RETRIAL NETWORKS WITH FINITE BUFFERS AND THEIR APPLICATION TO INTERNET DATA TRAFFIC

Kostia Avrachenkov* Uri Yechiali†

Abstract

Data on the Internet is sent by packets that go through a network of routers. A router drops packets either when its buffer is full or when it uses the Active Queue Management. Up to the present, majority of the Internet routers use a simple DropTail strategy. The rate at which a user injects the data into the network is determined by Transmission Control Protocol (TCP). However, most connections in the Internet consist only of few packets, and TCP does not really have an opportunity to adjust the sending rate. Thus, the data flow generated by short TCP connections appears to be some uncontrolled stochastic process. In the present work we try to describe the interaction of the data flow generated by short TCP connections with a network of finite buffers. The framework of retrial queues and networks seems to be an adequate approach for this problem. The effect of packet retransmission becomes essential when the network congestion level is high. We consider several benchmark retrial network models. In some particular cases explicit analytic solution is possible. If the analytic solution is not available or too entangled, we suggest to use a fixed point approximation scheme. In particular, we consider a network of one or two tandem $M/M/1/K$ queues with blocking and $M/M/1/\infty$ retrial (orbit) queue. We explicitly solve the models with $K = 1$, derive stability conditions, and present several graphs based on numerical results.

*INRIA Sophia Antipolis, France, e-mail: k.avrachenkov@sophia.inria.fr

†Tel Aviv University, Israel, e-mail: uriy@post.tau.ac.il

1 Introduction

Data on the Internet is sent by packets that go through a network of routers. A router drops packets either when its buffer is full or when the router uses the Active Queue Management (AQM) [7]. When the AQM is used, an incoming packet is dropped by the router with probability which is a function of the average queue size. The dropped packets are then retransmitted by the sender. The rate at which a user injects the data into the network is determined by Transmission Control Protocol (TCP) [1, 8]. However, most connections in the Internet consist only of few packets, and TCP does not get an opportunity to adjust the sending rate. Thus, the data flow generated by short TCP connections appears to be a stochastic noise that cannot be correctly represented by the utility function optimization approach [9, 11, 12]. In the present work we try to describe the data flow generated by short TCP connections with the help of retrial networks. We note that up to the present most of the work on retrial models has been done on a single retrial queue [2, 3, 5, 6].

Let us model the data network as a set of links L. Let I be a set of major data flows that traverse the network. These major flows can be interpreted for instance as the aggregation of flows that go from some Internet Service Provider (ISP) to some major Web site or portal. We assume that each data flow $i \in I$ follows a fixed path $\pi_i = \{l_1^i, ..., l_{n(i)}^i\}$. We also define $\pi_i(u) = \{v_1^i, ..., u\}$, that is, $\pi_i(u)$ corresponds to the part of the path π_i from the source link v_1^i up to link u. There is a buffer of size K_l associated with each link $l \in L$, where, if needed, the packets wait for transmission. We denote the transmission capacity of link l by μ_l. We assume that the packet transmission time is exponentially distributed. Of course, we are aware of the fact that the routing in the Internet is dynamic and that packets from the same TCP connection may follow different routes if some links in the network go down. We suppose that these deficiencies are not frequent and that the routing tables in the Internet routers do not change during long periods of time so that our assumptions can hold. This has been shown to be the case in the Internet [14] where more than 2/3 of the routes persist for days or even weeks. If a packet from flow i is lost in some router either because of the buffer overflow or because of the preventive drop by AQM, it is retransmited by the source after some random time. This random time can be modelled either by $M/M/1/\infty$ or by $M/M/\infty$ queue with retransmission rate μ_{0i}. We denote the nominal aggregated sending rate of flow $i \in I$ by λ_i. By the nominal aggregated sending rate we mean the rate of flow i not counting retransmited packets. Of course, the actual sending rate including the rate of retransmited packets is higher than the nominal rate.

The exact analysis of the above network model does not seem to be feasible for the general case. Therefore, we propose and study particular cases and approximation schemes. For instance, we can assume that packets are lost in buffers with some fixed probabilities. These probabilities can in particular be functions of the buffer length, the average load or the average queue length as is the case in the AQM routers. We call this approach a fixed point approximation. In the present work we consider the following two particular cases: (i) a single $M/M/1/K$ retrial queue with $M/M/1/\infty$ orbit queue, and (ii) two tandem $M/M/1/1$ queues with $M/M/1/\infty$ orbit queue. Even for these two basic network examples the exact calculations of system characteristics are involved. Having in hand the exact solutions for the basic network schemes helps us to determine the cases when the fixed point approximation works well.

In section 2 we study the case with a single $M/M/1/K$ primary queue and a $M/M/1/\infty$ orbit queue. We derive explicit expressions for various key probabilities in the cases where K=1 and K=2, and derive the stability condition for an arbitrary value of K. We further consider a fixed point approximation scheme where we assume that the drop probabilities are fixed (yet, depending on system parameters). We exhibit various graphs showing the regions where the approximation works well. In section 3 we consider a network with two M/M/1/1 queues in tandem and a $M/M/1/\infty$ orbit queue. We obtain explicit solutions for certain probabilities, and derive the (involved) stability condition. We then analyze our fixed point approximation (which coincides with the exact solution for the single node case with K=1). Finally, we calculate mean queue sizes and present graphs depicting the dependence of those sizes on system parameters.

2 $M/M/1/K$ primary queue with $M/M/1/\infty$ ordit queue

Let us consider a basic single node example of retrial networks. Namely, we consider the case of one $M/M/1/K$ primary queue with $M/M/1/\infty$ ordit queue. Customerss arriving to a full buffer in the primary queue are blocked and go to the orbit queue. Each orbit customer first waits in the ordit queue and then retries to enter the primary queue after exponentially distributed time. This models the process of packet retransmissions by the source after they are lost in the Internet routers. The transition rate diagram of the associated Markov chain is depicted in Figure 1, where the horizontal axis depicts the primary queue occupancy and the vertical axis depicts the number of jobs in orbit. The present model is a particular case of the more general retrial queue $BMAP/PH/N/N+R$ analysed in [10]. The authors of [10] use the matrix analytic technique [13]. In order to obtain explicit analytical results we have decided to perform a more detail analysis of the simpler model.

