Optimal routing among $\cdot$/M/1 queues with partial information *

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Abstract

When routing dynamically randomly arriving messages, the controller of a high-speed communication network very often gets the information on the congestion state of down stream nodes only after a considerable delay, making that information irrelevant at decision epochs. We consider the situation where jobs arrive according to a Poisson process and must be routed to one of two (parallel) queues with exponential service time distributions (possibly with different means), without knowing the congestion state in one of the queues. However, (the conditional) probability distribution of the state of the unobservable queue can be computed by the router. We derive the joint probability distribution of the congestion states in both queues as a function of the routing policy. This allows us to identify optimal routing schemes for two types of frameworks: global optimization, in which the weighted sum of average queue lengths is minimized, and individual optimization, in which the goal is to minimize the expected delay of individual jobs.

Keywords: communication networks, performance optimization, stochastic modeling, queueing, dynamic routing, partial information, non-cooperative game, Nash equilibrium

1 Introduction

In many applications in which routing decisions have to be made, the information relevant to the decision maker is only partially available. This is true, in particular, in high speed communication networks in which information about the state of a down stream node may be subject to a considerable delay, thus making that information irrelevant at decision epochs. This situation may occur even in case the down stream link is quite close but — links being unidirectional — the effect of routing decisions on the down stream node appears much quicker than the return of information about that node's congestion state.

We analyze dynamic routing choices between two paths. We model path delays using two $\cdot$/M/1 queues, with state dependent arrivals. The latter dependence is due to the fact that the arrival to any queue depends on the routing decision, which, in turn, depends on the congestion state of the path. Specifically, we consider the problem in which the controller taking the routing decisions can observe only one of the queues. In spite of lacking precise information about the second queue, the controller can probabilistically evaluate the influence of its routing strategies on the distribution of the congestion state of the second queue.

We concentrate on routing strategies of a random threshold type. These are characterized by two parameters: (n, r). If the number of jobs in the first path X_1 is less than n at an instance of arrival, the arriving

*Most of this work was done at INRIA during a sabbatical stay of Prof. Yechiali and a visit of Dr. R. Núñez-Queija. The latter was funded by the Netherlands Organization for Scientific Research (NWO) under grant R 61-451.

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job is sent to path 1. If $X_1 = n$, then it is routed to path 1 with probability r and to path 2 with the complementary probability. If $X_1 > n$, then it is routed to path 2.

We study and analyze two types of dynamic routing control methods. In the first, we wish to determine the threshold parameters that minimize the average expected delay in the system. We call this the **Socially optimal routing** problem. Secondly, we consider **Individually optimal routing**. In this type of dynamic control we wish to obtain a routing strategy that minimizes the expected delay of each individual job. (The minimization of the expected delay for a given job does not take into account the impact of the individual routing decision on the delays that will be experienced by future jobs.) This type of routing approach has been advocated in the context of ad-hoc networks [**?**]. Although the routing decisions may be taken by a single controller, this control problem can be formulated as an equivalent game-theoretic model in which each job chooses its own route. We shall use the basic game-theoretic optimality concept of Nash equilibria [**?**]. Here, a Nash equilibrium is a set of routing strategies for all jobs such that no single job can decrease its weighted expected delay by deviating from its routing strategy.

The individual optimization problem was also considered in [**?**] in a different context: customers have to make a choice between two gas stations on a highway. Following a game theoretic analysis, a numerical solution is presented in [**?**] to obtain the performance for given candidate solutions. In our paper we provide a verifiable necessary condition to check whether an equilibrium indeed exists within the class of threshold policies (in fact this is a condition under which the best response to a threshold policy of all other users is also a threshold policy). When the service rates at the two stations are equal to each other, it is verified numerically (for $n = 1, 2, \ldots, 18$) that a threshold equilibrium exists, which is in agreement with the results in [**?**]. However, when the service rates at the two stations differ, such an equilibrium need no longer exist, as is shown in an example.

The structure of the paper is as follows. In Section **??** we formulate — for a fixed (n, r) policy — the underlying Quasi Birth and Death process and derive the balance equations for the two-dimensional steady state probability distribution. The marginal distribution of X_2, the number of jobs in queue 2, is derived in Section **??** for the case when $r = 1$. In Section **??** we develop expressions for the joint Probability Generating Functions (PGF) of the queue lengths for any $r \in (0, 1]$. These results are then used in the following sections to solve the routing problems in the two types of frameworks. The first, in Section **??**, focuses on a global (social) optimization framework in which the average weighted sum of queue lengths (or equivalently, waiting times) is to be minimized. Then in Section **??** we study the individual optimization in a game theoretic setting, in which the weighted expected delay of individual jobs is minimized. In all that follows, the delay of a job shall be the sum of its waiting time and its service time, i.e., a job's delay is its response time, which is the *total time spent in the system*. From the individual job's perspective it is optimal to join the queue with the smallest weighted expected delay. From a supervising controller's point of view, the effect of a single routing decision on the delay of future jobs must be taken into account.

We focus on the case $r = 1$ and use the obtained expressions to show (Lemma **??**) that if a job is routed to the second queue, then its expected delay, conditional on the number of customers in the first queue, say i, is convex in i. This implies (Corollary **??**) that if the objective is to minimize the expected delay of an individual job, the optimal strategy (for that individual job) will be a two-threshold policy, the job being routed to queue 2 if the number of customers in the first queue is between the two thresholds (see Figure **??**). Still, such a policy could be a 'normal' single-threshold policy, if the larger threshold value is not smaller than $n + 1$. Theorem **??** gives a condition on the parameters for which there exist values of the service rates such that no single-threshold policy is optimal in response to a fixed (n, r) policy.

2 Model description and stability condition

The system is modelled as two queues with exponential service rates μ_1 and μ_2, respectively. Arrivals of jobs to queue 1 occur according to a Poisson process with rate λ. The routing policy is of a random threshold type (n, r), as described in the Introduction.

The model gives rise to a so-called Quasi Birth and Death process. Such models are known to have a matrix-geometric equilibrium solution [?]. Following the approach in that monograph, we denote the state of the system by a pair of two integers (i, j), which indicates that there are $X_1 = i$ $(i = 0, 1, 2, ..., n+1)$ jobs in the first queue and $X_2 = j$ $(j = 0, 1, 2, 3, ...)$ jobs in the second. A transition rate diagram is depicted in Figure **??**, where the horizontal (resp., vertical) axis represents the number of customers in queue 1 (resp., 2). For the matrix-geometric approach the states are ordered lexicographically, such that (i_1, j_1) precedes

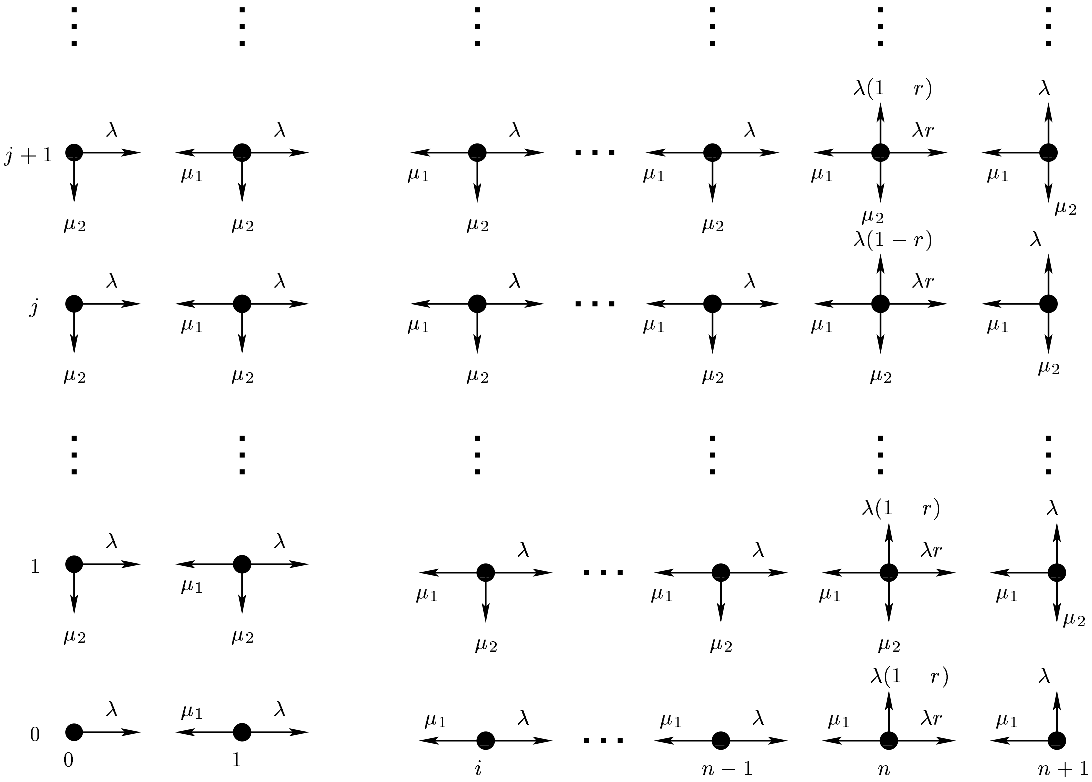

Figure 1: Transition diagram

(i_2, j_2) if and only if $j_1 < j_2$, or $\{j_1 = j_2$ and $i_1 < i_2\}$. Using this ordering of the states, the generator of the process is given by

