A TWO-ECHELON MULTI-SERVER MARKOVIAN QUEUEING SYSTEM WITH A LIMITED INTERMEDIATE BUFFER

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To the Memory of Micha Yadin

ABSTRACT

We consider a two-echelon multi-server tandem queueing system fed by a Poissonian stream of arrivals. There are S_k servers at stage k where service times are exponentially distributed with parameter μ_k $(k = 1,2)$. The first stage has an unlimited waiting capacity, while the second stage has a finite buffer of size $M - S_2$. Upon service completion at the first stage, a customer leaves the system with probability q, or moves on to request additional service at the second stage. If the intermediate buffer is full, the customer leaves the system with probability 1.

Such a model describes, for example, a telephone information system where a customer, after being served by a regular operator, may request a higher-echelon-service from a supervisor.

We formulate the problem as a two-dimensional continuous-time Markov chain and derive a set of linear equations, $A(z)\cdot\underline{\Pi}(z) = \underline{b}(z)$, where $\underline{\Pi}(z)$ is an $M+1$-dimensional vector of unknown partial generating functions which determine the steady-state probabilities of the system. The tridiagonal matrix $A(z)$, which contains the parameters, is transformed into an Hessenberg matrix whose determinants yield polynomials with interesting interlacing properties. These properties are exploited to obtain a set of equations in "boundary" probabilities that are required for the complete specification of the generating functions. An example and numerical results are presented.

Keywords: Queues in Tandem, Multi-server, Limited Buffer, Hessenberg Matrix.

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* Supported by a Grant from the France-Israel Scientific Cooperation in Computer Science and Engineering between the Israeli Ministry of Science and Technology and the French Ministry of Research and Technology, Grant Number 3321190.

1. INTRODUCTION

We consider a system of two multi-server queues in tandem with limited intermediate buffer. External arrival is Poissonian with rate λ, and service times of individual customers at queue k are exponential with parameter μ_k. There are S_k identical servers at stage k ($k = 1, 2$). Upon service completion at the first queue (lower echelon), a customer may request (with probability $p = 1-q$) an additional service at the second stage (higher echelon). If such a service is requested and the intermediate buffer is full, the customer is forced to leave the system immediately.

Such a model describes, for example, a telephone information system where a customer, after being served by a regular operator, may request a higher-echelon-service from a supervisor.

Systems with a sequence of queues in tandem, where intermediate buffers are limited, have been studied in the literature from the early stages of queueing theory. One of the earliest works was a study by Avi-Itzhak and Yadin [2] of a sequence of two *single-server* stations with no intermediate queue. They derived the moment generating function of the steady-state number of customers in the various parts of the system under the assumption of Poisson arrivals and arbitrarily distributed service times in both stations. The results were then extended to the case of a finite intermediate buffer. Avi-Itzhak [3] further extended this work to a sequence of several stations, each of which consists of the same number of servers, with deterministic service times. He derived several properties regarding waiting times, which enable one to obtain waiting time distributions of such queueing processes. Neuts [16] analyzed a two-stage *single-server* network with a finite intermediate buffer and *blocking*, where service times have a general distribution function at the first stage, and exponential distribution at the second stage. He emphasized equilibrium conditions and obtained several limit theorems of practical interest. Konheim and Reiser [13] studied a two-stage *single-server* queueing network with *feedback* and a finite intermediate waiting room, where the first server is *blocked* whenever M requests are present in the second stage. Under Markovian assumptions they provided an algorithmic solution for the state probabilities of the network. For the above model, apart from blocking, Bocharov and Al'bores [4] allowed for two additional disciplines: loss of the blocked customer, and repeated service by the first server. When there is no feedback and the loss discipline is considered, this model coincides with ours whenever $S_1 = S_2 = 1$ and $q = 0$.

A brief literature review (up to 1980) of the considerable number of papers devoted to the study of systems consisting of *two queues* in tandem, separated by a finite intermediate buffer, where the physical phenomenon of primary interest is *blocking*, is given in Chapter 5 of Neuts [17]. The excellent exposition indicates various results and emphasizes the relationships between these blocking models and their matrix-geometric solutions.

Gershwin [6] considered a system of k *single-server* stations in tandem with limited storage capacity between any two nodes. He approximated the single k-machine line by a set of $k - 1$ two-machine lines and developed an algorithm to calculate the parameters of the modified system.

Altiok [1] studied a system of *single-server* queues in series with finite waiting-room capacities and blocking, where service times have phase-type distributions. He presented an approximation method, indicating that exact results for the steady-state queue-length distributions are generally not attainable.

Langaris [14] considered a two-stage service system with a *single server* at each stage, *finite* waiting space in *both* stages and blocking. He developed the finite set of balance equations required for the solution of the two-dimensional steady-state probabilities, and showed how these equations can be modified to present the case of multiple servers in both stages.