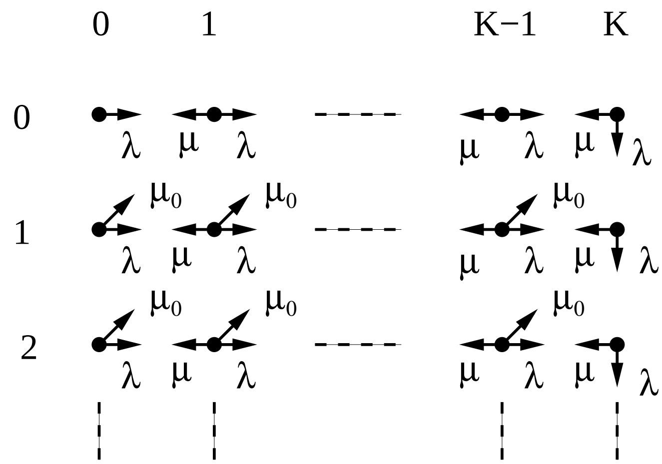

Figure 1: Transition rate diagram

We denote the steady state probabilities for this system by $P_i(n)$, $i = 0, 1, ..., K$, $n = 0, 1, ...$, where the index i corresponds to the number of jobs in the primary queue and the index n corresponds to the number of jobs in the orbit queue. Also, we denote the input rate to the system by λ, the service rate of the primary queue by μ and the service rate of the orbit queue by μ_0. We can write the following sets of balance equations.

- For $i = 0$:

$$\lambda P_0(0) = \mu P_1(0),$$

$$(\lambda + \mu_0)P_0(n) = \mu P_1(n), \quad n \geq 1,$$

- For $0 < i < K$:

$$(\lambda + \mu)P_i(0) = \lambda P_{i-1}(0) + \mu P_{i+1}(0) + \mu_0 P_{i-1}(1),$$

$$(\lambda + \mu + \mu_0)P_i(n) = \lambda P_{i-1}(n) + \mu P_{i+1}(n) + \mu_0 P_{i-1}(n+1), \quad n \geq 1,$$

- For $i = K$:

$$(\lambda + \mu)P_K(0) = \lambda P_{K-1}(0) + \mu_0 P_{K-1}(1),$$

$$(\lambda + \mu)P_K(n) = \lambda P_{K-1}(n) + \mu_0 P_{K-1}(n+1) + \lambda P_K(n-1), \quad n \geq 1,$$

- The set of balance equations for the transitions of the number of jobs in the orbit queue between levels n and $n+1$ is given by

$$\lambda P_K(n) = \mu_0 \sum_{i=0}^{K-1} P_i(n+1), \quad n \geq 0.$$

Using the above sets of balance equations, we derive a system of equations for the generating functions $G_i(z) = \sum_{n=0}^{\infty} z^n P_i(n)$, $i = 0, 1, ..., K$,

$$(\lambda + \mu_0)G_0(z) - \mu_0 P_0(0) = \mu G_1(z),$$

$$(\lambda + \mu + \mu_0)G_i(z) - \mu_0 P_i(0) = \lambda G_{i-1}(z) + \mu G_{i+1}(z) + \frac{\mu_0}{z}(G_{i-1}(z) - P_{i-1}(0)), \quad 0 < i < K,$$

$$(\lambda + \mu)G_K(z) = \lambda G_{K-1}(z) + \frac{\mu_0}{z}(G_{K-1}(z) - P_{K-1}(0)) + \lambda z G_K(z),$$

$$\lambda G_K(z) = \frac{\mu_0}{z} \sum_{i=0}^{K-1} (G_i(z) - P_i(0)).$$

In fact, one of the last two equations is redundant.

Next, we study the particular cases of $K = 1$ and $K = 2$. Consider first $K = 1$. If we set $z = 1$ in the first and the last of the above equations for the generating functions, denote $P_i(\cdot) = G_i(1)$ and add the normalization condition, we get the following system of equations

$$(\lambda + \mu_0)P_0(\cdot) - \mu P_1(\cdot) = \mu_0 P_0(0),$$

$$-\mu_0 P_0(\cdot) + \lambda P_1(\cdot) = -\mu_0 P_0(0),$$

$$P_0(\cdot) + P_1(\cdot) = 1,$$

which results in

$$P_0(\cdot) = 1 - \frac{\lambda}{\mu},$$

$$P_1(\cdot) = \frac{\lambda}{\mu},$$

$$P_0(0) = 1 - \frac{\lambda}{\mu} - \frac{\lambda}{\mu_0}\frac{\lambda}{\mu} = 1 - \frac{\lambda}{\mu}(1 + \frac{\lambda}{\mu_0}).$$

Furthermore, given $P_0(0)$, $G_0(z)$ and $G_1(z)$ are easily calculated from the above set of equations for the generating functions. Since the empty system is a regeneration point, $P_0(0) > 0$ is a necessary and sufficient condition for the system stability. Note that the primary queue is always stable. The stability issue is with regard to the orbit queue. The condition $P_0(0) > 0$ is equivalent to

$$\rho < \frac{1}{1 + \rho_0}, \tag{1}$$

with $\rho = \lambda/\mu$ and $\rho_0 = \lambda/\mu_0$. That is, $\lambda < (\sqrt{\mu_0(\mu_0 + 4\mu)} - \mu_0)/2$. One can see that the stability condition for the system composed of $M/M/1/1$ retrial queue and $M/M/1/\infty$ orbit queue is more restrictive than the stability condition for the classical queue with infinite buffer. We also note that as $\mu_0 \to \infty$, $\rho_0 \to 0$, and the above stability condition becomes the standard condition $\rho < 1$.