$$\mathcal{Q} := \begin{bmatrix} Q_d^{(0)} & \Lambda & 0 & \dots & & \\ M & Q_d & \Lambda & 0 & \dots & \\ 0 & M & Q_d & \Lambda & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where Λ, M, $Q_d^{(0)}$ and Q_d are $(n+2) \times (n+2)$ matrices. $\Lambda = [\Lambda_{k,l}]$, $k, l = 0, 1, \dots, n+1$, is the matrix with $\Lambda_{n,n} = \lambda(1-r)$, $\Lambda_{n+1,n+1} = \lambda$, and all other elements equal to zero. M is also a diagonal matrix with all diagonal elements equal to μ_2. Finally, $Q_d^{(0)} = Q^{(Y)} - \Lambda$ and $Q_d = Q^{(Y)} - \Lambda - M$, where $Q^{(Y)}$ is the generator of the Markov process describing the number of jobs in the first queue,

$$Q^{(Y)} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ \mu_1 & -\lambda-\mu_1 & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & \mu_1 & -\lambda-\mu_1 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mu_1 & -\mu_1 - r\lambda & r\lambda \\ 0 & 0 & 0 & 0 & \dots & 0 & \mu_1 & -\mu_1 \end{pmatrix}.$$

Remark 2.1 The joint steady-state distribution of the numbers of jobs in both queues can also be derived for the following class of more general policies [**?**, Ch. 2]. Let $r_i \in [0,1]$ be the probability of joining queue 1 if the number of jobs in that queue equals i. Let $K < \infty$ be the maximum number of jobs in queue 1. The matrices Λ, $Q^{(Y)}$ and M become of dimension $K+1$. The diagonal matrix Λ has diagonal elements $(1-r_i)\lambda$, $i = 0, 1, \dots, K-1$. The matrix $Q^{(Y)}$ has elements $r_i\lambda$ just above the diagonal and (as before) all elements just below the diagonal are equal to μ_1. The matrix M is still μ_2 times the identity matrix.

Let $P_{ij} = \mathbf{P}\{X_1 = i, X_2 = j\}$ be the steady-state probability that the system is in state (i,j) and let the $(n+2)$-dimensional (row) vector $\overline{P}_j := (P_{0j}, P_{1j}, \dots, P_{n+1,j})$ contain the steady-state probabilities corresponding to *level* j, i.e., the subset of states in which there are j jobs in the second queue. Then the sequence $\overline{P}_j$, $j = 0, 1, \dots$, satisfying the balance equations

$$\overline{P}_0 Q_d^{(0)} + \overline{P}_1 M = \overline{0}, \tag{1}$$

$$\overline{P}_{j-1}\Lambda + \overline{P}_j Q_d + \overline{P}_{j+1} M = \overline{0}, \quad j = 1, 2, \dots, \tag{2}$$

(where $\overline{0}$ is the (n+2)-dimensional vector with all elements equal to 0) is known to satisfy the matrix-geometric relation

$$\overline{P}_j = \overline{P}_0 R^j \tag{3}$$

where the *rate* matrix R is the minimal non-negative solution to the quadratic matrix equation,

$$\Lambda + RQ_d + R^2 M = 0, \tag{4}$$

cf. [**?**, Theorem 3.1.1]. Relation (**??**) is not suitable to determine R in a stable and efficient way. Instead, a 'dual' quadratic matrix equation can be solved from which R is found, see [**?**]. Here we follow an alternative approach and determine the steady-state probabilities through a spectral characterization of the matrix R using generating functions. We refer to [**?**] for a general exposition of this method applied to the more general classes of Markov chains of M/G/1 and G/M/1 types. Computational advantages of the spectral expansion technique compared to the matrix-geometric solution were mentioned in [**?**, **?**].

First we note that the marginal probabilities of the number of jobs in queue 1 are easily obtained, since the rates of arrivals into and departures from queue 1 are independent of the state of queue 2. Define, for $0 \le i \le n+1$ and $j \ge 0$,

$$P_{i\bullet} \triangleq \sum_{j=0}^{\infty} P_{ij}, \qquad P_{\bullet j} \triangleq \sum_{i=0}^{n+1} P_{ij},$$

and the row vector

$$\overline{p} := (P_{0\bullet}, \dots, P_{n+1,\bullet}) = \sum_{j=0}^{\infty} \overline{P}_j.$$

We further define $\rho_1 = \lambda/\mu_1$, which is the traffic load on the first queue if all jobs are to be routed there. Clearly, the vector $\overline{p}$ satisfies

$$\overline{p}Q^{(Y)} = \overline{0}.$$

This follows, for instance, by summing the balance equations (??) and (??) over j. The marginal probabilities $P_{i\bullet}$ are now easily obtained:

$$\begin{aligned} P_{i\bullet} &= {\rho_1}^i P_{0\bullet}, \qquad 0 \le i \le n && (5)\\ P_{n+1,\bullet} &= r{\rho_1}^{n+1} P_{0\bullet}. && (6) \end{aligned}$$

Since the entries of $\overline{p}$ constitute a probability distribution we have the normalization condition $\langle \overline{p}, \overline{1} \rangle = 1$, where the column vector $\overline{1}$ has all entries equal to 1 and $\langle \cdot, \cdot \rangle$ denotes the inner product of a row vector and a column vector. We find:

$$P_{0\bullet} = \left[\sum_{i=0}^{n} {\rho_1}^i + r{\rho_1}^{n+1} \right]^{-1}$$

and when $\rho_1 \neq 1$ this sum is given by

$$P_{0\bullet} = \frac{1-\rho_1}{1-(1-r){\rho_1}^{n+1} - r{\rho_1}^{n+2}}. \tag{7}$$

The stability condition (cf. [?, Theorem 3.1.1]) is given by

$$\langle \overline{p}, \Lambda\overline{1} \rangle < \langle \overline{p}, M\overline{1} \rangle,$$

or, equivalently, by

$$\lambda(1-r)P_{n\bullet} + \lambda P_{n+1,\bullet} < \mu_2. \tag{8}$$

Evidently, it is the actual arrival rate into the second queue that determines whether the system is stable or not. Since $P_{n+1,\bullet} = P_{n\bullet} r\rho_1$, the stability condition becomes

$$\lambda P_{n\bullet} \left[(1-r) + r\rho_1 \right] < \mu_2,$$

i.e.,

$$\lambda \left(\rho_1\right)^n \left[(1-r) + r\rho_1\right] \left[\frac{1-\rho_1}{1-(1-r)(\rho_1)^{n+1} - r(\rho_1)^{n+2}} \right] < \mu_2. \tag{9}$$

3 Marginal distribution of X_2 when $r = 1$

The detailed analysis that we shall present in Section ?? will allow us to obtain the joint probability distributions of the two queues. This is in particular important for the application to the *individually optimal routing* problem, studied in Section ??. The decision whether a job should be routed to queue 2 depends on the expected delay of the job in that queue, conditioned on the state at queue 1. On the other hand, for the *socially optimal routing* problem, only the *marginal* distributions are needed, since the objective will be to minimize the (weighted) average expected delay in both queues. Thus, in addition to the marginal distribution of the (number of jobs in the) first queue, we need also the marginal distribution of the second queue. Here we derive it for the case $r = 1$. For that case, observe that consecutive times between routing instants to the second queue are i.i.d. This readily implies that the second queue is a GI/M/1 queue.

The distribution of the inter-arrival times to queue 2 is obtained as follows. Consider an M/M/1 queue with arrival rate λ and service rate μ_1. For $m = 0, 1, 2, ...$, let Y_m be the elapsing time from the moment the queue length is m until it is (for the first time) $m+1$. Due to the memoryless property, the time between consecutive arrivals to queue 2, given that all jobs are using policy $(n,1)$ is distributed like Y_{n+1}.

