On the $M^X/G/1$ Queue with a Waiting Server and Vacations

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Abstract

We analyze the $M^X/G/1$ queue with server vacations and an *additional* feature, reflecting various real-life situations, in which the server, upon finding an empty system at the end of a vacation, activates a timer of duration T and *waits* dormant. If a batch arrives during the dormant period, a new busy period starts, but if no arrivals occur, the server waits no more and takes another vacation. The $M^X/G/1$ queues with multiple or with single vacations become limiting cases of the above model when $T \to 0$ or $T \to \infty$, respectively.

Keywords and phrases: $M^X/G/1$ queue; Vacations; Waiting server

1 Introduction

We study a batch-arrival $M^X/G/1$ system with server vacations, and with the *additional* feature, reflecting many real-life situations, that when the server returns from a vacation and finds an empty queue, he decides to *wait* for some time T (called a Timer), before taking further action. If a batch arrives before T expires, a busy period starts immediately. However, if there are no arrivals within T, the server leaves for a new vacation. This extension, besides modeling a common human behaviour, generalizes models of $M^X/G/1$ queue with server vacations: when T shrinks to 0, we obtain the $M^X/G/1$ queue with *multiple* vacations, while when T extends to infinity, the result is the $M^X/G/1$ queue with *single* vacations.

The batch-arrival single-class $M^X/G/1$ queue without vacations and FIFO regime was analyzed by Burke (1975), and the corresponding queue with multiple vacations by Baba

(1986). Takagi and Takahashi (1991) investigated priority queues with batch Poisson arrivals under the FIFO service regime, with either multiple or single vacations. Rosenberg and Yechiali (1993) analyzed the $M^X/G/1$ queue with LIFO service regime under three versions: without server vacations, with multiple vacations and with single vacations. They derived explicit formulae for the Laplace-Stieltjes transform (LST), mean and second moment of the waiting time W_{LIFO} of an arbitrary customer and showed that in each case $E[W^2_{\mathrm{LIFO}}] = E[W^2_{\mathrm{FIFO}}]/(1-\rho)$, where ρ is the traffic load. Shomrony and Yechiali (2001) studied the $M^X/G/1$ queue with vacations under the Randomly Timed Gated (RTG) protocol, introduced by Eliazar and Yechiali (1998), by which, whenever the server *starts a busy period*, a timer with a random duration is activated. If the server empties the queue before the timer expires, he leaves for another vacation. Otherwise, if there are still customers in the system when the timer expires, the server either completes service to the customer being served and leaves for a vacation (version 1), or leaves immediately (version 2). Recently, G. Choudhury studied the $M^X/G/1$ queue with multiple vacations and with a setup period (2000), and further investigated the batch-arrival queue with single vacations (2002).

The 'wait' option of the server when finding an empty queue was introduced and studied by Boxma, Schlegel and Yechiali (2002) for the *regular* $M/G/1$ queue. In the current work we extend the analysis to the batch-arrival queue and derive the probability generating functions (PGFs) of the queue size not only at service completion epochs (section 3) but also at a start of a busy period (section 7). We further derive the LST, mean and second moment of the busy period duration (section 4), the LST and mean of the (total) vacation period (section 5) and of the sum of dormant times within a vacation period (section 6). In section 8 we obtain the LST and mean of the cycle time and in section 9 we derive the LST and mean of the *waiting time* of an *arbitrary customer*. It should be indicated that, differently from the $M/G/1$ scenario where PASTA applies and the derivation of the LST of the waiting time follows a standard argument relating this LST with the PGF of the queue size at arrival (and departure) instants, in the batch-arrival case this argument can not be directly applied and a careful consideration of the waiting time *within* a batch is required. Finally, in each section we obtain the corresponding results for the $M^X/G/1$ queue with multiple or with single vacations by letting $T \to 0$ or by letting $T \to \infty$, respectively.

2 The Model

We consider an $M^X/G/1$ queueing system where i.i.d random batches of customers arrive according to a Poisson process $\{A(t), t \geq 0\}$ with intensity λ. Each batch-size, X, has a probability mass function $P(X = m) = f_m$ $(m = 1, 2, 3, \ldots)$ with probability generating function (PGF), $\widehat{X}(z) = E[z^X] = \sum_{m=1}^{\infty} f_m z^m$. We let $f = f^{(1)} = E[X]$, $f^{(2)} = E[X(X-1)]$ and $f^{(3)} = E[X(X-1)(X-2)]$, where $f^{(k)} = d^k\widehat{X}(z)/dz^k\big|_{z=1}$. Customers are served one at a time by a single server and service times, B, of individual customers are i.i.d. random variables with LST $B^*(s) = E[e^{-sB}]$, mean $E[B] = b^{(1)} = b$ and k-th moment $E[B^k] = b^{(k)}$. Similar notation is used for other random variables introduced in the sequel. The traffic load is denoted by $\rho = \lambda E[X]E[B] = \lambda f b$, and the system is stable when $\rho < 1$. The 'residual' service time, R_B, has an LST $R_B^*(s) = [1 - B^*(s)]/\big[sE[B]\big]$ with mean $E[R_B] = E[B^2]/\big(2E[B]\big)$. Batches are admitted to service according to their order of arrival, and within a batch, individual customers are served according to their inner order (FIFO regime). Service of customers is non-preemptive. At the termination of a busy period, when the queue becomes empty, the server takes a random vacation U. At the end of a vacation U the server returns to the main queue. If upon return from a vacation there are $N > 0$ customers in the system (i.e. at least one batch has arrived during U) the server starts servicing exhaustively until the first moment thereafter at which the system becomes empty again. At that moment the server goes for another vacation. However, if $N = 0$, the server activates a random timer T and *waits*. If a batch arrives before T expires, the server immediately starts servicing (one by one) the just arrived customers and continues working until the end of the busy period, before taking another vacation U. If no batches arrive during the timer's duration (i.e. the timer's length is shorter than the inter-arrival time of batches), the server does not wait any more and leaves for a vacation U. We invoke the usual independence assumptions between inter-arrival times, batch sizes, service times, vacation lengths and timer durations. The distributions of B, U and T are assumed to be general.