Next, let us analyze the case $K = 2$. Using again the equations for the generating functions and setting $z = 1$, we obtain the following system of linear equations for $P_0(\cdot), P_1(\cdot), P_2(\cdot), P_0(0)$ and $P_1(0)$

$$(\lambda + \mu_0)P_0(\cdot) - \mu_0 P_0(0) = \mu P_1(\cdot),$$

$$(\lambda + \mu + \mu_0)P_1(\cdot) - \mu_0 P_1(0) = \lambda P_0(\cdot) + \mu P_2(\cdot) + \mu_0(P_0(\cdot) - P_0(0)),$$

$$\lambda P_2(\cdot) = \mu_0(P_0(\cdot) - P_0(0)) + \mu_0(P_1(\cdot) - P_1(0)),$$

$$\lambda P_0(0) = \mu P_1(0),$$

$$P_0(\cdot) + P_1(\cdot) + P_2(\cdot) = 1.$$

The solution of the above system is given by

$$P_0(\cdot) = 1 - \frac{\lambda}{\mu},$$

$$P_1(\cdot) = \frac{\lambda\mu(\mu + \mu_0) - \lambda^2(\mu_0 - \mu) - \lambda^3}{\mu^2(2\lambda + \mu + \mu_0)},$$

$$P_2(\cdot) = \frac{\lambda^2(\lambda + \mu + \mu_0)}{\mu^2(2\lambda + \mu + \mu_0)},$$

$$P_0(0) = \frac{\mu\mu_0(\mu + \mu_0) - \lambda\mu_0(\mu_0 - \mu) - 2\lambda^2\mu_0 - \lambda^3}{\mu\mu_0(2\lambda + \mu + \mu_0)},$$

$$P_1(0) = \frac{\lambda\mu\mu_0(\mu + \mu_0) - \lambda^2\mu_0(\mu_0 - \mu) - 2\lambda^3\mu_0 - \lambda^4}{\mu^2\mu_0(2\lambda + \mu + \mu_0)}.$$

As in the case $K = 1$, the stability condition is given by

$$P_0(0) > 0,$$

which is equivalent to

$$\mu\mu_0(\mu + \mu_0) - \lambda\mu_0(\mu_0 - \mu) - 2\lambda^2\mu_0 - \lambda^3 > 0.$$

Sunstituting into the above inequality $\mu = \lambda/\rho$ and $\mu_0 = \lambda/\rho_0$, we obtain the stability condition in terms of ρ and ρ_0 as follows:

$$-(1 + \rho_0)^2\rho^2 + (1 + \rho_0)\rho + \rho_0 > 0. \tag{2}$$

Solving the above quadratic inequality, we get

$$\rho < \frac{1 + \sqrt{1 + 4\rho_0}}{2(1 + \rho_0)}. \tag{3}$$

Since $1 + \sqrt{1 + 4\rho_0} > 2$, the above condition (3) gives a larger region of stability than the one in condition (1). As one could expect, the increase of the buffer space in the primary queue improves the stability of the system. In the case of fast retransmissions ($\rho_0 \to 0$), condition (1) can be approximated as

$$\rho < 1 - \rho_0 + \mathrm{o}(\rho_0),$$

and condition (3) can be approximated as

$$\rho < 1 - \rho_0^2 + \mathrm{o}(\rho_0^2).$$

In Figure 2 we plot the values of the right hand sides of conditions (1) and (3) for small and medium values of ρ_0. Clearly, the stability region increases significantly when moving from $K = 1$ to $K = 2$.

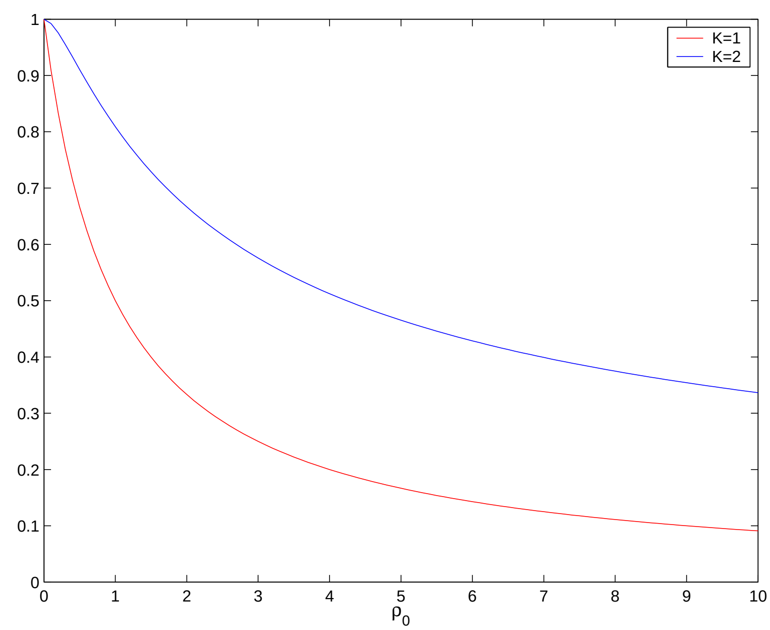

Figure 2: Comparison of stability conditions for $K = 1$ and $K = 2$ (right hand sides of inequalities (1) and (3)).

Next, let us derive the stability condition for arbitrary K. Let us enumerate the system states (n, i) in lexicographic order. Then, the Markov chain generator for the system has the block structure

$Q = \{Q_{ij}\}_{i,j\geq 0}$ where

$$Q_{i,i-1} = \begin{bmatrix} 0 & \mu_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu_0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad i \geq 1,$$

$$Q_{i,i} = \begin{bmatrix} -\lambda-(1-\delta_{i,0})\mu_0 & \lambda & 0 & \cdots & 0 & 0 \\ \mu & -\lambda-\mu-(1-\delta_{i,0})\mu_0 & \lambda & \cdots & 0 & 0 \\ 0 & \mu & -\lambda-\mu-(1-\delta_{i,0})\mu_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda-\mu-(1-\delta_{i,0})\mu_0 & \lambda \\ 0 & 0 & 0 & \cdots & \mu & -\mu \end{bmatrix},$$

$$i \geq 0,$$

$$Q_{i,i+1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}, \quad i \geq 0,$$

and the other blocks are zeros. All blocks are of dimension $(K+1)\times(K+1)$. Let us denote $Q_0 = Q_{i,i-1}$, $Q_1 = Q_{i,i}$, and $Q_2 = Q_{i,i+1}$ for $i \geq 1$. Then, the necessary and sufficient condition for the system stability is given by the following inequalilty [4, 13]

$$xQ_2e < xQ_0e, \tag{4}$$

where x is an $1\times(K+1)$ vector which is the unique solution to the system

$$\begin{cases} x(Q_0+Q_1+Q_2) = 0, \\ xe = 1, \end{cases} \tag{5}$$

and where e is the vector of ones and its dimension is $(K+1)\times 1$. The system (5) has an explicit solution

$$x_r = x_0\left(\frac{\lambda+\mu_0}{\mu}\right)^r = x_0\rho^r\left(\frac{1+\rho_0}{\rho_0}\right)^r, \quad r = 0, ..., K, \tag{6}$$

with

$$x_0 = \left(\sum_{r=0}^{K} \left(\frac{\lambda + \mu_0}{\mu} \right)^r \right)^{-1}.$$

Substituting (6) into (4), we obtain the following stability condition in terms of ρ and ρ_0.