To obtain the distribution of $Y_m, m \leq n$, let T denote an inter-arrival time and V a service time in the first queue. Then

$$\begin{aligned} Y_m &\triangleq T \cdot 1\{T < V\} + (V + Y_{m-1} + Y_m) \cdot 1\{V \leq T\} \\ &\triangleq \min\{T, V\} + (Y_{m-1} + Y_m) \cdot 1\{V \leq T\}. \end{aligned}$$

Thus, as T and V are exponentially distributed with means $1/\lambda$ and $1/\mu$, respectively, the Laplace transform $\tilde{Y}_m(s) := E[\exp(-sY_m)]$ is given by

$$\tilde{Y}_m(s) = \frac{\lambda + \mu_1}{\lambda + \mu_1 + s} + \tilde{Y}_{m-1}(s)\tilde{Y}_m(s)\frac{\mu_1}{\lambda + \mu_1},$$

which yields

$$\tilde{Y}_m(s) = \frac{\lambda + \mu_1}{\lambda + \mu_1 + s}\left(1 - \frac{\mu_1}{\lambda + \mu_1}\tilde{Y}_{m-1}(s)\right)^{-1}.$$

This allows us to compute $\tilde{Y}_m(s)$ recursively. We have, in particular, $Y_0 \triangleq T$ so that

$$\tilde{Y}_0(s) = \frac{\lambda}{\lambda + s},$$

$$\tilde{Y}_1(s) = \frac{(\lambda + \mu_1)^2(\lambda + s)}{(\lambda + \mu_1 + s)[(\lambda + \mu_1)s + \lambda^2]}$$

$$\tilde{Y}_2(s) = \frac{(\lambda + \mu_1)[(\lambda + \mu_1)s + \lambda^2]}{(\lambda + \mu_1)s^2 + \lambda(2\lambda + \mu_1)s + \lambda(\lambda^2 - (\mu_1)^2)}.$$

Thus $EY_0 = \lambda^{-1}$, and EY_m, $m > 0$ satisfies

$$E[Y_m] = \frac{1}{\lambda + \mu_1} + \Big(E[Y_{m-1}] + E[Y_m]\Big)\frac{\mu_1}{\lambda + \mu_1}.$$

This gives the recursion $E[Y_m] = (1 + \mu_1 E[Y_{m-1}])/\lambda$ and, hence,

$$E[Y_m] = \frac{1}{\lambda}\left(1 + \frac{\mu_1}{\lambda} + \dots + \left(\frac{\mu_1}{\lambda}\right)^m\right) \overset{\lambda \neq \mu_1}{=} \frac{1}{\lambda}\left(\frac{1 - (\mu_1/\lambda)^{m+1}}{1 - \mu_1/\lambda}\right). \tag{10}$$

Remark 3.1 An alternative way to obtain the distribution or expectation of Y_m is through the observation that Y_m has the same distribution as the busy period in an M/M/1/$m+1$ queue with *interchanged* arrival and service rates of the first queue: the arrival rate is μ_1 and the service rate is λ.

Note that the second queue is stable if and only if $E[Y_{n+1}] > 1/\mu_2$. This gives

$$\mu_2 \frac{1}{\lambda}\left(\frac{1 - (\mu_1/\lambda)^{n+2}}{1 - \mu_1/\lambda}\right) > 1.$$

which agrees with (**??**). For the case $n = 0$ and $\mu_1 = \mu_2 = \mu$ the stability condition reads

$$\frac{\lambda}{\mu} < \frac{1 + \sqrt{5}}{2},$$

which is the well known Golden Ratio.

4 The joint Probability Distribution Function

We now derive the joint probability distribution of the numbers of jobs in both queues, which is needed for the analysis of the individually optimal routing. We define the Partial Generating Functions (PGFs)

$$G_i(z) = \sum_{j=0}^{\infty} P_{ij} z^j, \qquad i = 0, 1, 2, ..., n+1.$$

Multiplying the equation corresponding to j by z^j in (**??**) and (**??**), and summing over all j, we obtain:

$$\overline{G}(z)A(z) = (1-z)\mu_2 \overline{G}(0), \tag{11}$$

where $\overline{G}(z)$ ($\overline{G}(0)$, respectively) is the row vector whose ith entry ($i = 0, ..., n+1$) is $G_i(z)$ (P_{i0}, respectively), and $A(z)$ is the matrix

$$A(z) = z^2\Lambda + zQ_d + M.$$

More precisely, $A(z)$ is given by

$$\begin{pmatrix}
\mu_2 - \beta z & \lambda z & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\mu_1 z & \mu_2 - \alpha z & \lambda z & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & \mu_1 z & \mu_2 - \alpha z & \lambda z & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_1 z & \mu_2 - \alpha z & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \mu_1 z & \mu_2 - \alpha z & \lambda z & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & \mu_1 z & \mu_2 - \alpha z + \lambda(1-r)z^2 & \lambda r z \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \mu_1 z & \mu_2 - \alpha z + \lambda z^2
\end{pmatrix}$$

where $\alpha \triangleq \lambda + \mu_1 + \mu_2$ and $\beta \triangleq \lambda + \mu_2$.

The solution of (**??**) is given by

$$\overline{G}(z) = \frac{(1-z)\mu_2}{|A(z)|}\overline{G}(0)\text{cof}\,[A(z)], \quad |z| < 1, \tag{12}$$

where, by Cramer's rule, the element (i,k) of the *cofactor* matrix cof $[A(z)]$ is equal to $(-1)^{i+k}$ times the determinant of the matrix obtained from $A(z)$ after deleting the k-th row and the i-th column.

In order to fully determine $\overline{G}(z)$ it thus remains to determine the $n+2$ unknowns in the vector $\overline{P}_0 = \overline{G}(0)$. One way to proceed is to note that since $\overline{G}(z)$ is analytic in the unit disk, for all $|z| < 1$ for which $|A(z)| = 0$, we also have

$$\overline{G}(0)\text{cof}\,[A(z)] = \overline{0}. \tag{13}$$

This provides a set of equations for $\overline{G}(0)$. It can be shown that, up to multiplication by a scalar (it is clear that multiplying any solution to the above equations by a scalar yields another solution), the boundary probabilities constituting the vector $\overline{G}(0)$ are uniquely determined by (**??**). Without further details we remark that this procedure ultimately would require solving a (full) set of $n+2$ equations to determine $\overline{G}(0)$. Below (Theorems **??** and **??**) we show that this can be avoided, reducing the problem to a set of 2 (even 1, in case $r = 1$) equations. We shall need the following result.

Lemma 4.1 *Assume that the stability condition holds and that $r \in (0,1]$. Then all roots $\{z_k\}$ of the determinant $|A(z)|$ are distinct, real and positive. If $r = 1$ there is a single root which is greater than 1, one root equal to 1, and there are $n+1$ roots in the interval $(0,1)$. That is, $0 < z_1 < ... < z_{n-1} < z_n < z_{n+1} < z_{n+2} = 1 < z_{n+3}$. If $r \in (0,1)$ then there is an additional root (z_{n+4}) larger than 1.*

Proof The proof is based on Theorem 2.4.5 from [?], which is formulated for the more general case where Λ and M are arbitrary (non negative) diagonal matrices. For $z \neq 0$, the matrix $T(z)$ in that reference satisfies $A(z) = z^2 T(1/z)$. ∎

We now obtain the steady-state distributions for the cases $r \in (0,1)$ and $r = 1$ in two separate theorems.

Theorem 4.2 *Suppose that $r \in (0,1)$. Let $\zeta_1 < \zeta_2$ be the two roots of $|A(z)|$ for $z \in (1,\infty)$ and $\overline{v}_1 = (v_{1,0},\ldots,v_{1,n+1})$ and $\overline{v}_2 = (v_{2,0},\ldots,v_{2,n+1})$ the corresponding left null (row) vectors of $A(\zeta_1)$ and $A(\zeta_2)$, respectively. Furthermore, let the constants c_1 and c_2 be the unique pair solving*

$$\begin{aligned} c_1 \frac{v_{1,n}\zeta_1}{\zeta_1 - 1} + c_2 \frac{v_{2,n}\zeta_2}{\zeta_2 - 1} &= \frac{{\rho_1}^n}{\sum_{k=0}^n {\rho_1}^k + r\rho_1^{n+1}}, \\ c_1 \frac{v_{1,n+1}\zeta_1}{\zeta_1 - 1} + c_2 \frac{v_{2,n+1}\zeta_2}{\zeta_2 - 1} &= \frac{r{\rho_1}^{n+1}}{\sum_{k=0}^n {\rho_1}^k + r\rho_1^{n+1}}. \end{aligned} \tag{14}$$

Then,

$$\overline{P}_j = c_1 (1/\zeta_1)^j \overline{v}_1 + c_2 (1/\zeta_2)^j \overline{v}_2, \qquad j \geq 1, \tag{15}$$

or, equivalently,

$$P\{X_1 = i, X_2 = j\} = c_1 v_{1,i} (1/\zeta_1)^j + c_2 v_{2,i} (1/\zeta_2)^j, \qquad 0 \leq i \leq n+1,\ j \geq 1. \tag{16}$$