3 Number of Customers

3.1 Law of Motion

In this section we derive the PGF and mean of the number of customers in the system at *service completion (departure) epochs.*

Let L_n denote the number of customers left behind by the n-th departing customer. Then, the law of motion of the system's state L at departure epochs is given as follows:

If $L_n > 0$, then

$$L_{n+1} = L_n - 1 + \sum_{j=1}^{A(B)} X_j \ , \tag{3.1}$$

where $A(B)$ stands for the number of batches that arrived during the service time B_{n+1} of the $(n+1)$-th customer, and $X_1, X_2, \ldots$ are i.i.d. random variables, all distributed as X.

If $L_n = 0$, then

$$L_{n+1} = \begin{cases} \sum_{j=1}^{\xi} X_j - 1 + \sum_{i=1}^{A(B)} X_i & \text{w.p.} \quad \frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)} = \alpha \\ X - 1 + \sum_{i=1}^{A(B)} X_i & \text{w.p.} \quad \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)} = 1-\alpha \ , \end{cases} \tag{3.2}$$

where $\xi = A(U)\big|_{A(U)\geq 1}$, $T^*(\lambda) = P(\text{no arrivals in } T) = \int_0^\infty e^{-\lambda t} dP(T \leq t)$, and $U^*(\lambda) = P(\text{no arrivals in } U) = P(A(U) = 0)$.

The explanation of (3.2) is similar to the one given in [2]: when the server takes a vacation, the probability of no batch arrivals during U is $\int_0^\infty e^{-\lambda t} dP(U \leq u) = U^*(\lambda)$. Then, upon finding an empty system, the server activates a timer T. The probability of no arrivals during T is $T^*(\lambda)$, and the server takes another vacation, etc. This combined process repeats itself $k \geq 0$ times with probability $[U^*(\lambda)T^*(\lambda)]^k$, until, after k repetitions, there is at least one batch arrival during U. This last event occurs with probability $1 - U^*(\lambda)$ and the server then finds $\xi = A(U)\big|_{A(U)\geq 1}$ waiting *batches* with $\sum_{j=1}^{\xi} X_j$ individual jobs. Thus, the next departure will leave behind $\sum_{j=1}^{\xi} X_j - 1 + \sum_{i=1}^{A(B)} X_i$ waiting customers with probability $\alpha = (1 - U^*(\lambda)) \sum_{k=0}^{\infty} [U^*(\lambda)T^*(\lambda)]^k = \frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}$. The other possibility is that after k repeated pairs of U and T without batch arrivals, there will be another vacation with no arrivals, but with a batch arrival during the following T. This occurs with probability $U^*(\lambda)(1 - T^*(\lambda)) \sum_{k=0}^{\infty} [U^*(\lambda)T^*(\lambda)]^k = \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)} = 1 - \alpha$. Then, the server starts servicing when there is *exactly one batch* in the system, and the first departure thereafter leaves behind $X - 1 + \sum_{i=1}^{A(B)} X_i$ customers.

3.2 Generating Function of the Queue Size at a Service Completion (departure) Epoch

Using relations (3.1) and (3.2) we derive

$$\begin{aligned} E[z^{L_{n+1}}] &= E\left[z^{(L_n - 1 + \sum_{j=1}^{A(B)} X_j)}\big| L_n > 0\right] \cdot P(L_n > 0) \\ &\quad + \left\{ E\left[z^{(\sum_{j=1}^{\xi} X_j - 1 + \sum_{i=1}^{A(B)} X_i)}\right] \cdot \alpha \right. \\ &\quad \left. + E\left[z^{(X - 1 + \sum_{i=1}^{A(B)} X_i)}\right] \cdot (1-\alpha) \right\} \cdot P(L_n = 0) \ . \end{aligned} \tag{3.3}$$

We consider the system in steady state ($\rho < 1$) where $L_n \to L$, with $\widehat{L}(z) = \lim_{n\to\infty} E[z^{L_n}]$, and denote the probability that the system is empty at a service completion instant by

$P_0 := P(L=0)$. Then

$$E[z^L|L>0]\cdot P(L>0) = \left(\sum_{k=1}^{\infty} z^k \cdot \frac{P(L=k)}{P(L>0)}\right)\cdot P(L>0) = \widehat{L}(z) - P_0 \ . \tag{3.4}$$

The PGF of the total number of customers arriving within a service time B is given by

$$\begin{aligned} E\left[z^{(\sum_{j=1}^{A(B)} X_j)}\right] &= E_{A(B)}\left[E\left[z^{\sum_{j=1}^{A(B)} X_j}\middle|A(B)\right]\right] = E_{A(B)}\left[\left(\widehat{X}(z)\right)^{A(B)}\right] \\ &= \int_{t=0}^{\infty}\sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left(\widehat{X}(z)\right)^k \cdot dP(B\le t) \\ &= \int_{t=0}^{\infty} e^{-\lambda(1-\widehat{X}(z))t} dP(B\le t) = B^*\left[\lambda\left(1-\widehat{X}(z)\right)\right] \end{aligned} \tag{3.5}$$