Proposition 1 *The $M/M/1/K$ retrial queue with $M/M/1/\infty$ orbit queue is stable if and only if*

$$1 + \sum_{r=1}^{K-1} \rho^r \left(\frac{1+\rho_0}{\rho_0} \right)^r - \rho_0 \rho^K \left(\frac{1+\rho_0}{\rho_0} \right)^K > 0. \tag{7}$$

First, we note that the conditions (1) and (2) are indeed particular cases of the above general condition (7). Then, let us study the limiting case $K \to \infty$. Denote $\xi = \rho \frac{1+\rho_0}{\rho_0}$. Using this new notation, the condition (7) can be rewritten as follows:

$$1 + \sum_{r=1}^{K-1} \xi^r - \rho_0 \xi^K > 0,$$

or, equivalently,

$$\frac{\xi^K - 1}{\xi - 1} - \rho_0 \xi^K > 0. \tag{8}$$

Next, consider two cases $0 < \xi < 1$ and $\xi > 1$. We note that the conditions $0 < \xi < 1$ and $\xi > 1$ are equivalent to the conditions $\rho < \frac{\rho_0}{1+\rho_0}$ and $\rho > \frac{\rho_0}{1+\rho_0}$, respectively. When $0 < \xi < 1$, the condition (8) is equivalent to

$$\xi^K - 1 - \rho_0 \xi^K (\xi - 1) < 0.$$

Since $0 < \xi < 1$, one can always find a sufficiently large K such that the above condition is satisfied. Thus, the system is always stable in the case $\rho < \frac{\rho_0}{1+\rho_0}$. In the second case $\xi > 1$, the condition (8) is equivalent to

$$\xi^K - 1 - \rho_0 \xi^K (\xi - 1) > 0.$$

We can rewrite the above condition as follows:

$$(1 + \rho_0) - \rho_0 \xi - \frac{1}{\xi^K} > 0.$$

For the limiting case $K \to \infty$, the above condition reduces to

$$\xi < \frac{1+\rho_0}{\rho_0}.$$

or, equivalently,

$$\rho < 1.$$

Since $\frac{\rho_0}{1+\rho_0} < 1$, we can combine the two separate cases $0 < \xi < 1$ and $\xi > 1$ to produce a single stability condition: $\rho < 1$. This condition is expected as in the system with large buffer size K the retrials are seldom and the system behaves similarly to the standard $M/M/1$ queue.

Now let us consider an approximate model where we assume that packets are dropped with a fixed probability P. It was shown in [3] that the approximate model with fixed probabilities approximates well the exact retrial model when the nominal load is small. First, we consider the case of arbitrary value of K and then we study in more detail the cases $K = 1$ and $K = 2$.

Taking into account retransmissions, the actual load on the primary queue is given by the following formula

$$\hat{\rho} = \frac{\rho}{1-P}, \tag{9}$$

where $\rho = \lambda/\mu$ is the nominal load.

Assuming that the packets arrive to the primary queue according to a Poisson process, one can use for the approximation the classical $M/M/1/K$ queueing model. Thus, the drop probability is given by

$$P = \frac{\hat{\rho}^K(1-\hat{\rho})}{(1-\hat{\rho}^{K+1})}, \tag{10}$$

where K is the total buffer size in packets.

Proposition 2 *If $\rho < 1$, the system of equations* (9) *and* (10) *has a unique solution. This solution can be found by fixed point iterations*

$$\hat{\rho}^{(n+1)} = \rho \frac{1-(\hat{\rho}^{(n)})^{K+1}}{1-(\hat{\rho}^{(n)})^K} \tag{11}$$

which converges for any initial value $\hat{\rho}^{(0)} \in [0, \infty)$.

PROOF: We substitute the expression for the packet loss probability (10) into (9) to get

$$\hat{\rho} = \frac{\rho(1-\hat{\rho}^{K+1})}{1-\hat{\rho}^K}.$$

We can rewrite it as follows:

$$\frac{\hat{\rho}(1-\hat{\rho}^K)}{1-\hat{\rho}^{K+1}} = \rho,$$

$$\frac{\hat{\rho}+\hat{\rho}^2+...+\hat{\rho}^K}{1+\hat{\rho}+\hat{\rho}^2+...+\hat{\rho}^K} = \rho,$$

$$1 - \frac{1}{1+\hat{\rho}+\hat{\rho}^2+...+\hat{\rho}^K} = \rho.$$

Now let us consider the left hand side of the above equation. Clearly this function of $\hat{\rho}$ is strictly increasing on the interval $[0,\infty)$. Moreover, it has a horizontal asymptote $y=1$. Hence, if $\rho \geq 1$, there is no solution and if $\rho < 1$ there is a unique solution.

Next we show that the fixed point iterations (11) converge to the solution of (9) and (10). Since the above considerations demonstrate that there is a unique fixed point (of course, we are now interested only in the case $\rho < 1$), we only need to prove that the fixed point iterations converge. To prove this, it is enough to show that the derivative of the right hand side of (11) is less than one.

$$\frac{\mathrm{d}}{\mathrm{d}\hat{\rho}}\left[\rho\frac{1-\hat{\rho}^{K+1}}{1-\hat{\rho}^K}\right] = \frac{\mathrm{d}}{\mathrm{d}\hat{\rho}}\left[\rho\left(1+\frac{\hat{\rho}^K}{1+\hat{\rho}+...+\hat{\rho}^{K-1}}\right)\right] =$$

$$\rho\frac{K\hat{\rho}^{K-1}+(K-1)\hat{\rho}^K+...+2\hat{\rho}^{2K-3}+\hat{\rho}^{2K-2}}{(1+\hat{\rho}+...+\hat{\rho}^{K-1})^2} =$$

$$\rho\frac{K\hat{\rho}^{K-1}+(K-1)\hat{\rho}^K+...+2\hat{\rho}^{2K-3}+\hat{\rho}^{2K-2}}{1+2\hat{\rho}+...+K\hat{\rho}^{K-1}+(K-1)\hat{\rho}^K+...+2\hat{\rho}^{2K-3}+\hat{\rho}^{2K-2}} < \rho < 1$$

This completes the proof.