Gün and Makowski [11] analyzed a two-node tandem queueing system with a *single server* at each node and phase-type service times. Buffers are limited and feedback is allowed at both stages. They showed that the invariant probability vector of the underlying *finite-state* quasi-birth-and-death process admits a matrix-geometric representation for all values of the arrival parameter λ. A closed-form expression for the corresponding rate matrix provides the basis for an efficient calculation of the invariant probability vector.

Recently, Chao and Pinedo [5] studied a system with two *single-server* stations in tandem, with no intermediate buffer, where customers arrive in *batches*. Assuming arbitrary service times at the two stations, they derived an expression for the expected time in system, and discussed its various properties.

Multi-server systems with two types of servers were considered by Green [7,8]. There are primary servers and auxiliary servers, and two types of customers: those who are satisfied with a service given by a primary server, and those who request a *combined* service of a primary server and an auxiliary one working together. Clearly, such a customer is being served only when two different servers (primary and auxiliary) are available simultaneously.

Another model analyzed by Green [9,10] is also characterized by two types of servers: the general-use server may perform any job, whereas the limited-use server may perform only specific jobs. Accordingly, there are two types of customers: those who request the general service, and those who are content with a limited service.

Our model differs from these two studies in two main points: (i) a customer is served *separately* at each queue, and (ii) a customer is forced to leave the system if he is blocked before the second stage.

We formulate our model as a two-dimensional continuous-time markov chain, where each of the coordinates of the system-state (i,j) represents the number of customers waiting and being served at the corresponding stage. By following the method used by Levy and Yechiali [15] and by Bocharov and Al'bores [4], we obtain expressions for the partial generating functions $\pi_j(z) = \sum_{i=0}^{\infty} p_{i,j} z^i$ determining the steady-state distribution of the system $\{p_{i,j}\}$ $(i = 0,1,\ldots;\ j = 0,1,\ \ldots,M)$, where M is the maximal total number of customers at stage 2. The determination of the partial generating functions $\{\pi_j(z)\}_{j=0}^{M}$ is based on a detailed analysis of an Hessenberg matrix obtained from the set of linear equations in the unknown generating functions. This analysis exploits interlacing properties of the roots of the determinant of the above Hessenberg matrix.

In Section 2 the model is described in detail, and the set of balance equations for the steady-state probabilities $\{p_{i,j}\}$ is derived. In Section 3 we obtain a linear set of equations $A(z) \cdot \underline{\Pi}(z) = \underline{b}(z)$, where $\underline{\Pi}(z)$ is an $M+1$-dimensional vector of the above generating functions; $\underline{b}(z)$ is a vector whose entries are comprised of the parameters of the system $(\lambda, \mu_1, \mu_2, S_1, S_2, p)$ and a set of $S_1(M+1)$ so-called "boundary" probabilities $\{p_{i,j}\}$ $(i = 0,1,\ldots,S_1-1;\ j = 0,1,\ldots,M)$; and $A(z)$ is a tridiagonal matrix involving only the parameters of the system.

In Section 4, $M+1$ roots of the determinant of $A(z)$, lying in the interval $(0,1]$, are used to obtain additional equations in the probabilities $\{p_{i,j}\}$, the knowledge of which determines completely the set of generating functions $\{\pi_j(z)\}_{j=0}^{M}$. An example is presented to show how these equations can be solved.

In Section 5 we investigate the properties of an Hessenberg matrix, derived from $A(z)$ by elementary row operations, and show that its determinant possesses $2M+2$ *distinct real* roots. This is done by exploring the interlacing structure resulting from the special form of the matrix. Furthermore, we show that $M+1$ roots lie in the interval $(0,1]$, and the remaining $M+1$ in the interval $(1,\infty)$.

In the Appendix we present a few numerical results of the actual values of the roots of the equation $|A(z)|=0$, for various values of the parameters $(\lambda, \mu_1, \mu_2, S_1, S_2, p)$.

2. THE MODEL AND BALANCE EQUATIONS

Consider a two-echelon multi-server tandem queueing system, where customers arrive to the first (lower) echelon according to a Poisson process with rate λ, and are served there by S_1 identical servers. Service times at the first stage are exponentially distributed with parameter μ_1, and the waiting room is unlimited. After being served at the lower echelon, a customer leaves the system (i.e., service is complete) with probability q, or moves on and requests additional service at the second stage (higher echelon) with probability $p = 1-q$. There are S_2 identical servers at the second stage, and service times of individual customers are exponentially distributed with parameter μ_2. However, the intermediate buffer between the two stages is limited to $M - S_2$ (at most M customers at the second stage), such that a customer who requests service at the higher echelon and finds the buffer full, leaves the system with probability 1.

We formulate the system as a two-dimensional birth-and-death process, and write down the balance equations for the steady-state probabilities of the system. Clearly, a necessary and sufficient condition for ergodicity is $\lambda < S_1\mu_1$, as the first stage is simply an $M/M/S_1$ queue, and the second stage, being a finite-capacity buffer queue, regulates itself and is ergodic for any finite arrival and service rates. However, the first and second queues are *dependent*, as the input to stage 2 depends on the output of stage 1.