The PGF of the total number of customers arriving within a vacation U, given that there was at least one batch arrival, is

$$\begin{aligned} E\left[z^{(\sum_{j=1}^{\xi} X_j)}\right] &= E_\xi\left[\left(\widehat{X}(z)\right)^\xi\right] = \widehat{\xi}\left(\widehat{X}(z)\right) = E\left[\widehat{X}^{A(U)}(z)\middle|A(U)\ge 1\right] \\ &= \int_{t=0}^{\infty}\sum_{k=1}^{\infty}\left(\widehat{X}(z)\right)^k\cdot e^{-\lambda t}\frac{(\lambda t)^k}{k!} dP(U\le t)/P\left(A(U)\ge 1\right) \\ &= \frac{1}{1-U^*(\lambda)}\int_{t=0}^{\infty} e^{-\lambda t}\left(e^{\lambda t\widehat{X}(z)}-1\right)dP(U\le t) \\ &= \frac{U^*[\lambda(1-\widehat{X}(z))]-U^*(\lambda)}{1-U^*(\lambda)} \ . \end{aligned} \tag{3.6}$$

Similarly,

$$E[z^\xi] = \frac{U^*[\lambda(1-z)]-U^*(\lambda)}{1-U^*(\lambda)}, \quad \text{resulting in} \quad E[\xi] = \frac{\lambda E[U]}{1-U^*(\lambda)} \ . \tag{3.7}$$

Thus, utilizing equations (3.4), (3.5) and (3.6), we get,

$$\begin{aligned} \widehat{L}(z) &= \left(\widehat{L}(z)-P_0\right)z^{-1}B^*\left[\lambda\left(1-\widehat{X}(z)\right)\right] \\ &\quad + \left\{\left[\frac{U^*[\lambda(1-\widehat{X}(z))]-U^*(\lambda)}{1-U^*(\lambda)}\right]z^{-1}\cdot B^*\left[\lambda\left(1-\widehat{X}(z)\right)\right]\cdot\frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}\right. \\ &\quad \left. +\widehat{X}(z)\cdot z^{-1}B^*\left[\lambda\left(1-\widehat{X}(z)\right)\right]\cdot\frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)}\right\}P_0 \ . \end{aligned} \tag{3.8}$$

Rearranging terms and writing $\delta := \lambda(1-\widehat{X}(z))$ we obtain

$$\widehat{L}(z) = P_0\frac{B^*(\delta)[(1-\widehat{X}(z))U^*(\lambda)(1-T^*(\lambda))+1-U^*(\delta)]}{[B^*(\delta)-z][1-U^*(\lambda)T^*(\lambda)]} \ . \tag{3.9}$$

To calculate P_0 we use $\widehat{L}(1)=1$ and $B^*(\delta)\big|_{z=1}=1$. This leads to

$$1=\frac{P_0}{1-U^*(\lambda)T^*(\lambda)}\lim_{z\to 1}\left[\frac{U^*(\lambda)(1-T^*(\lambda))(1-\widehat{X}(z))-U^*(\delta)+1}{B^*(\delta)-z}\right] .$$

Applying L'Hospital's rule, we obtain (with $\rho:=\lambda f b$)

$$P_0=(1-\rho)\frac{1-U^*(\lambda)T^*(\lambda)}{f[\lambda E(U)+U^*(\lambda)(1-T^*(\lambda))]} . \tag{3.10}$$

Finally

$$\widehat{L}(z)=\frac{(1-\rho)B^*(\delta)[U^*(\lambda)(1-T^*(\lambda))(1-\widehat{X}(z))-U^*(\delta)+1]}{[B^*(\delta)-z]f[\lambda E[U]+U^*(\lambda)(1-T^*(\lambda))]} . \tag{3.11}$$

When $X\equiv 1$, equation (3.11) reduces to equation (3.5) in Boxma, Schlegel and Yechiali (2002). Note that, in such a case, $\widehat{L}(z)$ is also the PGF of the number of customers in the system at an arbitrary moment.

When $T=\infty$ (i.e. $T^*(\lambda)=0$), the timer model reduces to the $M^X/G/1$ queue with *single* vacations (SV). In this case, equation (3.11) reduces to

$$\widehat{L}(z)_{M^X/G/1+\mathrm{SV}}=\frac{(1-\rho)B^*(\delta)[1-U^*(\delta)+U^*(\lambda)(1-\widehat{X}(z))]}{f\cdot[B^*(\delta)-z][\lambda E[U]+U^*(\lambda)]} . \tag{3.12}$$

Equation (3.12) coincides with equation (5.5) in Choudhury (2002).
If $T=0$ (with $T^*(\lambda)=1$), we obtain the $M^X/G/1$ queue with *multiple* vacations (MV). Equation (3.11) then reduces to

$$\begin{aligned}\widehat{L}(z)_{M^X/G/1+\mathrm{MV}} &= \frac{(1-\rho)B^*(\delta)[1-U^*(\delta)]}{f\cdot[B^*(\delta)-z]\lambda E[U]}\\ &= \frac{(1-\rho)B^*(\delta)(1-\widehat{X}(z))}{f[B^*(\delta)-z]}\cdot\frac{1-U^*(\delta)}{\lambda E[U](1-\widehat{X}(z))}\\ &= \widehat{L}(z)_{M^X/G/1}\cdot\frac{1-U^*(\delta)}{\delta E[U]} \qquad (3.13)\end{aligned}$$

(see equation (4.20) in Shomrony and Yechiali (2001)). Denoting by R_U the 'residual' part of the vacation time U, then $R_U^*(\delta)=(1-U^*(\delta))/(\delta E[U])$ expresses the PGF of the total number of customers arriving during R_U. Thus, equation (3.13) reveals the decomposition property that the number of customers at a service completion instant is the sum of the number of customers in the $M^X/G/1$ queue plus the number of customers arriving during the 'remaining time' of a vacation, R_U.