□

Let us now consider the particular case $K=1$. Solving the system of fixed point equations

$$P = \frac{\hat{\rho}}{1+\hat{\rho}},$$

$$\hat{\rho} = \frac{\rho}{1-P},$$

we obtain

$$P = \rho,$$

$$\hat{\rho} = \frac{\rho}{1-\rho}.$$

Now recall that in the case $K=1$ we have $P_1(\cdot)=\rho$. Hence, in this particular case the approximate packet drop probability P matches exactly $P_1(\cdot)$.

Next, we consider the particular case $K=2$. By solving the set of fixed point approximation equations

$$P = \frac{\hat{\rho}^2}{1+\hat{\rho}+\hat{\rho}^2},$$

$$\hat{\rho} = \frac{\rho}{1-P},$$

we obtain

$$P = \frac{1}{4}(\sqrt{1+3\rho} - \sqrt{1-\rho})^2.$$

First, we note that in contrast to the case $K = 1$ the approximate packet drop probability P does not match exactly $P_2(\cdot)$. There are two important limiting cases:

$$P_2(\cdot) = \frac{\lambda^2(\lambda+\mu+\mu_0)}{\mu^2(2\lambda+\mu+\mu_0)} \to \rho^2, \quad \text{as} \quad \mu_0 \to \infty,$$

$$P_2(\cdot) = \frac{\lambda^2(\lambda+\mu+\mu_0)}{\mu^2(2\lambda+\mu+\mu_0)} \to \frac{\rho^2(1+\rho)}{1+2\rho}, \quad \text{as} \quad \mu_0 \to 0.$$

In Figures 3 and 4 we plot the relative error $|P - P_2(\cdot)|/P_2(\cdot)$ as a function of ρ for the limiting cases $\mu_0 \to \infty$ and $\mu_0 \to 0$, respectively.

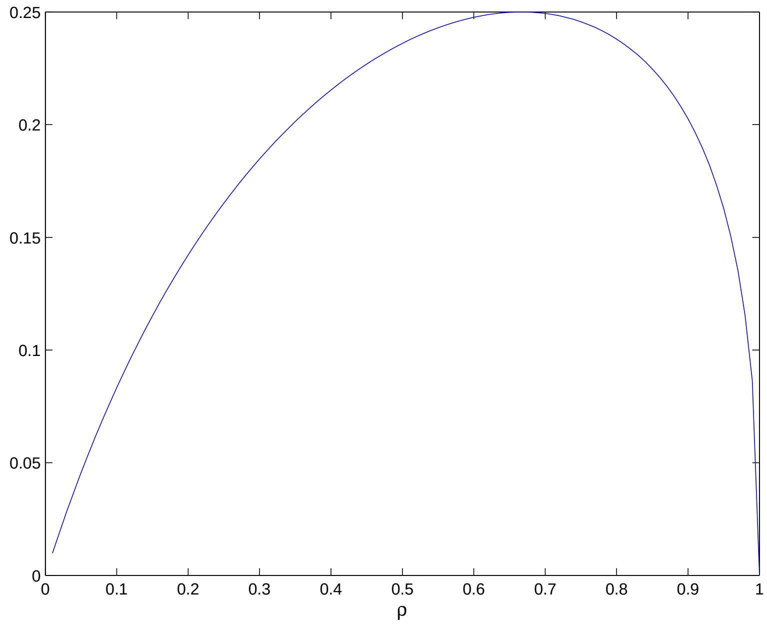

Figure 3: The relative error $|P - P_2(\cdot)|/P_2(\cdot)$ in the limiting case $\mu_0 \to \infty$.

As one can see, P approximates $P_2(\cdot)$ well when ρ and μ_0 are both small.

3 Two M/M/1/1 queues in tandem with M/M/1/∞ ordit queue

As another particular case of the network of finite queues with retrials, let us consider two M/M/1/1 queues in tandem with M/M/1/∞ orbit queue. We denote the steady state probabilities for this system by $P_{ij}(n)$, $i, j = 0, 1$ and $n = 0, 1, \ldots$, where the indicies i and j correspond to the number

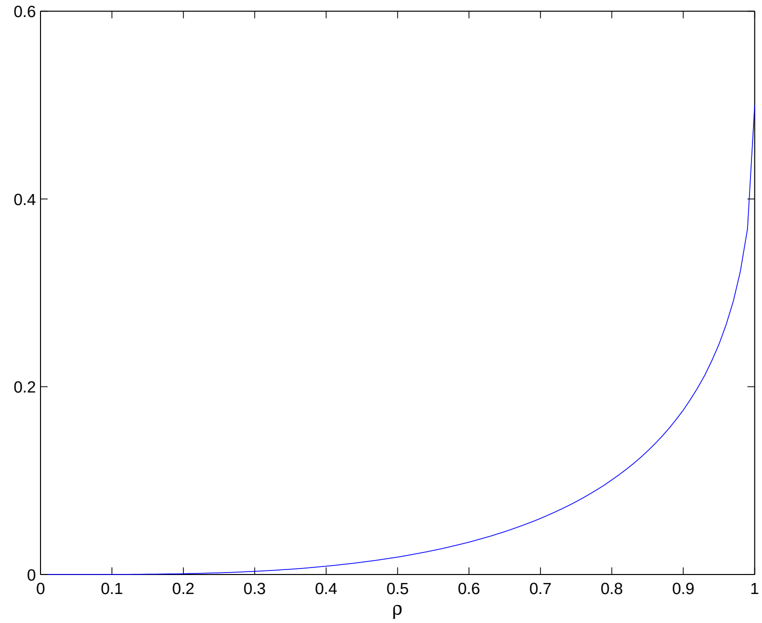

Figure 4: The relative error $|P - P_2(\cdot)|/P_2(\cdot)$ in the limiting case $\mu_0 \to 0$.

of jobs in the first and the second queues, respectively, and n corresponds to the number of jobs in the ordit queue. We denote the input rate to the system by λ, the service rate of queue k by μ_k, $k = 1, 2$, and the service rate of the orbit queue by μ_0. The transition rate diagram is depicted in Figure 5. We can write the following five sets of balance equations:

- For the $(0, 0)$ column:

$$\lambda P_{00}(0) = \mu_2 P_{01}(0), \tag{12}$$

$$(\lambda + \mu_0) P_{00}(n) = \mu_2 P_{01}(n), \quad n \geq 1,$$

- For the $(1, 0)$ column:

$$(\lambda + \mu_1) P_{10}(0) = \lambda P_{00}(0) + \mu_0 P_{00}(1) + \mu_2 P_{11}(0),$$

$$(\lambda + \mu_1) P_{10}(n) = \lambda P_{00}(n) + \mu_0 P_{00}(n + 1) \lambda P_{10}(n - 1) + \mu_2 P_{11}(0), \quad n \geq 1,$$

- For the $(0, 1)$ column:

$$(\lambda + \mu_2) P_{01}(0) = \mu_1 P_{10}(0),$$

$$(\lambda + \mu_2 + \mu_0) P_{01}(n) = \mu_1 P_{10}(n) + \mu_1 P_{11}(n - 1), \quad n \geq 1,$$

- For the $(1,1)$ column:

$$(\lambda + \mu_1 + \mu_2)P_{11}(0) = \lambda P_{01}(0) + \mu_0 P_{01}(1),$$

$$(\lambda + \mu_1 + \mu_2)P_{11}(n) = \lambda P_{01}(n) + \mu_0 P_{01}(n+1) + \lambda P_{11}(n-1), \quad n \geq 1,$$

- Taking a "cut" between levels n and $n+1$ of the number of jobs in orbit yields:

$$\lambda P_{10}(n) + (\lambda + \mu_1)P_{11}(n) = \mu_0 P_{00}(n+1) + \mu_0 P_{01}(n+1), \quad n \geq 0.$$

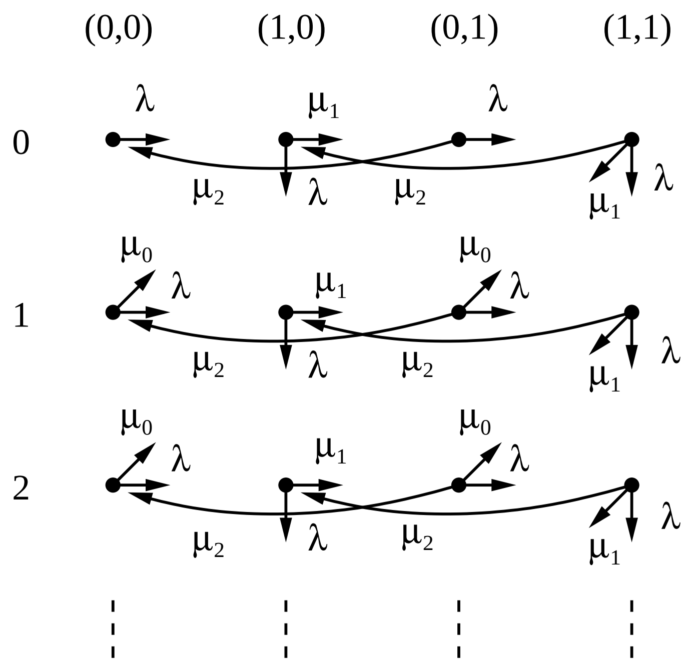

Figure 5: Transition rate diagram

Using the above sets of balance equations, we derive a system of equations for the generating functions $G_{ij}(z) = \sum_{n=0}^{\infty} z^n P_{ij}(n)$, $i, j = 0, 1$.

$$(\lambda + \mu_0)G_{00}(z) - \mu_2 G_{01}(z) = \mu_0 P_{00}(0), \tag{13}$$

$$(\lambda z + \mu_0)G_{00}(z) - z(\lambda(1-z) + \mu_1)G_{10}(z) + \mu_2 z G_{11}(z) = \mu_0 P_{00}(0), \tag{14}$$

$$-\mu_1 G_{10}(z) + (\lambda + \mu_2 + \mu_0)G_{01}(z) - \mu_1 z G_{11}(z) = \mu_0 P_{01}(0), \tag{15}$$

$$(\lambda z + \mu_0)G_{01}(z) - z(\lambda(1-z) + \mu_1 + \mu_2)G_{11}(z) = \mu_0 P_{01}(0), \tag{16}$$

$$\mu_0 G_{00}(z) - \lambda z G_{10}(z) + \mu_0 G_{01}(z) - (\lambda + \mu_1)z G_{11}(z) = \mu_0 P_{00}(0) + \mu_0 P_{01}(0). \tag{17}$$

Next, we denote $P_{ij}(\cdot) = G_{ij}(1)$ for $i, j = 0, 1$, and set $z = 1$ in equations (13),(14),(15),(17), add equation (12) and the normalization equation $P_{00}(\cdot) + P_{10}(\cdot) + P_{01}(\cdot) + P_{11}(\cdot) = 1$. This results in a system of linear independent equations:

$$(\lambda + \mu_0)P_{00}(\cdot) - \mu_2 P_{01}(\cdot) = \mu_0 P_{00}(0),$$

$$(\lambda + \mu_0)P_{00}(\cdot) - \mu_1 P_{10}(\cdot) + \mu_2 P_{11}(\cdot) = \mu_0 P_{00}(0),$$

$$-\mu_1 P_{10}(\cdot) + (\lambda + \mu_2 + \mu_0)P_{01}(\cdot) - \mu_1 P_{11}(\cdot) = \mu_0 P_{01}(0),$$

$$\mu_0 P_{00}(\cdot) - \lambda P_{10}(\cdot) + \mu_0 P_{01}(\cdot) - (\lambda + \mu_1)P_{11}(\cdot) = \mu_0 P_{00}(0) + \mu_0 P_{01}(0),$$

$$\lambda P_{00}(0) = \mu_2 P_{01}(0),$$

$$P_{00}(\cdot) + P_{10}(\cdot) + P_{01}(\cdot) + P_{11}(\cdot) = 1.$$

The solution of the above system of linear equations leads to the following result