The probabilities on the boundary $j = 0$ are given by:

$$P\{X_1 = i, X_2 = 0\} = \frac{{\rho_1}^i}{\sum_{k=0}^n {\rho_1}^k + r\rho_1^{n+1}} - \frac{c_1 v_{1,i}}{\zeta_1 - 1} - \frac{c_2 v_{2,i}}{\zeta_2 - 1}, \qquad 0 \leq i \leq n-1, \tag{17}$$

$$P\{X_1 = i, X_2 = 0\} = c_1 v_{1,i} + c_2 v_{2,i}, \qquad i = n, n+1, \tag{18}$$

Proof Let $\overline{1}_i$ be the (row) vector with i-th entry equal to 1 and all other entries equal to 0. Then the vectors $\overline{v}_1, \overline{v}_2, \overline{1}_0, \overline{1}_1, \overline{1}_2, \ldots, \overline{1}_{n-1}$, are the left eigenvectors of the matrix R (see [?, Corollary 2.3.3 and Lemma 2.3.4]), and therefore are an independent basis for $\mathbb{R}^{n+2}$, which proves that there are unique coefficients $c_1, c_2, d_0, d_1, \ldots, d_{n-1}$ such that:

$$\overline{P}_0 = \overline{G}(0) = c_1 \overline{v}_1 + c_2 \overline{v}_2 + \sum_{i=0}^{n-1} d_i \overline{1}_i. \tag{19}$$

As an immediate consequence we have (**??**), where we still need to prove that c_1 and c_2 are indeed determined by (**??**). We shall do this at the end of the proof. Now rewrite (**??**) as:

$$\overline{G}(z) = \frac{\mu_2}{|A(z)|/(z-1)} \overline{G}(0) \mathrm{cof}[A(z)], \quad |z| < 1.$$

Note that each entry of the matrix $\mathrm{cof}[A(z)]$ is a polynomial in z of degree at most $n+3$ and, hence, so is the vector $\overline{G}(0)\mathrm{cof}[A(z)]$. The degree of $|A(z)|/(z-1)$ is *exactly* $n+3$. We know that each of the $n+1$ roots of $|A(z)|$ in $(0,1)$ is also a root of each entry of $\overline{G}(0)\mathrm{cof}[A(z)]$. Cancelling these roots, makes all entries of $\overline{G}(z)$ rational functions with the denominator of degree 2 with roots z_{n+3} and z_{n+4} (both larger than 1), and the numerator of degree at most 2. Now, since for $z > 1$,

$$\frac{1}{z-1} = \sum_{j=1}^{\infty} \left(\frac{1}{z}\right)^j,$$

and taking into account (??), we arrive at (??), which by definition is equivalent to (??).

If, for $i = n$, we sum $P\{X_1 = n, X_2 = j\}$ over all $j \geq 0$, then using the marginal distribution of X_1 given in (??) together with (??) and (??) we get the first part of (??). Similarly, using (??) for $i = n+1$, we obtain the second part of (??).

It can be argued that if (??) admits more than 1 solution, then the representation (??) is not unique, which contradicts the above. Now, for $0 \leq i \leq n-1$, using $P_{i\bullet} = \rho_1^i P_{0\bullet}$ from (??), and applying (??) and (??), we obtain, after summing $P\{X_1 = i, X_2 = j\}$ over all $j \geq 1$, the desired equation (??). ∎

Theorem 4.3 *Assume $r = 1$. Let ζ^* be the unique singularity point of $A(z)$ which is larger than 1 and let*

$$\overline{v}^* = (v_0^*, v_1^*, \ldots, v_{n+1}^*)$$

be the corresponding left null row vector of $A(\zeta^*)$, which is unique up to multiplication by a scalar. Then for $0 \leq i \leq n+1$:

$$\begin{aligned}
P\{X_1 = i, X_2 = j\} &= \left(1 - \frac{1}{\zeta^*}\right)\left(\frac{1}{\zeta^*}\right)^j \frac{{\rho_1}^{n+1} v_i^*/v_{n+1}^*}{\sum_{k=0}^{n+1} {\rho_1}^k}, \quad j \geq 1, \\
P\{X_1 = i, X_2 = 0\} &= \frac{{\rho_1}^i}{\sum_{k=0}^{n+1} {\rho_1}^k} - \frac{1}{\zeta^*} \frac{{\rho_1}^{n+1} v_i^*/v_{n+1}^*}{\sum_{k=0}^{n+1} {\rho_1}^k}.
\end{aligned}$$

In particular

$$E[X_2|X_1 = i] = \frac{v_i^*/v_{n+1}^*}{\zeta^* - 1} {\rho_1}^{n+1-i}. \tag{20}$$

Proof The proof proceeds along the same lines as that of Theorem ??. If $r = 1$ then the term with coefficient c_2 vanishes in both expressions (??) and (??), and in (??) an additional term $d_n \overline{1}_n$ appears. This proves that

$$\begin{aligned}
P\{X_1 = i, X_2 = j\} &= c v_i^* \left(\frac{1}{\zeta^*}\right)^j, \quad 0 \leq i \leq n+1, j \geq 1, \\
P\{X_1 = n+1, X_2 = 0\} &= c v_{n+1}^*.
\end{aligned}$$

(In the second expression we use, as before, the fact that the last entry of the $\overline{1}_k$ in (??) equals 0.) Summing $P\{X_1 = n+1, X_2 = j\}$ over all $j \geq 0$ we have:

$$c = \left(1 - \frac{1}{\zeta^*}\right) \frac{P\{X_1 = n+1\}}{v_{n+1}^*}.$$

Substituting this result for c in the expression for $P\{X_1 = i, X_2 = j\}$ above, and using the value of $P(X_1 = n+1) = P_{n+1,\bullet}$, as given by (??), we obtain the first assertion of the theorem. Then, using the value of $P_{i\bullet}$ as given in (??), when $r = 1$, we obtain the second assertion of the theorem.

Finally, note that

$$E[X_2|X_1 = i] = \alpha \frac{v_i^*}{{\rho_1}^i}, \qquad \text{where} \qquad \alpha := c \frac{\zeta^*}{(\zeta^* - 1)^2} \frac{1 - {\rho_1}^{n+2}}{1 - \rho_1} = \frac{1}{\zeta^* - 1} \frac{{\rho_1}^{n+1}}{v_{n+1}^*}.$$

∎

We indicate that the steady-state joint probabilities in (??) and (??) are computed for given threshold policies with parameter (n, r). We shall add this parameter explicitly to the notation in the following sections.

5 Socially optimal routing

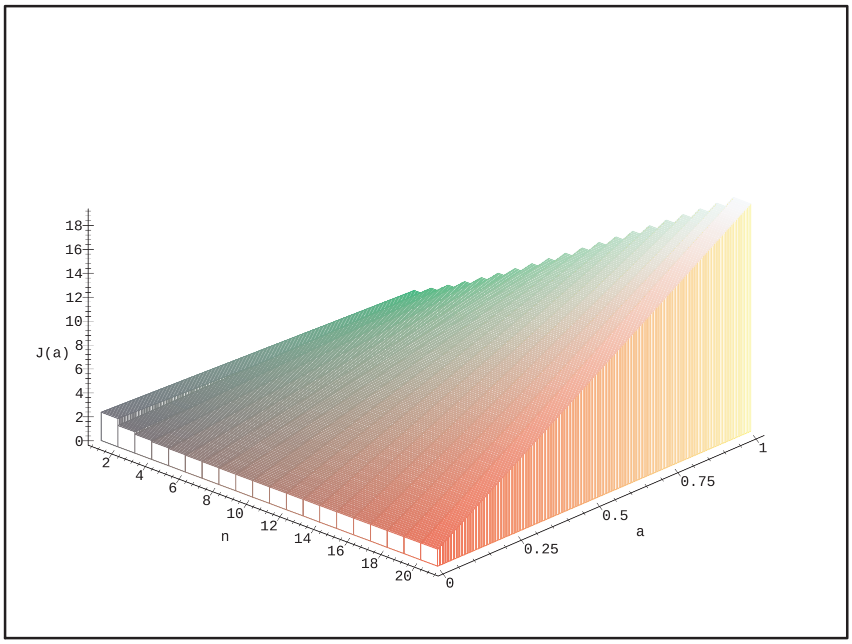

Figure 2: The function $J^{(n,1)}(a)$ for different values of a and n

In this section we discuss the computation of a threshold policy that minimizes a weighted sum of the mean queue lengths in both queues. To be specific, we consider the optimization problem of minimizing $J^{(n,r)}(a)$ where

$$J^{(n,r)}(a) := aE^{(n,r)}[X_1] + (1-a)E^{(n,r)}[X_2],$$

for some weight factor $a \in [0,1]$. Let

$$J^*(a) := \min_n J^{(n,1)}(a)$$

In general, the minimization can be taken over the parameters (n,r) in case we allow for randomized policies, but we shall restrict ourselves to the case $r=1$ and minimize over the parameter n only.