When $U=0$, equation (3.13) reduces to the PGF for the $M^X/G/1$ queue at departure epochs

$$\widehat{L}(z)_{M^X/G/1}=\frac{(1-\rho)B^*(\delta)(1-\widehat{X}(z))}{f[B^*(\delta)-z]} . \tag{3.14}$$

This follows since, when $U \to 0$, $[1 - U^*(\lambda(1 - \widehat{X}(z)))]/(\lambda E[U]) \to (1 - \widehat{X}(z))$.

Equation (3.14) coincides with equation (3.18) in Shomrony and Yechiali (2001) and with equation (2.10) in Cohen ((1982), page 386). Clearly, when $X \equiv 1$, we have $\widehat{X}(z) = z$, $\delta = \lambda(1-z)$ and $f = 1$, so that equation (3.14) reduces to the Khintchine-Pollazcek formula for the classical $M/G/1$ queue (see e.g. Levy and Yechiali (1975), Takagi (1991))

$$\widehat{L}(z)_{M/G/1} = \frac{(1-\rho)(1-z)B^*[\lambda(1-z)]}{B^*[\lambda(1-z)] - z} \; . \tag{3.15}$$

4 The Busy Period

A busy period starts either with $\xi = A(U)\big|_{A(U)\geq 1}$ batches that arrived during a vacation U, or with a batch of size X arriving within the timer's duration, T. As in equation (3.2), the former event occurs with probability α and the latter with probability $1-\alpha$. Consider first a busy period that starts with the arrival of a batch of size X. Denote its duration by θ_X. The total service time of all jobs belonging to this batch is $Y = \sum_{i=1}^{X} B_i$, where B_i are i.i.d, all distributed like B.

Clearly, $E[Y] = E[X] \cdot E[B] = f \cdot E[B] = fb$, while the LST of Y is given by (see Rosenberg and Yechiali (1993), equation (1))

$$Y^*(s) = E\big[e^{-s(\sum_{i=1}^{X} B_i)}\big] = \sum_{n=1}^{\infty} f_n \big[B^*(s)\big]^n = \widehat{X}\big(B^*(s)\big) \; . \tag{4.1}$$

Differentiating, the second and third moments of Y are derived:

$$E[Y^2] = f^{(2)}b^2 + fb^{(2)}, \quad E[Y^3] = f^{(3)}b^3 + 3f^{(2)}b^{(2)}b + fb^{(3)} \; . \tag{4.2}$$

Considering the service time of a batch as a service time of a 'super' customer in a regular $M/G/1$ queue with service times Y and utilizing (4.1), the LST of the busy period starting with an arrival of a batch is given by (see Rosenberg and Yechiali (1993), equation (2); Shomrony and Yechiali (2001), equation (2.11)).

$$\theta_X^*(s) = Y^*\big(s + \lambda - \lambda\theta_X^*(s)\big) = \widehat{X}\left(B^*\big(s + \lambda - \lambda\theta_X^*(s)\big)\right) \; . \tag{4.3}$$

Thus, by differentiating $Y^*(\cdot)$, we get

$$E[\theta_X] = E[Y]\big(1 + \lambda E[\theta_X]\big) = \frac{E[Y]}{1 - \lambda E[Y]} = \frac{E[Y]}{1-\rho} = \frac{fb}{1-\rho} = \frac{1}{\lambda} \cdot \frac{\rho}{1-\rho} \; . \tag{4.4}$$

Alternatively, by differentiating $\widehat{X}(\cdot)$ from the RHS of (4.3), we get

$$E[\theta_X] = E[X]b\big(1 + \lambda E[\theta_X]\big) = \frac{fb}{1-\rho} \; .$$

This follows since $1+\lambda E[\theta_X] = 1+\frac{\rho}{1-\rho} = (1-\rho)^{-1}$.

The second moment of θ_X is derived by differentiating $\theta_X^*(s)$ twice:

$$E[\theta_X^2] = E[Y^2]\big(1+\lambda E[\theta_X]\big)^2 + E[Y]\lambda E[\theta_X^2] = \frac{E[Y^2](1+\lambda E[\theta_X])^2}{1-\rho} = \frac{E[Y^2]}{(1-\rho)^3} \ . \qquad (4.5)$$

Now, consider a busy period that starts with ξ batches, and denote its duration by θ_ξ. Then, $\theta_\xi = \sum_{j=1}^{\xi}(\theta_X)_j$ where $(\theta_X)_j$ are i.i.d. all distributed like θ_X. Thus,

$$\theta_\xi^*(s) = E\left[\big(\theta_X^*(s)\big)^\xi\right] = \widehat{\xi}\big(\theta_X^*(s)\big) \ . \qquad (4.6)$$