Proposition 3 *The probabilities $P_{ij}(\cdot)$ and $P_{ij}(0)$ for $i, j = 0, 1$ are given by*

$$P_{00}(\cdot) = 1 - \frac{\lambda}{\mu_1} - \frac{\lambda}{\mu_2},$$

$$P_{10}(\cdot) = \frac{\lambda}{\mu_1},$$

in particular, implying that

$$P_{01}(\cdot) + P_{11}(\cdot) = \frac{\lambda}{\mu_2},$$

$$P_{01}(\cdot) = \frac{\lambda(\mu_1\mu_2(\mu_1 + \mu_2 + \mu_0) - \lambda(\mu_0(\mu_1 + \mu_2) - \mu_1\mu_2) - \lambda^2(\mu_1 + \mu_2))}{\mu_1\mu_2^2(2\lambda + \mu_1 + \mu_2 + \mu_0)},$$

$$P_{11}(\cdot) = \frac{\lambda^2(\lambda(\mu_1 + \mu_2) + \mu_0(\mu_1 + \mu_2) + \mu_1\mu_2)}{\mu_1\mu_2^2(2\lambda + \mu_1 + \mu_2 + \mu_0)},$$

$$P_{00}(0) = \frac{1}{\mu_0\mu_1\mu_2(2\lambda + \mu_1 + \mu_2 + \mu_0)} \left(\mu_0\mu_1\mu_2(\mu_1 + \mu_2 + \mu_0) - \lambda[\mu_0^2(\mu_1 + \mu_2) + \mu_0(\mu_1^2 + \mu_2^2)] \right.$$
$$\left. -\lambda^2[\mu_1^2 + \mu_2^2 + \mu_1\mu_2 + 2\mu_0(\mu_1 + \mu_2)] - \lambda^3[\mu_1 + \mu_2]\right),$$

$$P_{01}(0) = \frac{\lambda}{\mu_2} P_{00}(0),$$

$$P_{10}(0) = \frac{\lambda(\lambda + \mu_2)}{\mu_1\mu_2} P_{00}(0),$$

$$P_{11}(0) = \frac{\lambda^2(\lambda^2 + \lambda(\mu_1 + \mu_2 + \mu_0) + \mu_0(\mu_1 + \mu_2) + \mu_1\mu_2)}{\mu_1\mu_2^2(2\lambda + \mu_1 + \mu_2 + \mu_0)} P_{00}(0).$$

Again, as in the case of a single $M/M/1/1$ retrial queue, once $P_{00}(0)$ and $P_{01}(0)$ are determined, the generating functions $G_{ij}(z)$ are uniquely determined by the system of linear equations (13)-(16).

Let us now investigate the stability of the system. We argue that the empty system is a regeneration point, and hence $P_{00}(0) > 0$ is a necessary and sufficient condition for the system stability. The condition $P_{00}(0) > 0$ is equivalent to

$$\mu_0\mu_1\mu_2(\mu_1 + \mu_2 + \mu_0) - \lambda[\mu_0^2(\mu_1 + \mu_2) + \mu_0(\mu_1^2 + \mu_2^2)]$$

$$-\lambda^2[\mu_1^2 + \mu_2^2 + \mu_1\mu_2 + 2\mu_0(\mu_1 + \mu_2)] - \lambda^3[\mu_1 + \mu_2] > 0. \quad (18)$$

As the above expression seems to be too complex to analyse in the general case, we consider two important particular cases: the case of large μ_0 and the case of $\mu_1 = \mu_2$.

We note that for large values of μ_0 the probability $P_{00}(0)$ has a simple asymptotics. Namely, we have

$$P_{00}(0) \to 1 - \frac{\lambda}{\mu_1} - \frac{\lambda}{\mu_2},$$

as $\mu_0 \to \infty$. Thus, if

$$\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} < 1, \quad (19)$$

then there always exists a large enough value of μ_0 that ensures the stability of the system.

Next, let us consider the case $\mu_1 = \mu_2 =: \mu$. Without loss of generality, we can take $\lambda = 1$. The stability condition (18) becomes

$$(\mu - 2)\mu_0^2 + 2[\mu(\mu - 1) - 2]\mu_0 - 3\mu - 2 > 0. \quad (20)$$

We first note that if $\mu < 2$, then $\mu - 2 < 0$ and $\mu(\mu - 1) - 2 < 0$, and consequently, the stability condition cannot be satisfied by any choice of μ_0. Thus, a necessary condition for stabililty in this particular case is $\mu > 2$. Once this condition is guaranteed, the parameter μ_0 has to be chosen greater than the positive root of the left hand side of (20). Namely,

$$\mu_0 > \frac{\sqrt{\mu^3(\mu - 2)} - [\mu(\mu - 1) - 2]}{\mu - 2}.$$

We note that the right hand side of the above inequality has the asymptotics $3/(2\mu)$ as $\mu \to \infty$. Indeed, when μ becomes very large, (almost) every job passes through queues 1 and 2 without blocking. If there are a few jobs that get blocked, a (very) small value of $\mu_0 > 3/(2\mu)$ will ensure stability.

Next, let us analyze the fixed point approximation model which is described by the following set of equations

$$\hat{\rho}_1 = \frac{\rho_1}{(1-P_1)(1-P_2)}, \tag{21}$$

$$\hat{\rho}_2 = \frac{\rho_2}{(1-P_2)}, \tag{22}$$

$$P_1 = \frac{\hat{\rho}_1}{1+\hat{\rho}_1}, \tag{23}$$

$$P_2 = \frac{\hat{\rho}_2}{1+\hat{\rho}_2}. \tag{24}$$

Note that equations (22) and (24) correspond exactly to the single node case with $K = 1$. Hence, we have

$$P_2 = \rho_2,$$

and

$$\hat{\rho}_2 = \frac{\rho_2}{1-\rho_2}.$$

Then, to get P_1 and $\hat{\rho}_1$, we substitute $P_2 = \rho_2$ and P_1 from equation (23) into equation (21),

$$\hat{\rho}_1 = \frac{\rho_1}{(1-P_1)(1-\rho_2)}, \tag{25}$$

which results in

$$\hat{\rho}_1 = \frac{\rho_1}{1-\rho_1-\rho_2},$$

and, consequently, we have

$$P_1 = \frac{\rho_1}{1-\rho_2}.$$

Note that the system of equations (21)-(24) has a unique and positive solution if and only if $\rho_1 + \rho_2 < 1$. It is interesting that this condition coincides with the stability condition (19) for the limiting case when $\mu_0 \to \infty$.