By (**??**) and (**??**), we have

$$E^{(n,r)}[X_1] = P_{0\bullet}\rho_1 \left(\frac{n{\rho_1}^n(\rho_1 - 1) + 1 - {\rho_1}^n}{(1-\rho_1)^2} + r(n+1){\rho_1}^n \right),$$

where $P_{0\bullet}$ is given in (**??**). When $\rho_1 = 1$ (where $\rho_1 = \lambda/\mu_1$) the above expression is defined by its limit.

From Theorem **??** we have for $r=1$ and $j \geq 1$,

$$P\{X_2 = j\} = C_1 \left(\frac{1}{\zeta^*}\right)^j \quad \text{where} \quad C_1 = \left(1 - \frac{1}{\zeta^*}\right) \frac{\left({\rho_1}^{n+1} \sum_{i=0}^{n+1} v_i^*\right)/v_{n+1}^*}{\sum_{k=0}^{n+1} {\rho_1}^k}.$$

Thus,

$$E^{(n,r)}[X_2] = \frac{C_1 \zeta^*}{(\zeta^* - 1)^2}.$$

For $r < 1$ and $j > 0$, we have from Theorem **??**,

$$P\{X_2 = j\} = c_1 \left(\sum_{i=0}^{n+1} v_{1,i} \right) (1/\zeta_1)^j + c_2 \left(\sum_{i=0}^{n+1} v_{2,i} \right) (1/\zeta_2)^j .$$

Thus,

$$E^{(n,r)}[X_2] = \frac{c_1 \left(\sum_{i=0}^{n+1} v_{1,i} \right) \zeta_1}{(\zeta_1 - 1)^2} + \frac{c_2 \left(\sum_{i=0}^{n+1} v_{2,i} \right) \zeta_2}{(\zeta_2 - 1)^2}.$$

With these formulas we can evaluate the objective function for different choices of n and r and obtain their optimal values numerically.

To illustrate this approach, we consider a numerical example in which we chose $\mu_1 = \mu_2 = \lambda = 0.05$. We plot in Figure **??** the value $J^{(n,1)}(a)$ as a function of the weighting factor a and the threshold n (without randomization, i.e. for $r = 1$). We see clearly from the figure that for each a, there exists a well defined global minimum (as a function of n). The optimal threshold n is given in Figure **??** as a function of a. (The function $\mathcal{R}^*(a)$ displayed in the figure is defined and studied below.) Evidently, the optimal threshold

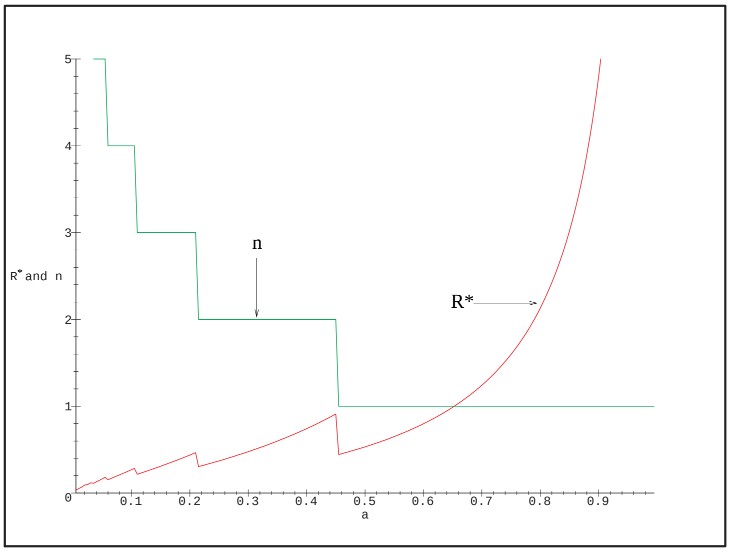

Figure 3: The optimal threshold and the corresponding value $\mathcal{R}^*(a)$ for different a's

decreases with increasing values of a, since by increasing a we penalize for joining queue 1. Note that the optimal threshold is a step function.

We now investigate in more detail the influence of a on the different components of $J^{(n,1)}(a)$, i.e. on the expected length of each of the two queues. We are motivated by the following question: what is the contribution of the congestion in each queue to the global cost, when using an optimal threshold? We note that for the non-symmetrical case, the benefit of node 1 is $aE^{(n,1)}[X_1]$ and the benefit for node 2 is $(1-a)E^{(n,1)}[X_2]$. We thus define

$$\mathcal{R}(a, n) := \frac{aE^{(n,1)}[X_1]}{(1-a)E^{(n,1)}[X_2]},$$

and we further define $\mathcal{R}^*(a)$ to be the value of $\mathcal{R}(a, n)$ evaluated at the threshold value $(n, 1)$ that minimizes $J^{(n,1)}(a)$. $\mathcal{R}^*(a)$ is depicted in Fig. **??** as a function of a (which is increased at steps of 0.005). We see that

$\mathcal{R}^*(\frac{1}{2}) < 1$. We conclude in this example that under symmetric weighting and symmetric service rates, the node for which the state information is available at the routing instants obtains a lower benefit at the socially optimal conditions, i.e., $aE^{(n,1)}[X_1] < (1-a)E^{(n,1)}[X_2]$. Note also that when a tends to zero then $\mathcal{R}^*$ tends to 0, and when a tends to 1 then $\mathcal{R}^*$ tends to ∞. Note finally that $\mathcal{R}^*$ is not monotone everywhere: it has irregularities at the points where the optimal threshold jumps. Indeed, since the optimal policy is piecewise constant, then, in between changes of the optimal policy, $\mathcal{R}^*(a)$ can be written as

$$\mathcal{R}^*(a) = const \times \frac{a}{1-a} = const\Big(\frac{1}{1-a} - 1\Big),$$

so in these intervals $\mathcal{R}^*(a)$ grows like an hyperbola. At the right end-points of those intervals $\mathcal{R}^*(a)$ may have negative jumps.

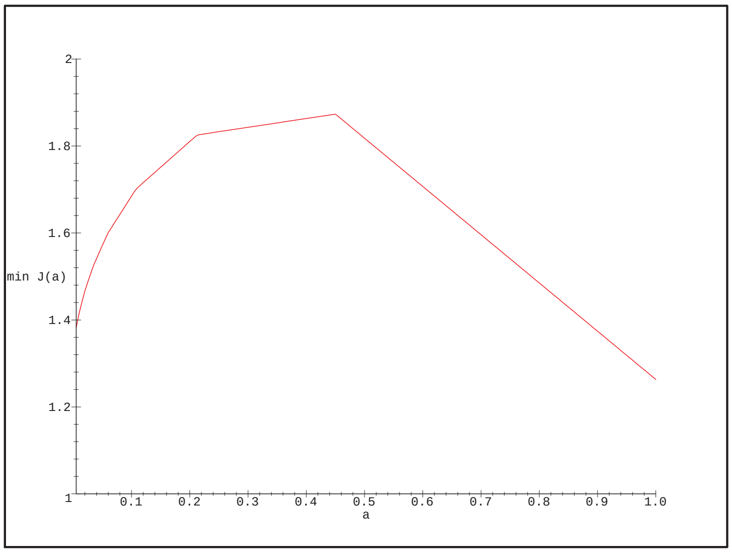

Figure 4: The optimal value for different a's

In Fig. **??** we plot the optimal value $J^*(a)$ for different values of a. Let us shortly comment on the behavior of $J^*(a)$ for $a \downarrow 0$ and $a \to 1$. We start with $a \downarrow 0$. Clearly, $E^{(n,1)}[X_1] \le n+1$. (If $\rho_1 < 1$ then $E^{(n,1)}[X_1]$ is even uniformly bounded by $E^{(\infty,1)}[X_1] = \rho_1/(1-\rho_1)$.) Choosing $n(a) = \lfloor\sqrt{1/a}\rfloor$ we have

$$0 \le J^*(a) \le J^{(n(a),1)}(a) \le a(n(a)+1) + E^{(n(a),1)}E[X_2] \longrightarrow 0+0, \quad a \downarrow 0.$$

Now let $a \to 1$. We choose $n' := \min\big\{n : E^{(n,r)}[X_2] < \infty\big\}$. This minimum shall be taken over $n \ge -1$, where the artificial choice $n = -1$ corresponds to the policy where all customers are routed to queue 2, We then have

$$J^*(a) \le J^{(n',1)}(a) = aE^{(n',1)}[X_1] + (1-a)E^{(n',1)}[X_2] \longrightarrow E^{(n',1)}[X_1], \quad a \to 1.$$

It is easy to see that, in the limit as $a \to 1$, this inequality can not be strict: the optimal choice of n can never be smaller than n', and for any choice $n \ge n'$ we have $J^{(n,1)}(a) \ge aE^{(n,1)}[X_1]$ and, hence,

$$J^*(a) \ge \min_{n \ge n'} aE^{(n,1)}[X_1] = aE^{(n',1)}[X_1] \to E^{(n',1)}[X_1], \quad a \to 1.$$

In particular, if $\lambda < \mu_2$ and if we allow $n = -1$ then $n' = -1$ and $J^*(a) \to 0$ when $a \to 1$.