Let $p_{i,j}$ be the steady-state probability of the system, where i and j denote the number of customers (waiting and in-service) at the first and second echelon, respectively ($i = 0, 1, 2, ...;\ j = 0, 1, 2, ..., M$).

The balance equations, representing the various distinct transitions between the states, take the form:

For $j = 0$:

$$\lambda p_{0,0} = \mu_2 p_{0,1} + q\mu_1 p_{1,0} \tag{2.1}$$

$$p_{i,0}(\lambda + i\mu_1) = \lambda p_{i-1,0} + \mu_2 p_{i,1} + q(i+1)\mu_1 p_{i+1,0} \qquad 1 \le i \le S_1 - 1 \tag{2.2}$$

$$p_{i,0}(\lambda + S_1\mu_1) = \lambda p_{i-1,0} + \mu_2 p_{i,1} + qS_1\mu_1 p_{i+1,0} \qquad S_1 \le i < \infty \tag{2.3}$$

For $1 \le j \le S_2 - 1$:

$$p_{0,j}(\lambda + j\mu_2) = (j+1)\mu_2 p_{0,j+1} + q\mu_1 p_{1,j} + p\mu_1 p_{1,j-1} \tag{2.4}$$

$$p_{i,j}(\lambda + j\mu_2 + i\mu_1) = \lambda p_{i-1,j} + (j+1)\mu_2 p_{i,j+1} + q(i+1)\mu_1 p_{i+1,j} + p(i+1)\mu_1 p_{i+1,j-1} \qquad 1 \le i \le S_1 - 1 \tag{2.5}$$

$$p_{i,j}(\lambda + j\mu_2 + S_1\mu_1) = \lambda p_{i-1,j} + (j+1)\mu_2 p_{i,j+1} + qS_1\mu_1 p_{i+1,j} + pS_1\mu_1 p_{i+1,j-1} \qquad S_1 \le i < \infty \tag{2.6}$$

For $S_2 \le j \le M - 1$:

$$p_{0,j}(\lambda + S_2\mu_2) = S_2\mu_2 p_{0,j+1} + q\mu_1 p_{1,j} + p\mu_1 p_{1,j-1} \tag{2.7}$$

$$p_{i,j}(\lambda + S_2\mu_2 + i\mu_1) = \lambda p_{i-1,j} + S_2\mu_2 p_{i,j+1} + q(i+1)\mu_1 p_{i+1,j} + p(i+1)\mu_1 p_{i+1,j-1} \qquad 1 \le i \le S_1 - 1 \tag{2.8}$$

$$p_{i,j}(\lambda + S_2\mu_2 + S_1\mu_1) = \lambda p_{i-1,j} + S_2\mu_2 p_{i,j+1} + qS_1\mu_1 p_{i+1,j} + pS_1\mu_1 p_{i+1,j-1} \qquad S_1 \le i < \infty \tag{2.9}$$

For $j = M$:

$$p_{0,M}(\lambda + S_2\mu_2) = \mu_1 p_{1,M} + p\mu_1 p_{1,M-1} \tag{2.10}$$

$$p_{i,M}(\lambda + S_2\mu_2 + i\mu_1) = \lambda p_{i-1,M} + (i+1)\mu_1 p_{i+1,M} + p(i+1)\mu_1 p_{i+1,M-1} \qquad 1 \le i \le S_1 - 1 \tag{2.11}$$

$$p_{i,M}(\lambda + S_2\mu_2 + S_1\mu_1) = \lambda p_{i-1,M} + S_1\mu_1 p_{i+1,M} + pS_1\mu_1 p_{i+1,M-1} \qquad S_1 \le i < \infty \tag{2.12}$$

3. GENERATING FUNCTIONS

A closed-form solution for equations (2.1) - (2.12) is not known. It turns out that the solution of the above equations depends on knowledge of the values of the "boundary" probabilities $\{p_{i,j}\}$ $(i = 0, \ldots, S_1 - 1;\ j = 0, \ldots, M)$.

In order to be able to calculate these probabilities, we define the following partial generating functions: $\pi_j(z) = \sum_{i=0}^{\infty} p_{i,j} z^i$ for $0 \leq j \leq M$ and $|z| \leq 1$. $\pi_j(z)$ is the marginal generating function of the number of customers at stage 1, when there are j customers at stage 2 $(0 \leq j \leq M)$, so that $\pi_j(1) = \sum_{i=0}^{\infty} p_{i,j} \equiv p_{\cdot j}$ is the marginal probability of j customers at the second stage.