But,

$$\widehat{\xi}(z) = E[z^{A(U)|A(U)\geq 1}] = \frac{U^*[\lambda(1-z)] - U^*(\lambda)}{1-U^*(\lambda)} \ . \qquad (4.7)$$

Thus, finally,

$$\theta_\xi^*(s) = \frac{U^*[\lambda(1-\theta_X^*(s))] - U^*(\lambda)}{1-U^*(\lambda)} \ . \qquad (4.8)$$

Differentiating, we get

$$E[\theta_\xi] = \frac{E[U]\cdot(\lambda E[\theta_X])}{1-U^*(\lambda)} = \frac{\lambda E[U] f b}{(1-\rho)(1-U^*(\lambda))} = \frac{\rho}{1-\rho}\cdot\frac{E[U]}{1-U^*(\lambda)} \qquad (4.9)$$

$$E[\theta_\xi^2] = \frac{1}{1-U^*(\lambda)}\Big[E[U^2]\big(\lambda E[\theta_X]\big)^2 + E[U]\lambda E[\theta_X^2]\Big] \ . \qquad (4.10)$$

Combining (4.3) and (4.8) we obtain the LST of the busy period θ:

$$\begin{aligned}
\theta^*(s) &= (1-\alpha)\theta_X^*(s) + \alpha\theta_\xi^*(s) \\
&= \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)}\cdot\widehat{X}\left(B^*\big(s+\lambda-\lambda\theta_X^*(s)\big)\right) \\
&\quad + \frac{U^*[\lambda(1-\theta_X^*(s))] - U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)} \ . \qquad (4.11)
\end{aligned}$$

Now,

$$\begin{aligned}
E[\theta] &= (1-\alpha)E[\theta_X] + \alpha E[\theta_\xi] \\
&= \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)}\cdot\frac{fb}{1-\rho} + \frac{E[U]}{1-U^*(\lambda)T^*(\lambda)}\cdot\frac{\rho}{1-\rho} \\
&= \frac{\rho\big[U^*(\lambda)(1-T^*(\lambda))/\lambda + E[U]\big]}{(1-\rho)[1-U^*(\lambda)T^*(\lambda)]} \\
&= E[\theta_X]\left[\frac{U^*(\lambda)(1-T^*(\lambda)) + \lambda E[U]}{1-U^*(\lambda)T^*(\lambda)}\right] \qquad (4.12)
\end{aligned}$$

The second moment of θ is given by

$$E[\theta^2] = (1-\alpha)E[\theta_X^2] + \alpha E[\theta_\xi^2] \tag{4.13}$$

where $E[\theta_X^2]$ and $E[\theta_\xi^2]$ are given by (4.5) and (4.10), respectively.

When $X \equiv 1$ ($M/G/1$ with vacations and a timer), equation (4.11) reduces to equation (5.1) of Boxma, Schlegel and Yechiali (2002).

For the multiple vacation case ($T^*(\lambda) = 1$), equation (4.11) yields

$$\theta^*(s)_{M^X/G/1+\mathrm{MV}} = \frac{U^*[\lambda(1-\theta_X^*(s)] - U^*(\lambda)}{1-U^*(\lambda)} \tag{4.14}$$

with

$$E\left[\theta_{M^X/G/1+\mathrm{MV}}\right] = \frac{E[U]}{1-U^*(\lambda)} \cdot \frac{\rho}{1-\rho} = E[\theta_\xi] \ . \tag{4.15}$$

For the single vacation (SV) case ($T = \infty$ and $T^*(\lambda) = 0$), equation (4.11) reduces to equation (3.2) of Choudhury (2002):

$$\theta^*(s)_{M^X/G/1+\mathrm{SV}} = U^*(\lambda)\left[\widehat{X}\left(B^*\big(s+\lambda-\lambda\theta_X^*(s)\big)\right) - 1\right] + U^*\left[\lambda\big(1-\theta_X^*(s)\big)\right] \tag{4.16}$$

and

$$\begin{aligned} E[\theta_{M^X/G/1+\mathrm{SV}}] &= \frac{U^*(\lambda)fb + E[U]\cdot\rho}{1-\rho} = \frac{\rho}{1-\rho}\left[\frac{U^*(\lambda)}{\lambda} + E(U)\right] \\ &= E[\theta_X]\big(U^*(\lambda) + \lambda E[U]\big) \ . \end{aligned} \tag{4.17}$$

5 Vacation Period

Let I be the Exponential(λ) inter-arrival time, with LST $I^*(s) = \frac{\lambda}{s+\lambda}$. Denote by V_P the *vacation period*, i.e. the time interval beginning at the end of an active busy period and extending to the start of the next busy period. Let $\{U_i,\ i = 1,2,3,...\}$ and $\{T_i,\ i = 1,2,3,...\}$ be two sequences of i.i.d. random variables having LST $U^*(\cdot)$ and $T^*(\cdot)$, respectively.

For particular values of $U_1, U_2, \ldots, U_{k+1}, T_1, T_2, \ldots, T_k$, and I we have

$$V_P = \begin{cases} (U_1+T_1) + \cdots + (U_k+T_k) + U_{k+1} & \text{w.p.} \quad e^{-\lambda\sum_{i=1}^k (U_i+T_i)}(1-e^{-\lambda U_{k+1}}) \\ (U_1+T_1) + \cdots + (U_k+T_k) + U_{k+1} + I & \text{w.p.} \quad e^{-\lambda\sum_{i=1}^k (U_i+T_i)}e^{-\lambda U_{k+1}}\overline{F}_{T_{k+1}}(I) \end{cases}$$

where $\overline{F}_{T_{k+1}}(I) = P[T_{k+1} > I]$.

