has not been terminated before. Define also the following events:

$N_i =$ none of the units is hit after i rounds in a game $(1 \le i \le m)$.

$A_i(B_i) =$ the game terminates with A (B) hitting B (A) on the i^{th} round (and A(B) is not being hit).

$D_i =$ the game is terminated at the i^{th} round with *both* units hit simultaneously.

Now, let $\widetilde{A} = \bigcup_{i=1}^m A_i$, $\widetilde{B} = \bigcup_{i=1}^m B_i$, $\widetilde{D} = \bigcup_{i=1}^m D_i$. It readily follows that

$$P(N_i) = \prod_{j=1}^{i}(1-p_j)(1-q_j), 1 \le i \le m; \qquad P(A_i) = p_i(1-q_i)P(N_{i-1});$$

$$P(B_i) = q_i(1-p_i)P(N_{i-1}); \qquad P(D_i) = p_i q_i P(N_{i-1}); \text{ and}$$

$$P(\widetilde{A}) = \sum_{i=1}^{m} P(A_i); \qquad P(\widetilde{B}) = \sum_{i=1}^{m} P(B_i); \qquad P(\widetilde{D}) = \sum_{i=1}^{m} P(D_i) .$$

Expected Number of Wins Until a Loss

Let $\theta(m)$ denote the probability that in a given game unit A is hit. Then, $\theta(m) = P(\widetilde{B}) + P(\widetilde{D}) = \sum_{i=1}^{m} q_i P(N_{i-1})$. Let $N(m)$ denote the number of games elapsed until A is hit. Clearly, $P[N(m) = n] = [1-\theta(m)]^{n-1}\theta(m)$, $n = 1, 2, 3, \ldots$. If $N(m) = n$ the probability that at the first $n-1$ games A hits ℓ units of B $(0 \le \ell \le n-1)$ is given by

$$b(\ell; n-1) = \binom{n-1}{\ell}[P(\widetilde{A})/(1-\theta(m))]^{\ell}[P(N_m)/(1-\theta(m))]^{n-1-\ell} .$$

Let $R(m)$ be the expected number of hits by unit A until it is hit. Then,

$$R(m) = \sum_{n=1}^{\infty} \theta(m)[1-\theta(m)]^{n-1} \sum_{\ell=0}^{n-1} b(\ell; n-1) \left[\ell \cdot \frac{P(\widetilde{B})}{\theta(m)} + (\ell+1)\frac{P(\widetilde{D})}{\theta(m)} \right]$$
$$= [P(\widetilde{A}) + P(\widetilde{D})]/\theta(m) = q(m)/\theta(m), \qquad (13)$$

where $q(m) = P(\widetilde{A}) + P(\widetilde{D}) = \sum_{i=1}^{m} p_i P(N_{i-1})$ is the probability that B is hit in a given game.

Result (13) has an intuitive explanation similar to that of Theorem 1: $R(m)$ equals the expected number of games until A is hit $[= 1/\theta(m)]$ times the expected number of hits by A in a game $[= P(\widetilde{A}) + P(\widetilde{D})]$.

Improving Versus Constant Hit Probabilities

For any two (empirical of analytic) discrete distributions one can use equation (13) to calculate and compare values of $R(m)$. As an example we present the case where unit B has the same hitting probability in all rounds, while unit A improves its hitting probability from one round to the next.

Specifically, we assume that $p_i = 1 - e^{-ip}, 1 > p > 0, i = 1, 2, \ldots$, while $q_i = q$ for all i. Then, using (13) and noting that $P(N_{i-1}) = \prod_{j=1}^{i-1} (1-p_j)(1-q_j) = (1-q)^{i-1} \exp[-pi(i-1)/2]$, we derive

$$R(m) = \frac{\sum_{i=1}^{m} (1-e^{-ip})(1-q)^{i-1} \exp[-pi(i-1)/2]}{q \sum_{i=1}^{m} (1-q)^{i-1} \exp[-pi(i-1)/2]} .$$

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