Consider first $j = 0$. For each i in equations (2.1), (2.2) and (2.3), multiply the corresponding equation by z^i, and sum over all i. This results, after some algebraic manipulations, in

$$\pi_0(z)\left[\lambda(1-z) + S_1\mu_1\left(1 - \frac{q}{z}\right)\right] - \mu_2\pi_1(z) = b_0(z) \tag{3.1}$$

where

$$b_0(z) = \left(\frac{q}{z} - 1\right)\mu_1 \sum_{i=1}^{S_1-1} i p_{i,0} z^i + \left(1 - \frac{q}{z}\right) S_1\mu_1 \sum_{i=0}^{S_1-1} p_{i,0} z^i \tag{3.2}$$

Similarly, for $j = 1, 2, \ldots, S_2 - 1$, we obtain

$$\frac{-pS_1\mu_1}{z}\pi_{j-1}(z) + \pi_j(z)\left[\lambda(1-z) + j\mu_2 + S_1\mu_1\left(1 - \frac{q}{z}\right)\right] - (j+1)\mu_2\pi_{j+1}(z) = b_j(z) \tag{3.3}$$

where

$$b_j(z) = \left(\frac{q}{z} - 1\right)\mu_1 \sum_{i=1}^{S_1-1} i p_{i,j} z^i + \left(1 - \frac{q}{z}\right) S_1\mu_1 \sum_{i=0}^{S_1-1} p_{i,j} z^i$$
$$+ \frac{p\mu_1}{z} \sum_{i=1}^{S_1-1} i p_{i,j-1} z^i - \frac{pS_1\mu_1}{z} \sum_{i=0}^{S_1-1} p_{i,j-1} z^i \tag{3.4}$$

In the same manner, for $j = S_2, \ldots, M - 1$, we derive

$$\frac{-pS_1\mu_1}{z}\pi_{j-1}(z) + \pi_j(z)\left[\lambda(1-z) + S_2\mu_2 + S_1\mu_1\left(1 - \frac{q}{z}\right)\right] - S_2\mu_2\pi_{j+1}(z) = b_j(z) \tag{3.5}$$

where

$$b_j(z) = \left(\frac{q}{z} - 1\right) \mu_1 \sum_{i=1}^{S_1-1} i p_{i,j} z^i + \left(1 - \frac{q}{z}\right) S_1 \mu_1 \sum_{i=0}^{S_1-1} p_{i,j} z^i$$

$$+ \frac{p\mu_1}{z} \sum_{i=1}^{S_1-1} i p_{i,j-1} z^i - \frac{pS_1\mu_1}{z} \sum_{i=0}^{S_1-1} p_{i,j-1} z^i \tag{3.6}$$

Finally, for $j = M$, the calculations lead to

$$\frac{-pS_1\mu_1}{z} \pi_{M-1}(z) + \pi_M(z) \left[\lambda(1-z) + S_2\mu_2 + S_1\mu_1(1 - \frac{1}{z})\right] = b_M(z) \tag{3.7}$$

where

$$b_M(z) = \left(\frac{1}{z} - 1\right) \mu_1 \sum_{i=1}^{S_1-1} i p_{i,M} z^i + \left(1 - \frac{1}{z}\right) S_1 \mu_1 \sum_{i=0}^{S_1-1} p_{i,M} z^i$$

$$+ \frac{p\mu_1}{z} \sum_{i=1}^{S_1-1} i p_{i,M-1} z^i - \frac{pS_1\mu_1}{z} \sum_{i=0}^{S_1-1} p_{i,M-1} z^i \tag{3.8}$$

The collection (3.1) - (3.8) comprises a set of $M+1$ equations in the $M+1$ unknown generating functions $\pi_j(z), 0 \le j \le M$, where the $S_1(M+1)$ "boundary" probabilities $\{p_{i,j}\}$ $(i = 0, \ldots, S_1 - 1; j = 0, \ldots, M)$ appearing in the expressions for the $b_j(z)$ are yet to be determined. Hence, if expressions for these $S_1(M+1)$ probabilities can be obtained, one is left with a linear set of $M+1$ equations in the $M+1$ unknown generating functions $\pi_j(z), \quad 0 \le j \le M$. Once the set $\{\pi_j(z)\}_{j=0}^{M}$ is obtained, the entire two-dimensional probability distribution $\{p_{i,j}\}$ $(0 \le i < \infty; 0 \le j \le M)$ can be calculated.

As a result of the above considerations, we first focus our attention on finding the values of the "boundary" probabilities $\{p_{i,j}\}$ $(0 \le i \le S_1 - 1; 0 \le j \le M)$.

For this purpose we make use of the balance equations (2.1), (2.4), (2.7) and (2.10), where $i = 0; 0 \le j \le M$, and equations (2.2), (2.5), (2.8) and (2.11), where $1 \le i \le S_1 - 2;$ $0 \le j \le M$. These equations comprise a set of $(S_1-1)(M+1)$ linear equations in $S_1(M+1)$ variables. We therefore need additional $M+1$ equations in the above unknown probabilities $\{p_{i,j}\}$.