It follows that the length of the vacation period in the current model is *identical* with the length of the vacation period in the regular $M/G/1$ queue with a waiting server and vacations.

Thus, as was shown in [2],

$$V_P^*(s) = \frac{U^*(s) - U^*(s+\lambda) + U^*(s+\lambda)\frac{\lambda}{\lambda+s}[1 - T^*(s+\lambda)]}{1 - U^*(s+\lambda)T^*(s+\lambda)} \ . \tag{5.1}$$

and

$$E[V_P] = \frac{E[U] + \frac{U^*(\lambda)}{\lambda}[1 - T^*(\lambda)]}{1 - U^*(\lambda)T^*(\lambda)} \ . \tag{5.2}$$

For the multiple vacation (MV) case, equations (5.1) and (5.2) reduce, respectively, to

$$V_P^*(s) = \frac{U^*(s) - U^*(s+\lambda)}{1 - U^*(s+\lambda)} \ ,$$

and

$$E[V_P] = \frac{E[U]}{1 - U^*(\lambda)} \ .$$

For the single vacation (SV) case, equations (5.1) and (5.2) yield, respectively,

$$V_P^*(s) = U(s) - U^*(s+\lambda) + U^*(s+\lambda)I^*(s)$$

and

$$E[V_P] = E[U] + \frac{U^*(\lambda)}{\lambda} \ .$$

6 Sum of Dormant ('Wait') Lengths within a Vacation Period

Denote by D the sum of dormant T times within a vacation period V_P. Similarly to the derivation of the LST of V_P we write

$$\begin{aligned} D^*(s) = E\left[e^{-sD}\right] &= \sum_{k=0}^{\infty} E\left[e^{-s(\sum_{i=1}^k T_i)} e^{-\lambda \sum_{i=1}^k (U_i+T_i)} (1 - e^{-\lambda U_{k+1}})\right] \\ &\quad + \sum_{k=0}^{\infty} E\left[e^{-s(\sum_{i=1}^k T_i)} e^{-\lambda \sum_{i=1}^k (U_i+T_i)} e^{-\lambda U_{k+1}}\right] \cdot E\left[e^{-sI}\overline{F}_T(I)\right] \\ &= \frac{1 - U^*(\lambda) + U^*(\lambda)\frac{\lambda}{\lambda+s}\left[1 - T^*(s+\lambda)\right]}{1 - U^*(\lambda)T^*(s+\lambda)} \ . \end{aligned} \tag{6.1}$$

By differentiation we obtain

$$E[D] = \frac{\frac{U^*(\lambda)}{\lambda}\ [1 - T^*(\lambda)]}{1 - U^*(\lambda)T^*(\lambda)} \ . \tag{6.2}$$

Then

$$E[V_P] - E[D] = \frac{E[U]}{1 - U^*(\lambda)T^*(\lambda)} \tag{6.3}$$

is the expected total time within a V_P in which the server is (only) on vacations U.

7 Queue Size Distribution at a Start of a Busy Period

As indicated before, a busy period starts either with $\xi = A(U)\big|_{A(U)\geq 1}$ batches that arrived during a vacation, or with a batch of size X arriving within a timer duration. Thus, the number of customers Q at a busy period initiation is

$$Q = \begin{cases} \sum_{j=1}^{\xi} X_j & \text{w.p. } \alpha \\ X & \text{w.p. } 1-\alpha \ . \end{cases} \tag{7.1}$$

It follows that

$$\begin{aligned} \widehat{Q}(z) = E[z^Q] &= \alpha E\left[z^{(\sum_{j=1}^{\xi} X_j)}\right] + (1-\alpha)E[z^X] \\ &= \alpha\widehat{\xi}\big(\widehat{X}(z)\big) + (1-\alpha)\widehat{X}(z) \ . \end{aligned} \tag{7.2}$$

Using (4.7) and (3.2) we get

$$\widehat{Q}(z) = \frac{U^*(\delta) - U^*(\lambda)}{1 - U^*(\lambda)T^*(\lambda)} + \frac{U^*(\lambda)(1 - T^*(\lambda))}{1 - U^*(\lambda)T^*(\lambda)}\widehat{X}(z) \ . \tag{7.3}$$

From (7.1) and (7.3) we obtain

$$\begin{aligned} E[Q] = E[\xi]E[X]\alpha + E[X](1-\alpha) &= \left[\frac{\lambda E[U]}{1 - U^*(\lambda)T^*(\lambda)} + \frac{U^*(\lambda)(1 - T^*(\lambda))}{1 - U^*(\lambda)T^*(\lambda)}\right] E[X] \\ &= \lambda E[V_P]E[X] \ . \end{aligned} \tag{7.4}$$

For the SV case ($T^*(\lambda) = 0$), equation (7.3) reduces to

$$Q^*_{M^X/G/1+\text{SV}}(z) = U^*(\delta) - U^*(\lambda)\big(1 - \widehat{X}(z)\big) \ . \tag{7.5}$$

Equation (7.5) coincides with equation (2.2) of Choudhury (2002).