6 Individually optimal routing

Next we consider the question of individual optimization. An arriving job observes the length of the first queue and on the basis of this observation decides which queue to join, so as to minimize its sojourn time, possibly giving different weights to waiting in the two queues as in the global optimization of Section **??**.

Suppose that all previous jobs, in an infinitely long past, followed a common $(n, 1)$ policy. Hence, in particular, the first queue has evolved exactly like a pure M/M/1/K queue with $K = n+1$. Assume further that the system has reached steady state and that the pair of random variables X_1 and X_2 are, as before, jointly distributed as the queue lengths in queue 1 and queue 2, respectively. For notational convenience we normalize the time with respect to the service rate at the first queue. Note that the distribution of (X_1, X_2) and, hence, the routing decisions are not affected by such a normalization. We define as before $\rho_1 = \lambda/\mu_1$, and further define

$$s := \frac{\mu_2}{\mu_1}, \tag{21}$$

i.e., s is the relative service speed of the second queue with respect to that of the first. The time-normalized system is now characterized by the parameters n, r, ρ_1 and s. For later reference we rewrite the stability condition (**??**) as

$$\rho_1(1-r)P_{n\bullet} + \rho_1 P_{n+1,\bullet} < s. \tag{22}$$

$T^{n,r}_{\rho_1,s}(i,h)$ is defined as the expected sojourn time (normalized with respect to μ_1) in case a new job joins queue h given that the number of jobs in the first queue equals i:

$$\begin{aligned} T^{n,r}_{\rho_1,s}(i,1) &= i+1, \\ T^{n,r}_{\rho_1,s}(i,2) &= (1+E[X_2|X_1=i])/s. \end{aligned} \tag{23}$$

(The actual expected sojourn time is $T^{n,r}_{\rho_1,s}(i,h)/\mu_1$.) In the sequel, when $r = 1$ we commonly omit this parameter in the notation and write $T^{n}_{\rho_1,s}(i,h)$ instead of $T^{n,1}_{\rho_1,s}(i,h)$. An individual job will join queue 1 if $aT^{n,r}_{\rho_1,s}(i,1) \leq (1-a)T^{n,r}_{\rho_1,s}(i,2)$, where $a \in (0,1)$ is a weight parameter (as in Section **??**). When $a = 1/2$ each individual job just aims at minimizing its sojourn time.

6.1 Nash equilibria

A set of policies is called a Nash equilibrium if no job can improve his choice by a unilateral change of policy. Based on numerical results, for the case of two equally fast servers, the existence of a randomized threshold policy leading to a Nash equilibrium was postulated in [**?**]. Our results further support this assertion when $n = 1, \ldots, 18$, and strongly indicate that the same is true for $n \geq 19$ (more details are given in Section **??**). We show, however, that this is not necessarily true when the servers have different service speeds. In particular, we show that an individual job's best response to a common non randomized threshold policy of all other jobs is not always of threshold type. This implies a more intricate (numerical) investigation of the existence of a Nash equilibrium. (A threshold policy (n, r) can effectively be represented by the single parameter $n + r$, which is convenient for numerical analysis.) Such an investigation is beyond the scope of this paper. Our main contribution in that direction is to establish conditions on the parameters such that, for a given threshold policy $(n, 1)$, the optimal response is again a threshold policy. Our proof technique is partly analytic and does not seem to lend itself for a straightforward extension to include randomized (n, r) policies. If such an extension were realized, this would support the numerical investigation of the existence of Nash equilibria of threshold type along the lines of [**?**].

6.2 Optimality of two-threshold responses for $r = 1$

Numerical examples in [?] indicate that, when $s = 1$ (and $a = 1/2$), $T^{n,r}_{\rho_1,s}(i,1) - T^{n,r}_{\rho_1,s}(i,2)$ is increasing in i, implying that the optimal response is a threshold policy. Below it is shown that this is not necessarily true for general s and a. See for instance Figure ?? where, for the parameter values $n = 3$, $r = 1$, $\rho_1 = 1$ and $s = 0.56$, we plotted $T^{n,r}_{\rho_1,s}(i,1)$ and $T^{n,r}_{\rho_1,s}(i,2)$ for $i = 0, 1, \ldots, 4$. If we take $a = 0.5$, an individual job will join queue 2 if there are exactly 3 jobs in the first queue but it will join queue 1 when the queue length is *larger or smaller* than 3.

In the sequel we focus on non-randomized threshold policies by taking $r = 1$. Our main interest is in determining structural properties of optimal responses.

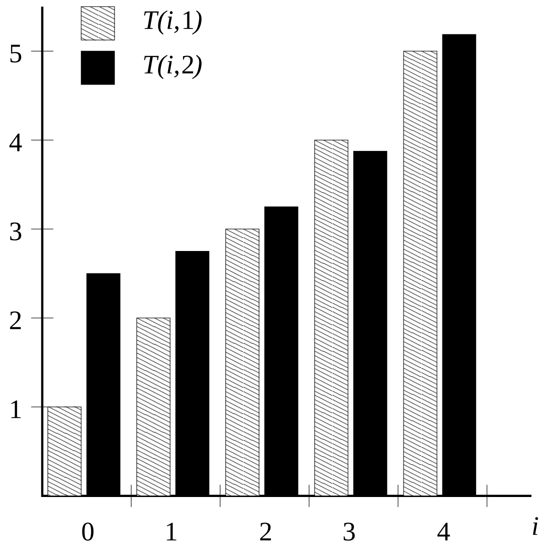

Figure 5: $T^{n,r}_{\rho_1,s}(i,1)$ and $T^{n,r}_{\rho_1,s}(i,2)$

Lemma 6.1 *For fixed n, ρ_1 and s, $T^{n}_{\rho_1,s}(i,2)$ is convex in i.*

Proof Let ζ^* and $\overline{v}^*$ be as in Theorem ??. From $\overline{v}^* A(\zeta^*) = \overline{0}$ we have for $0 < i < n+1$:

$$v^*_{i-1}\lambda\zeta^* + v^*_i\left(\mu_2 - (\lambda + \mu_1 + \mu_2)\,\zeta^*\right) + v^*_{i+1}\mu_1\zeta^* = 0.$$

Hence, if we normalize such that $v^*_0 = 1$ we may write:

$$v^*_i = \delta(w_1)^i + (1-\delta)(w_2)^i. \tag{24}$$

Here, $x = w_1$ and $x = w_2$ are the two roots of the polynomial

$$\lambda\zeta^* + \left(\mu_2 - (\lambda + \mu_1 + \mu_2)\,\zeta^*\right)x + \mu_1\zeta^* x^2.$$

We choose w_1 and w_2 such that

$$w_1 > \max\{1, \rho_1\} \geq \min\{1, \rho_1\} > w_2 > 0.$$

The coefficient δ is given by

$$\delta \;=\; \frac{\frac{\mu_2}{(\zeta^*)^2} - \frac{\lambda+\mu_2}{\zeta^*} + \frac{\mu_1}{\zeta^*}w_2}{\frac{\mu_1}{\zeta^*}(w_2 - w_1)} \;=\; \frac{w_2 - \rho_1}{w_2(w_2 - w_1)} \;>\; 0,$$

Similarly

$$1-\delta \quad = \quad \frac{\frac{1}{w_1}-\frac{\mu_1}{\lambda}}{\frac{1}{w_1}-\frac{1}{w_2}} > 0.$$

Hence, $0<\delta<1$. From (??) and (??) we have:

$$E\left[X_2|X_1=i\right] \quad = \quad \frac{(\rho_1)^{n+1}/v^*_{n+1}}{\zeta^*-1}\left(\delta\left(\frac{w_1}{\rho_1}\right)^i+(1-\delta)\left(\frac{w_2}{\rho_1}\right)^i\right).$$

Clearly, $E\left[X_2|X_1=i\right]$ and, hence, $T^n_{\rho_1,s}(i,2)$ are convex in i. ∎

For fixed n, ρ_1 and s let us define the set of states of queue 1 for which it is optimal to join queue 2:

$$S(n,\rho_1,s) := \left\{i : aT^n_{\rho_1,s}(i,1) \geq (1-a)T^n_{\rho_1,s}(i,2)\right\} \subset \{0,1,2,\ldots,n+1\}. \tag{25}$$