To this end we exploit the information concentrated within the generating functions. We can rewrite equations (3.1), (3.3), (3.5) and (3.7) in a matrix form as $A(z) \cdot \underline{\Pi}(z) = \underline{b}(z)$, where $\underline{\Pi}(z)$ is a vector of the $M+1$ generating functions $\pi_j(z)$, $0 \le j \le M$; $A(z)$ is

Figure 1: The Coefficient Matrix $A(z)$ in the Equation $A(z) \cdot \underline{\Pi}(z) = \underline{b}(z)$

an $M+1$-dimensional square matrix, and $\underline{b}(z)$ is an $M+1$-dimensional vector whose components are defined by equations (3.2), (3.4), (3.6) and (3.8). Figure 1 represents specifically the coefficients of the matrix $A(z)$ in the equation $A(z)\cdot\underline{\Pi}(z)=\underline{b}(z)$, where $q(z)\equiv\lambda(1-z)+S_1\mu_1(1-\frac{q}{z})$. Note that the columns of $A(z)$ are numbered from 0 to M.

By Cramer's rule, for each value of z such that $A(z)$ is non-singular,

$$|A(z)|\cdot\pi_j(z)=|A_j(z)|,\quad 0\leq j\leq M \tag{3.9}$$

where $A_j(z)$ is the matrix obtained from $A(z)$ by replacing column j by the right-hand-side vector $\underline{b}(z)$, and $|A|$ is the determinant of A.

Since for every $|z|\leq 1$ the system $A(z)\cdot\underline{\Pi}(z)=\underline{b}(z)$ always possesses a solution, it follows that whenever $A(z)$ is singular, so is $A_j(z)$, and (3.9) holds in this case as well. That is, for any root z_k of the equation $|A(z)|=0$, and for each $0\leq j\leq M$, we can write $|A_j(z_k)|=0$, which is a linear equation in the unknowns $\{p_{i,j}\}$ $(0\leq i\leq S_1-1;$ $0\leq j\leq M)$, appearing in the vector $\underline{b}(z)$.

It will be shown in Section 5 that the equation $|A(z)|=0$ has $M+1$ *distinct* roots in the interval $(0,1]$. M roots $\{z_k\}$ will be utilized in Section 4 to obtain M linear equations of the form $|A_M(z_k)|=0$, and the remaining root will result in a redundant equation. The last equation will be derived separately, and will complete the set of $S_1(M+1)$ equations in the desired "boundary" probabilities $\{p_{i,j}\}$.

4. SOLUTION OF THE SYSTEM

In this section we solve for the $S_1(M+1)$ unknowns $\{p_{i,j}\}$ $(0\leq i\leq S_1-1;$ $0\leq j\leq M)$, which are required for the complete knowledge of the generating functions $\{\pi_j(z)\}$, $0\leq j\leq M$. Clearly, by knowing the generating functions, one can derive the entire set of the two-dimensional stationary probabilities and various other parameters of the system.

In Section 2 we derived a set of (S_1-1) $(M+1)$ independent equations in the above $S_1(M+1)$ "boundary" probabilities. In order to obtain additional $M+1$ equations, we will investigate in the next section the equation in z, $|A(z)|=0$, and will show that the polynomial $|A(z)|$ has $M+1$ distinct roots in $(0,1]$, denoted $\big(z_1(M+1), z_2(M+1),\ldots,z_M(M+1),$

$z_{M+1}(M+1)=1\big)$, such that each root $z_k \equiv z_k(M+1)$ results (see (3.9)) in an equation $|A_M(z_\kappa)| = 0$, $1 \le k \le M+1$, in the unknowns $\{p_{i,j}\}$ $(0 \le i \le S_1 - 1;\ 0 \le j \le M)$.

Note, however, that the equation obtained from the root $z_{M+1}(M+1) = 1$ is redundant. This can easily be seen by examining the matrix $A_M(z)$, obtained from $A(z)$ by replacing its last column with the vector $\underline{b}(z)$. Indeed, for $z = 1$, the sum of all the terms in each column of $A(z)$ is 0, and so is the sum of the elements of $\underline{b}(1)$. Therefore, $|A_M(1)|$ is the zero polynomial, so that the equation $|A_M(z_{M+1})| = 0$ is redundant. Thus, we need an additional equation to be able to solve for the required $\{p_{i,j}\}$.

It was indicated in Section 2 that the first echelon comprises a regular $M/M/S_1$ queue whose stationary (one-dimensional) probabilities for i customers are the marginal probabilities of our two-dimensional system, i.e., $p_{i\cdot} \equiv \sum_{j=0}^{M} p_{i,j}$, $0 \le i < \infty$. It could easily be derived from equations (2.1) - (2.12) that for $i \ge 0$, $\lambda p_{i\cdot} = N_{i+1}\mu_1 p_{i+1,\cdot}$ as in the classical $M/M/S_1$ queue, where $N_i = \min(i,\ S_1)$. We therefore readily have the known result (cf. Kleinrock [12]) that

$$p_{0\cdot} = \left[\sum_{k=0}^{S_1-1} \frac{\left(\frac{\lambda}{\mu_1}\right)^k}{k!} + \frac{\left(\frac{\lambda}{\mu_1}\right)^{S_1}}{S_1!} \left(\frac{1}{1 - \frac{\lambda}{S_1\mu_1}} \right) \right]^{-1} \tag{4.1}$$

However, $p_{0\cdot} = \sum_{j=0}^{M} p_{0,j}$ so that equation (4.1), with $\{p_{0,j}\}_{j=0}^{M}$ on the left-hand side, completes the set of $S_1(M+1)$ equations in the $S_1(M+1)$ "boundary" probabilities $\{p_{i,j}\}$.