For the MV case ($T^*(\lambda) = 1$), we get

$$\widehat{Q}_{M^X/G/1+\text{MV}}(z) = \frac{U^*(\delta) - U^*(\lambda)}{1 - U^*(\lambda)} \tag{7.6}$$

8 Cycle Time

Let $C = V_P + \theta$ denote the cycle time. Then, for given $U_1, T_1, U_2, T_2, \ldots, U_k, T_k, U_{k+1}$ and I, and with $\xi := A(U_{k+1})\big|_{A(U_{k+1})\geq 1}$ we have

$$C = \begin{cases} \sum_{i=1}^{k}(U_i + T_i) + U_{k+1} + \theta_\xi & \text{w.p. } \ e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)} \cdot (1 - e^{-\lambda U_{k+1}}) \\ \sum_{i=1}^{k}(U_i + T_i) + U_{k+1} + I + \theta_X & \text{w.p. } \ e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)} e^{-\lambda U_{k+1}}\overline{F}_{T_{k+1}}(I) \end{cases} \tag{8.1}$$

where $\overline{F}_{T_{k+1}}(I) = P(T_{k+1} > I)$ and θ_ξ and θ_X are as defined in section 4.

Then, the LST of C is given by

$$\begin{aligned} C^*(s) &= \sum_{k=0}^{\infty} E\left[e^{-s\sum_{i=1}^{k}(U_i+T_i)} e^{-sU_{k+1}} e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}(1-e^{-\lambda U_{k+1}}) e^{-s\sum_{j=1}^{\xi}(\theta_X)_j}\right] \\ &\quad + \sum_{k=0}^{\infty} E\left[e^{-s\sum_{i=1}^{k}(U_i+T_i)} e^{-sU_{k+1}} e^{-sI} e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)} e^{-\lambda U_{k+1}} \overline{F}_{T_{k+1}}(I) e^{-s\theta_X}\right] \\ &= \sum_{k=0}^{\infty} \left[U^*(s+\lambda)T^*(s+\lambda)\right]^k \cdot E\left[e^{-sU_{k+1}}\left(1-e^{-\lambda U_{k+1}}\right)\left[\theta_X^*(s)\right]^{A(U_{k+1})\big|_{A(U_{k+1})\geq 1}}\right] \\ &\quad + \sum_{k=0}^{\infty} \left[U(s+\lambda)T^*(s+\lambda)\right]^k U^*(s+\lambda)\frac{\lambda}{\lambda+s}\left[1-T^*(s+\lambda)\right]\theta_X^*(s) \ . \end{aligned}$$

Thus,

$$C^*(s) = \frac{U^*[s+\lambda(1-\theta_X^*(s)] - U^*(s+\lambda) + U^*(s+\lambda)\frac{\lambda}{\lambda+s}[1-T^*(s+\lambda)]\theta_X^*(s)}{1-U^*(s+\lambda)T^*(s+\lambda)} \ . \qquad (8.2)$$

The mean cycle time is given by

$$E[C] = \frac{E[U] + \frac{U^*(\lambda)}{\lambda}[1-T^*(\lambda)]}{(1-\rho)[1-U^*(\lambda)T^*(\lambda)]} \ . \qquad (8.3)$$

It readily follows from (4.12) and (8.3) that the fraction of time that the server is busy or is non-operative is, as expected,

$$P_{\text{busy}} := \frac{E[\theta]}{E[C]} = \rho, \quad P_{\text{non-operative}} := \frac{E[V_P]}{E[C]} = 1-\rho \ .$$

Similarly, the proportion of time that the server is *dormant* is given by

$$P_D = \frac{E[D]}{E[C]} = \frac{\frac{U^*(\lambda)}{\lambda}\ [1-T^*(\lambda)](1-\rho)}{E[U] + \frac{U^*(\lambda)}{\lambda}\ [1-T^*(\lambda)]} \ .$$

Finally, the proportion of time the server spends on 'pure' vacations U is

$$P_U = \frac{E[V_P]-E[D]}{E[C]} = \frac{E[U](1-\rho)}{E[U] + \frac{U^*(\lambda)}{\lambda}\ \left[1-T^*(\lambda)\right]} \ .$$

9 Waiting Times

Let W := waiting (queueing) time of an arbitrary customer, J. W is composed of two components: (1) W_1 := the time from the arrival epoch of the batch to which J belongs until the moment at which the first customer of this batch starts service, and (2) $W_2 :=$ the time elapsing from the latter moment until customer J begins service.

Consider a random batch of size X as a 'super customer' with (total) service time $Y = \sum_{i=1}^{X} B_i$ (see section 4). Now, the LST of W_1 is the same as the LST of the waiting time of an arbitrary customer (derived by Boxma, Schlegel and Yechiali (BSY) (2002)) for an $M/G/1$ queue with a waiting server, timer and vacations, where the individual service time B there is replaced by Y here. Thus, using the expression for $W^*(s)$ in section 4 of BSY (2002), we have

$$\begin{aligned} W_1^*(s) &= \frac{s(1-\rho)}{s-\lambda+\lambda Y^*(s)} \cdot \frac{\lambda[1-U^*(s)]/s + U^*(\lambda)[1-T^*(\lambda)]}{U^*(\lambda)[1-T^*(\lambda)]+\lambda E[U]} \\ &= W^*_{M/G(Y)/1}(s)\left[\frac{E[V_P]-E[D]}{E[V_P]} R_U^*(s) + \frac{E[D]}{E[V_P]} \times 1\right] \end{aligned} \tag{9.1}$$

where $W_{M/G(Y)/1}$ is the waiting time of a customer in a *regular* $M/G/1$ queue where service times are Y rather than B. Thus,

$$E[W_1] = \frac{\lambda E[Y^2]}{2(1-\rho)} + \frac{\lambda E[U^2]}{2} \cdot \frac{1}{\gamma}\ , \tag{9.2}$$

where $\gamma = \lambda E[U] + U^*(\lambda)(1-T^*(\lambda))$.