Furthermore, let $t^-(n,\rho_1,s) := \inf S(n,\rho_1,s)$ and $t^+(n,\rho_1,s) := \sup S(n,\rho_1,s)$ (by convention $\inf\emptyset=+\infty$ and $\sup\emptyset=-\infty$). We also define the set of states for which both queues are equally attractive:

$$E(n,\rho_1,s) := \left\{i : aT^n_{\rho_1,s}(i,1) = (1-a)T^n_{\rho_1,s}(i,2)\right\} \subset \left\{t^-(n,\rho_1,s)\right\}\cup\left\{t^+(n,\rho_1,s)\right\}. \tag{26}$$

Corollary 6.2 *Let ρ_1 and s be fixed. The optimal responses to a non randomized $(n,1)$ threshold policy are two-threshold policies characterized by $t^-(n,\rho_1,s)$ and $t^+(n,\rho_1,s)$ as follows. It is optimal to route a job to queue 2 if $t^-(n,\rho_1,s)\leq X_1\leq t^+(n,\rho_1,s)$ and to queue 1 otherwise. If $X\in E(n,\rho_1,s)$, which may be the case at either threshold value, one may deviate from the above and route to queue 1 without losing optimality (in that case both routes are equally attractive).*

Proof If $S(n,\rho_1,s)=\emptyset$ it is always optimal to route to the first queue. Otherwise we have, from Lemma ?? and the fact that $T^n_{\rho_1,s}(i,1)$ is linear in i, that $S(n,\rho_1,s)$ is an uninterrupted sequence of integers. ∎

Remark 6.1 If $t^-(n,\rho_1,s)=0$ and $t^+(n,\rho_1,s)<n+1$, then the optimal response is effectively characterized by a single threshold, namely $t^+(n,\rho_1,s)$. However, we shall not refer to such a policy as being a threshold policy. We reserve that term for the 'natural' threshold policies introduced earlier, in which a new job joins the first queue if there is a 'small' number of jobs at that queue and the job joins the second queue otherwise. When $t^-(n,\rho_1,s)=0$ and $t^+(n,\rho_1,s)<n+1$, the opposite happens: a new job should join the first queue if there is a 'large' number of jobs at that queue.

Clearly, a non randomized threshold policy (note Remark ?? above) is the optimal response to a non randomized threshold policy if and only if $t^+(n,\rho_1,s)=n+1$. Because of the convexity of $T^n_{\rho_1,s}(i,2)$ (Lemma ??) and the linearity of $T^n_{\rho_1,s}(i,1)$, a sufficient condition for $t^+(n,\rho_1,s)=n+1$ is $(1-a)T^n_{\rho_1,s}(n+1,2)\leq aT^n_{\rho_1,s}(n+1,1)$. From (??) we have that

$$E\left[X_2|X_1=n+1\right]=\frac{1}{\zeta^*-1},$$

hence, by (??),

$$T^n_{\rho_1,s}(n+1,2)=\left(\frac{1}{\zeta^*-1}+1\right)\frac{1}{s}. \tag{27}$$

The next lemma shows that the condition $(1-a)T^n_{\rho_1,s}(n+1,2) \le aT^n_{\rho_1,s}(n+1,1)$ does not always hold. The properties mentioned in the lemma play a crucial role in our later analysis.

Lemma 6.3 *Let n, ρ_1 and $i \in \{0,1,2,\ldots,n+1\}$ be fixed. Let $\underline{s}(n,\rho_1) > 0$ be such that $s = \underline{s}(n,\rho_1)$ achieves equality in the stability condition (**??**). Then $T^n_{\rho_1,s}(i,2)$ is continuous and strictly decreasing in $s > \underline{s}(n,\rho_1)$. Moreover, if $s \to \infty$ then $T^n_{\rho_1,s}(i,2) \downarrow 0$ and if $s \downarrow \underline{s}(n,\rho_1)$ then $T^n_{\rho_1,s}(i,2) \to \infty$.*

Proof The proof can entirely be given using analytic arguments. However, to show the monotonicity we provide a more instructive proof using a coupling argument. Consider the system of two queues for two values of s, say s^- and s^+. Assume that $s^- < s^+$. Suppose that the sequences of inter-arrival times and service times at the first servers are identical for both systems. Note that the first queue behaves exactly the same in both systems. We couple the service times at the second server in the following way. Let $U_1, U_2, U_3, \ldots,$ be an i.i.d. sequence of exponentially distributed random variables with expectation 1. We construct the sequence of service times in both systems as follows. In the first system we let the service time of the k-th job arriving to the second queue be equal to U_k/s^-, while in the second system we set it equal to U_k/s^+. (Note that, since the first queue behaves identically in both systems, the arrival process at the second queue is also the same in both systems.) Clearly a job arriving into the second queue will be taken into service earlier or at the same time (if both second queues are empty) in the system with service rate s^+ than in the system with the lower service rate s^-. Hence, at any point in time, the number of jobs in the second queue is never larger in the s^+-system than in the s^--system.

The continuity of $T^n_{\rho_1,s}(i,2)$ in s can be proved using arguments as in [**?**, Section 2.6]. That $T^n_{\rho_1,s}(i,2) \downarrow 0$ as $s \to \infty$ can be seen by noting that the second queue is never larger than that of the M/M/1 queue that results from routing all jobs to the second queue, i.e., having arrival rate ρ_1 and service rate s. Finally, the unboundedness of $T^n_{\rho_1,s}(i,2)$ as s decreases and instability is approached, can be proved formally using that $1/\zeta^*$ is larger than $\rho_1 P_{n+1,\bullet}/s$, which is the traffic load on the second queue[1]. ∎

Corollary 6.4 *Let n, ρ_1 and i be fixed. Then $t^-(n,\rho_1,s)$ is non increasing and $t^+(n,\rho_1,s)$ is non decreasing in $s > \underline{s}(n,\rho_1)$. Moreover, for s large enough we have $t^-(n,\rho_1,s) = 0$ and $t^+(n,\rho_1,s) = n+1$, and for s small enough but larger than $\underline{s}(n,\rho_1)$ we have $t^-(n,\rho_1,s) = +\infty$ and $t^+(n,\rho_1,s) = -\infty$.*

Proof Directly implied by Lemmas **??** and **??**. ∎

6.3 Conditions for optimality of (natural) threshold responses

The next theorem essentially provides necessary and sufficient conditions for the optimal response to a threshold policy to be again a (possible different) threshold policy. Furthermore it shows that to check optimality of threshold responses for fixed n and ρ_1, it suffices to consider a specific value of s instead of all $s > \underline{s}(n,\rho_1)$.

Theorem 6.5 *Let n and ρ_1 be fixed and $s^\star(n,\rho_1) > \underline{s}(n,\rho_1)$ be the unique solution of*

$$T^n_{\rho_1,s^\star(n,\rho_1)}(n+1,2) = \frac{a}{1-a} T^n_{\rho_1,s^\star(n,\rho_1)}(n+1,1). \tag{28}$$

Then for all $s \ge s^\star(n,\rho_1)$ there exists a threshold policy which is an optimal response of an individual job to the $(n,1)$ policy. Furthermore, the following are equivalent:

[1]This property was shown, in a more general context, in the proof of Lemma 2.6.8 of [**?**, p. 52]. There the role of $1/\zeta^*$ is played by $\psi_{\epsilon,N}$ and the load on the second queue of our model equals ρ/c.

(i) for all values of $s > \underline{s}(n,\rho_1)$ there exists a threshold policy which is an optimal response,

(ii) $T^n_{\rho_1,s^\star(n,\rho_1)}(n,2) \geq \frac{a}{1-a} T^n_{\rho_1,s^\star(n,\rho_1)}(n,1) = \frac{a}{1-a}(n+1)$,

(iii) $s^\star(n,\rho_1) \geq \hat{s}(n,\rho_1,a) \stackrel{\mathrm{def}}{=} (1-a)\left(\rho_1 + \frac{1-a}{a(n+2)}\right)$,

(iv) either $\rho_1 \leq \frac{1}{n+2}$ *or*

$$(-1)^{n+2}\left|A\left(1+\frac{1/a}{\rho_1(n+2)-1}\right)\right| \leq 0,$$

where we substitute $s = \hat{s}(n,\rho_1,a)$ into the entries of the matrix $A(\cdot)$.

Proof The existence and uniqueness of $s^\star(n,\rho_1)$ follows directly from Lemma **??** and the fact that $T^n_{\rho_1,s}(n+1,1)$ is independent of s. For all $s \geq s^\star(n,\rho_1)$ we have that $t^+(n,\rho_1,s) = n+1$ (Lemma **??** and Corollary **??**) and, hence, there exists an optimal threshold response.