Once the "boundary" $\{p_{i,j}\}$ are determined and expressions for $\{\pi_j(z)\}_{j=0}^{M}$ are obtained, the mean total number of customers, L_1 and L_2, at the two echelons can be calculated:

$$L_1 = \sum_{i=1}^{\infty} i p_{i\cdot} = p_{0\cdot} \left[\frac{(\lambda/\mu_1)^{S_1}\, \rho_1}{S_1!(1-\rho_1)^2} \right] + \frac{\lambda}{\mu_1}$$

where $\rho_1 = \frac{\lambda}{S_1\mu_1}$ and $p_{0\cdot}$ is given by (4.1).

$$L_2 = \sum_{j=1}^{M} j p_{\cdot j} = \sum_{j=1}^{M} j \pi_j(1)\,.$$

Example The method of solution is illustrated for the case $S_1 = 2$, $S_2 = 1$, $M = 1$. In this case there are four unknown "boundary" probabilities $\{p_{i,j}\}$ $(i = 0,1;\ j = 0,1)$. Equations (2.1) and (2.10) get the form

$$\lambda p_{0,0} = \mu_2 p_{0,1} + q\mu_1 p_{1,0} \tag{4.2}$$

$$p_{0,1}(\lambda + \mu_2) = \mu_1 p_{1,1} + p\mu_1 p_{1,0} \tag{4.3}$$

Equation (4.1) is written as

$$p_{0,0} + p_{0,1} = \left[1 + \frac{\lambda}{\mu_1} + \frac{\lambda^2}{2\mu_1^2} \cdot \left(\frac{1}{1 - \frac{\lambda}{2\mu_1}}\right)\right]^{-1} \tag{4.4}$$

The fourth equation will be derived with the aid of the root z_0 $(0 < z_0 < 1)$ of $|A(z)| = 0$, where

$$A(z) = \left[\begin{array}{c:c} \lambda(1-z) + 2\mu_1\left(1 - \frac{q}{z}\right) & -\mu_2 \\ \hdashline \frac{-2p\mu_1}{z} & \lambda(1-z) + \mu_2 + 2\mu_1\left(1 - \frac{1}{z}\right) \end{array}\right] \tag{4.5}$$

Substituting $z = z_0$ in $\underline{b}(z)$ results in

$$b_0(z_0) = 2\mu_1\left(1 - \frac{q}{z_0}\right)p_{0,0} + \mu_1\left(1 - \frac{q}{z_0}\right)z_0 p_{1,0} \tag{4.6}$$

and

$$b_1(z_0) = \frac{-2p\mu_1}{z_0}p_{0,0} + 2\mu_1\left(1 - \frac{1}{z_0}\right)p_{0,1} - p\mu_1 p_{1,0} + \mu_1\left(1 - \frac{1}{z_0}\right)z_0 p_{1,1} \tag{4.7}$$

Replacing the second column of $A(z_0)$ by $\underline{b}(z_0)$ yields the matrix $A_1(z_0)$. The equation $|A_1(z_0)| = 0$ leads to the following equation in the four unknowns:

$$p_{0,0}\left[\lambda(1-z_0)\left(\frac{-2p\mu_1}{z_0}\right)\right] + p_{0,1}\left[\left(\lambda(1-z_0) + 2\mu_1\left(1 - \frac{q}{z_0}\right)\right)\left(2\mu_1\left(1 - \frac{1}{z_0}\right)\right)\right]$$
$$+ p_{1,0}\left[\lambda(1-z_0)(-p\mu_1)\right] + p_{1,1}\left[\left(\lambda(1-z_0) + 2\mu_1\left(1 - \frac{q}{z_0}\right)\right)\mu_1\left(1 - \frac{1}{z_0}\right)z_0\right] = 0 \tag{4.8}$$

Solution of the system of equations (4.2), (4.3), (4.4), and (4.8) gives the desired "boundary" probabilities for this case.

5. THE INTERLACING THEOREM

We transform the basic matrix $A(z)$ (see Figure 1) into an equivalent matrix $H(z)$ (possessing the same determinant) which is of the so-called Hessenberg form (cf. Wilkinson [18]). For this purpose we perform elementary operations on the rows of the matrix $A(z)$ as follows: we add the M^{th} row to the $M-1^{\text{st}}$ row, then we add the modified $M-1^{\text{st}}$ row to the $M-2^{\text{nd}}$ row, and continue in this manner up to the first (i.e. 0^{th}) row. The matrix thus obtained has its i^{th} row, $0 \le i \le M$, as the sum of the rows from i to M in the original matrix $A(z)$. The resulting matrix is characterized by the fact that the elements below the secondary diagonal (the one below the main diagonal) are all zero. Such a matrix is called an upper Hessenberg matrix, and is represented in Figure 2. Note that in our case all the elements of the 0^{th} row of the Hessenberg matrix obtained from $A(z)$ are the same, and equal to $p(z) \equiv \lambda(1-z) + S_1\mu_1(1-\frac{1}{z}) = q(z) - S_1\mu_1\frac{p}{z}$.