Equation (9.1) reveals that the queueing time of a batch is a composition of two independent variables: $W_1 = W_{M/G(Y)/1} + S$, where

$$S = \begin{cases} R_U & \text{with probability } (E[V_P]-E[D])/E[V_P] \\ 0 & \text{with probability } E[D]/E[V_P]\ . \end{cases}$$

When $X \equiv 1$, equation (9.2) reduces to equation (4.2) in BSY (2002).

Calculation of $W_2^*(s)$

Customer J is in the n-th position (within his batch) with probability $g_n = \frac{1}{f}\sum_{k=n}^{\infty} f_k$ (see Burke, 1975). Then

$$\begin{aligned} W_2^*(s) &= \sum_{n=1}^{\infty} E\left[e^{-sW_2} \middle| J \text{ is } n\text{-th}\right] g_n = \sum_{n=1}^{\infty} E\left[e^{-s(\sum_{i=1}^{n-1} B_i)}\right] g_n \\ &= \sum_{n=1}^{\infty} \left[B^*(s)\right]^{n-1} g_n = \sum_{n=1}^{\infty} \left[B^*(s)\right]^{n-1} \frac{1}{f}\sum_{k=n}^{\infty} f_k = \frac{1}{f}\sum_{k=1}^{\infty} f_k \sum_{n=1}^{k} \left[B^*(s)\right]^{n-1} \\ &= \frac{1}{f}\sum_{k=1}^{\infty} f_k\, \frac{1-[B^*(s)]^k}{1-B^*(s)} = \frac{1}{f(1-B^*(s))}\left[1-\widehat{X}\left(B^*(s)\right)\right]\ . \end{aligned} \tag{9.3}$$

Also,

$$
\begin{aligned}
E[W_2] &= \sum_{n=1}^{\infty} E[W_2 \big| J \text{ is } n\text{-th}] g_n = \sum_{n=1}^{\infty} (n-1) b g_n = \frac{b}{f} \sum_{k=1}^{\infty} f_k \sum_{n=1}^{k} (n-1) \\
&= \frac{b}{f} \sum_{k=1}^{\infty} f_k \frac{k(k-1)}{2} = \frac{b}{2f} \left[E[X^2] - f \right] = \frac{b f^{(2)}}{2f} \ . \qquad (9.4)
\end{aligned}
$$

Finally, since $W = W_1 + W_2$ while W_1 and W_2 are independent, we obtain

$$
W^*(s) = W_1^*(s) \cdot W_2^*(s) \ . \qquad (9.5)
$$

The mean queueing time is given by

$$
E[W] = \frac{\lambda E[Y^2]}{2(1-\rho)} + \frac{\lambda E[U^2]}{2} \cdot \frac{1}{\gamma} + \frac{b f^{(2)}}{2f} \ . \qquad (9.6)
$$

For the MV case, equation (9.5) reduces to Baba's (1986) equation (24) and Takagi's equation (3.20):

$$
W^*_{M^X/G/1+MV}(s) = \frac{s(1-\rho)\left[1 - \widehat{X}(B^*(s))\right]}{f\big(s - \lambda + \lambda \widehat{X}(B^*(s))\big)(1 - B^*(s))} \cdot \frac{1 - U^*(s)}{sE[U]} = W^*(s)_{M^X/G/1} \cdot R^*_U(s) \qquad (9.7)
$$

implying the decomposition $W = W_{M^X/G/1} + R_U$ (see equation (3.13) above). Clearly, (9.7) follows from (9.1) when there is no dormant time ($T = 0$ and $D = 0$) within a vacation period. Now,

$$
\begin{aligned}
E\left[W_{M^X/G/1+\mathrm{MV}}\right] &= \frac{\lambda E[Y^2]}{2(1-\rho)} + \frac{E[U^2]}{2E[U]} + \frac{b f^{(2)}}{2f} \\
&= \frac{\lambda f b^{(2)}}{2(1-\rho)} + \frac{b f^{(2)}}{2f(1-\rho)} + \frac{E[U^2]}{2E[U]} \ . \qquad (9.8)
\end{aligned}
$$

Equation (9.8) coincides with equation (18) of Rosenberg and Yechiali (1993), and with equation (25) of Baba (1986).

For the SV case,

$$
\begin{aligned}
E\left[W_{M^X/G/1+\mathrm{SV}}\right] &= \frac{\lambda E[Y^2]}{2(1-\rho)} + \frac{\lambda E[U^2]}{2(\lambda E[U] + U^*(\lambda))} + \frac{b f^{(2)}}{2f} \\
&= \frac{\lambda f b^{(2)}}{2(1-\rho)} + \frac{b f^{(2)}}{2f(1-\rho)} + \frac{\lambda E[U^2]}{2[\lambda E[U] + U^*(\lambda)]} \ . \qquad (9.9)
\end{aligned}
$$

Equation (9.9) is identical to equation (32) of Rosenberg and Yechiali (1993).

It is readily seen that $E[W_{M^X/G/1+\mathrm{SV}}] < E[W] < E[W_{M^X/G/1+\mathrm{MV}}]$.

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