Let us now prove the equivalence of (i) and (ii). Suppose (ii) holds. For $s \geq s^\star(n,\rho_1)$ we already proved the optimality of threshold policies in general. If $s \in (\underline{s}(n,\rho_1), s^\star(n,\rho_1))$ we have $S(n,\rho_1,s) = \emptyset$ because of Lemmas **??** and **??**. Hence, in this case it is always optimal to join queue 1. Therefore, (ii) implies (i). As for the converse, suppose that (ii) does not hold, i.e., $(1-a)T^n_{\rho_1,s^\star(n,\rho_1)}(n,2) < aT^n_{\rho_1,s^\star(n,\rho_1)}(n,1)$. By Lemma **??**, there exists an $s < s^\star(n,\rho_1)$ such that $(1-a)T^n_{\rho_1,s}(n+1,2) > aT^n_{\rho_1,s}(n+1,1)$ while still $(1-a)T^n_{\rho_1,s}(n,2) < aT^n_{\rho_1,s}(n,1)$. Therefore it is optimal to join queue 2 if $X_1 = n$ and join queue 1 if $X_1 = n+1$, which contradicts (i). This proves that (i) implies (ii).

Now we show that (iii) is equivalent with (ii). The following steps are justified below.

$$\begin{aligned}
&(1-a)T^n_{\rho_1,s^\star(n,\rho_1)}(n,2) \geq aT^n_{\rho_1,s^\star(n,\rho_1)}(n,1) \stackrel{1}{\Longleftrightarrow} (1-a)\left(\rho_1 \frac{v^*_n/v^*_{n+1}}{\zeta^*-1} + 1\right)\frac{1}{s^\star(n,\rho_1)} \geq a(n+1) \\
&\stackrel{2}{\Longleftrightarrow} \left((1-\zeta^*)\rho_1 + 1 + \frac{1-a}{a(n+2)}\right)\frac{1}{\zeta^*-1} + 1 \geq \frac{a}{1-a}(n+1)s^\star(n,\rho_1) \\
&\stackrel{3}{\Longleftrightarrow} -\rho_1 + 1 + \left(1+\frac{1-a}{a(n+2)}\right)\left(\frac{a(n+2)}{1-a}s^\star(n,\rho_1) - 1\right) \geq \frac{a}{1-a}(n+1)s^\star(n,\rho_1) \\
&\stackrel{4}{\Longleftrightarrow} s^\star(n,\rho_1) \geq (1-a)\left(\rho_1 + \frac{1-a}{a(n+2)}\right).
\end{aligned}$$

In step 1 we use (**??**) and (**??**). For step 2 we note that, after choosing $s = s^\star(n,\rho_1)$, the last column of the equation $\overline{v}^* A(\zeta^*) = \overline{0}$ gives:

$$v^*_n\zeta^*\rho_1 + v^*_{n+1}\left((\zeta^*)^2\rho_1 - \zeta^*(\rho_1+1+s^\star(n,\rho_1)) + s^\star(n,\rho_1)\right) = 0.$$

Hence,

$$\rho_1\frac{v^*_n}{v^*_{n+1}} = (1-\zeta^*)\rho_1 + 1 + \frac{\zeta^*-1}{\zeta^*}s^\star(n,\rho_1).$$

In the latter we also substitute

$$\frac{\zeta^*-1}{\zeta^*}s^\star(n,\rho_1) = \frac{1-a}{a(n+2)}, \tag{29}$$

which follows from (**??**) and (**??**). In step 3 we again use (**??**) and the last step concerns only elementary operations.

Finally, we show the equivalence of (iii) and (iv). Suppose that (iii) holds and that $\rho_1 > \frac{1}{n+2}$. From $\hat{s}(n,\rho_1,a) \leq s^\star(n,\rho_1)$ and Lemma ?? it follows that $(1-a)T^n_{\rho_1,\hat{s}(n,\rho_1,a)}(n+1,2) \geq a(n+2)$ which, using (??), is equivalent with $\zeta^* a\,(\rho_1(n+2)-1) \leq 1 + a\,(\rho_1(n+2)-1)$. Since $\rho_1 > \frac{1}{n+2}$ it must be that

$$\zeta^* \leq 1 + \frac{1}{a\,(\rho_1(n+2)-1)}.$$

Using that $z = \zeta^*$ is the unique root of $|A(z)|$ for $z > 1$ and $(-1)^{n+2}|A(z)| > 0$ for $z \in (1,\zeta^*)$ — see the proof of Lemma 2.4.4 in [?] — we now have $(-1)^{n+2}|A(z)| \leq 0$ for $z = 1 + \frac{1}{a(\rho_1(n+2)-1)}$. Thus, (iii) implies (iv). It remains to show the converse. This can be done by reversing the order of the previous steps: it follows that $\zeta^* a\,(\rho_1(n+2)-1) \leq 1 + a\,(\rho_1(n+2)-1)$ and, hence, $T^n_{\rho_1,\hat{s}(n,\rho_1,a)}(n+1,2) \geq \frac{a}{1-a}(n+2)$ so that $\hat{s}(n,\rho_1,a) \leq s^\star(n,\rho_1)$. ∎

Let us briefly discuss the consequences of Theorem ??. Part (ii) allows us to decide the optimality of threshold responses on the entire 3-dimensional parameter space (n,ρ_1,s) by computing $T^n_{\rho_1,s}(n,2)$ only for values $s = s^\star(n,\rho_1)$. Part (iii) improves this condition with respect to computational complexity: it requires a simpler condition on $s^\star(n,\rho_1)$ to be evaluated instead of computing $T^n_{\rho_1,s^\star(n,\rho_1)}(n,2)$. Finally, Part (iv) overcomes the inconvenience of having to evaluate $s^\star(n,\rho_1)$ from its implicit definition (??). Instead, the determinant of $A(z)$ needs only to be computed once at a specific value of z for each pair (n,ρ_1).

Corollary 6.6 *Let $a = 1/2$. For the cases $n = 0$, $n = 1$ and $n = 2$, the optimal response to an $(n,1)$ policy is a threshold policy (for any values of ρ_1 and s that make the system stable).*

Proof The corollary follows by Part (iv) of Theorem ??, involving only elementary (but tediously lengthy) matrix manipulations (which we omit). For the case $n = 0$, however, we give a (shorter) proof using Part (iii) of Theorem ??. As in the proof of Lemma ?? we use $1/\zeta^* < \rho_1 P_{n+1,\bullet}/s$, cf. [?, p. 52]. The right-hand side of this equation is the traffic load on the second queue. Using (??), for $r = 1$, and (??) we then have:

$$s^\star(n,\rho_1) > \left(\frac{1}{n+2}\right)\frac{{\rho_1}^{n+2}}{\sum_{i=0}^{n+1}{\rho_1}^i}. \tag{30}$$

When $n = 0$ it may be verified that for all values of ρ_1:

$$s^\star(0,\rho_1) > \frac{1}{2}\left(\frac{{\rho_1}^2}{1+\rho_1}\right) > \frac{1}{2}\left(\rho_1 + \frac{1}{2}\right).$$

For $n = 0$ this proves the optimality of threshold responses[2] (by Part (iii) of Theorem ??). ∎

7 Conclusion

We have studied optimal routing among two queues where the decisions are based on partial information. We derived the steady-state distributions for our model and identified how optimal routing strategies can be computed for social optimization. In the case of individual optimization we showed the structure of optimal responses of an individual customer to non-randomized threshold strategies $(n,1)$. Corollary ?? shows that for the cases $n = 0$, $n = 1$ and $n = 2$, the optimal response is always a non-randomized threshold policy (even for non-identical servers), provided the system is stable. This partly (the optimality of threshold responses

[2]The reader may verify that when $n = 1$ the bound (??) is already not sharp enough: the right-hand side may be smaller than $\frac{1}{2}\left(\rho_1 + \frac{1}{3}\right)$.

has not been shown for $r \neq 1$) justifies a numerical investigation of the existence of Nash equilibria along the lines of [?] when $n \leq 2$. On the other hand, we have used Part (iv) of Theorem **??** in a numerical investigation to determine values of (ρ_1, s) such that the optimal response is not a threshold policy but necessarily a two-threshold policy when $n = 3, 4, \ldots, 18$. The numerical results indicate that this is also the case for larger values of n, since the set of values of ρ_1 for which there exist choices of s leading to optimal *two-threshold* policies turns out to be increasing with n. For values of n between 3 and 18 the interval grows gradually with increasing n. When $n = 3$ such examples exist for $\rho_1 \in (0.4, 1.2)$ and when $n = 18$ we may take $\rho_1 \in (0.2, 1.8)$. In such cases, (numerical) investigation of the existence of Nash equilibria of threshold type requires further justification.