Let $B_n(z)$ denote the determinant of the square sub-matrix of $H(z)$ comprised of its first n rows and n columns (i.e. rows $i = 0, 1, \ldots, n-1$, and columns $j = 0, 1, \ldots n-1$).

Let $C_n(z)$ be the determinant of the above n-dimensional sub-matrix, with the exception that the lower right element is $p(z)$ instead of $p(z) + K_n\mu_2$, where $K_n = \min(n, S_2)$.

Now, calculating $B_n(z)$ and $C_n(z)$, $n = 1, 2, \ldots, M+1$, we write

$$B_n(z) = (p(z) + K_{n-1}\mu_2)B_{n-1}(z) + \frac{pS_1\mu_1}{z}C_{n-1}(z) \tag{5.1}$$

$$C_n(z) = p(z)B_{n-1}(z) + \frac{pS_1\mu_1}{z}C_{n-1}(z) \tag{5.2}$$

Subtracting (5.2) from (5.1), we have

$$C_n(z) = B_n(z) - K_{n-1}\mu_2 B_{n-1}(z) \tag{5.3}$$

Using (5.3) for $C_{n-1}(z)$ and substituting in (5.1) yields

$$B_n(z) = \left(p(z) + K_{n-1}\mu_2 + \frac{pS_1\mu_1}{z}\right)B_{n-1}(z) - \frac{pS_1\mu_1}{z}K_{n-2}\mu_2 B_{n-2}(z) \tag{5.4}$$

Figure 2: The Hessenberg Matrix $H(z)$

We rewrite equation (5.4) as

$$B_n(z) = f_{n-1}(z)B_{n-1}(z) - g_{n-2}(z)B_{n-2}(z) \tag{5.5}$$

where

$$f_n(z) = p(z) + K_n\mu_2 + \frac{pS_1\mu_1}{z} \tag{5.6}$$

and

$$g_n(z) = \frac{pS_1\mu_1}{z}K_n\mu_2 \tag{5.7}$$

By factoring out p(z) (see Figure 2), we write

$$B_n(z) = p(z)D_n(z), \qquad 1 \le n \le M+1 \tag{5.8}$$

where $D_n(z)$ is the determinant of the matrix determining $B_n(z)$, with the modification that all the elements of the first row equal 1. Thus, by dividing the recursion formula (5.5) by $p(z)$, a recursion formula for $D_n(z)$ is obtained:

$$D_n(z) = f_{n-1}(z)D_{n-1}(z) - g_{n-2}(z)D_{n-2}(z) \tag{5.9}$$

Our goal is to show that $|A(z)| = |H(z)| = B_{M+1}(z) = p(z)D_{M+1}(z)$ has $M+1$ *real* roots in the closed interval $(0, 1]$.

First note that $p(z)$ has two roots: $z_1 = 1$ and $z_2 = \frac{S_1\mu_1}{\lambda}$. As $z_2 > 1$ (stationary condition for the system), it follows that $p(z)$ has a single root in $(0, 1]$. Therefore, showing that $D_{M+1}(z)$ possesses M real roots in $(0, 1]$ is equivalent to showing that $B_{M+1}(z)$ has $M+1$ real roots in that interval.

Thus, we will inductively show that for every $1 \le n \le M+1$, $D_n(z)$ possesses $n-1$ real roots in $(0, 1]$. Moreover, it will be shown that all n roots are *distinct*.

$D_n(z)$ is a rational function with z^{n-1} as its denominator, and a polynomial $\widetilde{D}_n(z)$ of degree $2n-2$ as its numerator, such that $D_n(z) = \frac{\widetilde{D}_n(z)}{z^{n-1}}$.

In the sequel it will be proved that $\widetilde{D}_n(z)$ has $n-1$ of its roots in the open interval $(0, 1)$, and the remaining $n-1$ roots in $(1, \infty)$. The proof requires a few preliminary results.

Theorem 5.1. *For every n, $\widetilde{D}_n(z)$ is a polynomial with alternating signs. That is, $\widetilde{D}_n(z) = \sum_{i=0}^{2n-2}(-1)^i d_i(n) z^i$, where all $d_i(n)$ have the same sign. Furthermore, all $d_i(n)$ are non-zero, and for any two consecutive polynomials $\widetilde{D}_{n-1}(z)$ and $\widetilde{D}_n(z)$, the coefficients $d_0(n-1)$ and $d_0(n)$ have opposite signs.*

Proof: As $B_1(z) = p(z)$, we readily have $D_1(z) \equiv 1$.