Let us now study the average number of jobs in the system. Differentiating equations (13),(14),(15), and (17), then setting $z = 1$ and denoting $L_{ij} := G'_{ij}(1)$, $i, j = 0, 1$, we obtain a system of linear equations

$$(\lambda + \mu_0)L_{00} - \mu_2 L_{01} = 0, \tag{26}$$

$$(\lambda + \mu_0)L_{00} - \mu_1 L_{10} + \mu_2 L_{11} = -\lambda P_{00}(\cdot) - (\lambda - \mu_1)P_{10}(\cdot) - \mu_2 P_{11}(\cdot), \tag{27}$$

$$-\mu_1 L_{10} + (\lambda + \mu_2 + \mu_0)L_{01} - \mu_1 L_{11} = \mu_1 P_{11}(\cdot), \tag{28}$$

$$\mu_0 L_{00} - \lambda L_{10} + \mu_0 L_{01} - (\lambda + \mu_1)L_{11} = \lambda P_{10}(\cdot) + (\lambda + \mu_1)P_{11}(\cdot). \tag{29}$$

Proposition 4 *The average number of jobs in orbit is given by*

$$L_{orbit} = L_{00} + L_{10} + L_{01} + L_{11}, \tag{30}$$

the average number of jobs in the system is given by

$$L_{system} = L_{orbit} + P_{10}(\cdot) + P_{01}(\cdot) + 2P_{11}(\cdot), \tag{31}$$

and the average time spent by a job in the system is given by

$$T_{system} = L_{system}/\lambda, \tag{32}$$

where L_{ij}*,* $i, j = 0, 1$ *are provided by the solution of the system of equations* (26)-(29)*, and the probabilities* $P_{ij(\cdot)}$ *are given in Proposition 3. Moreover, denoting by* L_1 *and* L_2 *the mean queue size in the first and the second* $M/M/1/1$ *queues, respectively, we have*

$$L_1 = P_{10}(\cdot) + P_{11}(\cdot),$$

$$L_2 = P_{01}(\cdot) + P_{11}(\cdot),$$

and

$$L_{system} = L_{orbit} + L_1 + L_2.$$

Let us consider a numerical example. We fix $\lambda = 1$ and the total network capacity $\mu_1 + \mu_2 = 5$. In Figure 6, we plot the average number of jobs in the system L_{system} as a function of the first node capacity μ_1. We study two cases: $\mu_0 = 4$ and $\mu_0 = 7$. First, we observe that in the middle of the capacity value range, the value of L_{system} becomes less sensitive when μ_0 increases. Second, it seems that the minimal value of L_{system} is achieved when $\mu_1 = \mu_2$. This fact is further confirmed by the more detailed Figure 7, but, of course, requires additional analytic investigation.

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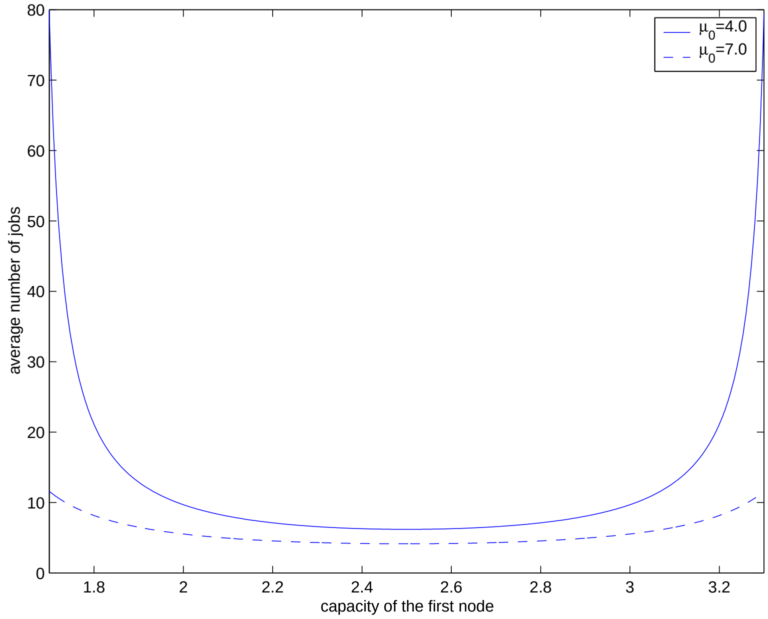

Figure 6: L_{system} as a function of μ_1 given $\mu_1 + \mu_2 = 5$

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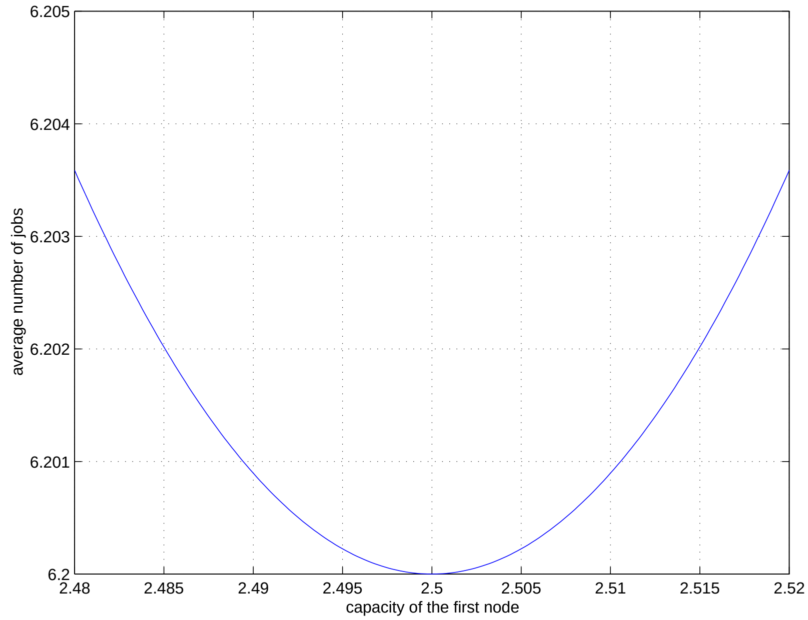

Figure 7: L_{system} as a function of μ_1 given $\mu_1 + \mu_2 = 5$ (detailed plot for $\mu_0 = 4.0$).

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