Also, since $B_2(z) = p(z) f_1(z)$, then

$$
\begin{aligned}
D_2(z) = f_1(z) &= p(z) + \mu_2 + \frac{pS_1\mu_1}{z} \\
&= \frac{-\lambda z^2 + z(\lambda + S_1\mu_1 + \mu_2) - qS_1\mu_1}{z} = \frac{\widetilde{D}_2(z)}{z}
\end{aligned}
$$

Clearly, $\widetilde{D}_2(z)$ has alternating signs, no zero coefficients, and $d_0(2) = -qS_1\mu_1$ has an opposite sign to that of $d_0(1) = 1$.

We now assume that Theorem 5.1 holds true for all values up to $n-1$, and prove its validity for $n > 2$.

Considering (5.9), concentrate first on the term $f_{n-1}(z)D_{n-1}(z)$. As was done for $f_1(z)$, equation (5.6) is rewritten as

$$
f_{n-1}(z) = \frac{-\lambda z^2 + z(\lambda + S_1\mu_1 + K_{n-1}\mu_2) - qS_1\mu_1}{z} \tag{5.10}
$$

The numerator in (5.10) is a polynomial with alternating signs. The numerator of the product $f_{n-1}(z)D_{n-1}(z)$ is also a polynomial with alternating signs, since by the induction assumption so is $\widetilde{D}_{n-1}(z)$, and the product of two such polynomials results in a polynomial with the same property. It is also clear that multiplying $D_{n-1}(z)$ by $f_{n-1}(z)$ changes the sign of $d_0(n-1)$ (the zero coefficient of $\widetilde{D}_{n-1}(z)$), since the zero coefficient of the numerator of $f_{n-1}(z)$ is negative. Furthermore, the denominator of the above product is $z \cdot z^{n-2} = z^{n-1}$.

Next, consider the second term of (5.9). As $g_{n-2}(z) = \frac{pS_1\mu_1}{z} K_{n-2}\mu_2$, the numerator of $-g_{n-2}(z)D_{n-2}(z)$ is again, by the induction assumption, a polynomial with alternating signs, and the sign of the zero coefficient of its numerator is different from that of $d_0(n-2)$. Its denominator is $z \cdot z^{n-3} = z^{n-2}$.

As $d_0(n-1)$ and $d_0(n-2)$ differ in sign, so do the zero coefficients of the numerators of $f_{n-1}(z)D_{n-1}(z)$ and $-g_{n-2}(z)D_{n-2}(z)$, as both $f_{n-1}(z)$ and $-g_{n-2}(z)$ have a negative zero coefficient. Recall that the denominator of the first term is z^{n-1}, while that of the second term is z^{n-2}. Multiplying both the numerator and the denominator of $-g_{n-2}(z)D_{n-2}(z)$ by z results in an expression with a numerator that is still a polynomial with alternating signs but with no zero coefficient. The sign of the coefficient of z^1 (previously the zero element) equals the sign of the coefficient of z^1 in the numerator of the product $f_{n-1}(z)D_{n-1}(z)$. Hence, the coefficient of each power comprises the sum of two non-zero expressions having the same sign.

It follows that $\widetilde{D}_n(z)$ has alternating signs, and that $d_0(n)$, which is in fact the zero coefficient of $f_{n-1}(z)D_{n-1}(z)$, is opposite in sign to $d_0(n-1)$. Q. E. D.

Corollary. *$\widetilde{D}_n(z) = \sum_{i=0}^{2n-2}(-1)^i d_i(n)z^i$ possesses no negative roots.*

Proof: As for every n all $d_i(n)$ have the same sign, we fix n and assume $d_i(n) > 0$ for all i. If $z < 0$, then $(-1 \cdot z)^i > 0$ for every i, so that $\widetilde{D}_n(z) = \sum_{i=0}^{2n-2}(-1)^i d_i(n)z^i > 0$. Similarly, if all $d_i(n) < 0$, then $\widetilde{D}_n(z) < 0$ for $z < 0$.

As a result, it is sufficient to consider only values of z in the domain $(0, \infty)$.

Remark: $\widetilde{D}_{n-1}(z)$ and $\widetilde{D}_n(z)$ have no common roots, for if they had one, then, by the recursion formula (5.9), $\widetilde{D}_{n-2}(z)$ would have had the same root. Continuing in this manner leads to $\widetilde{D}_1(z)$ having the same root. But $\widetilde{D}_1(z) \equiv 1$ has no roots, which is a contradiction. Hence, no two consecutive polynomials have common roots.

The Interlacing Theorem.

a) *The polynomial $\widetilde{D}_n(z)$ has $n-1$ distinct roots in the open interval $(0,1)$. Between any two roots of $\widetilde{D}_n(z)$ in this interval lies exactly one root of $\widetilde{D}_{n-1}(z)$. The smallest root of $\widetilde{D}_n(z)$ in $(0,1)$ is smaller than that of $\widetilde{D}_{n-1}(z)$, and the greatest root of $\widetilde{D}_n(z)$ in that interval is greater than that of $\widetilde{D}_{n-1}(z)$.*

b) *The same properties hold in the interval $(1, \infty)$.*

Proof: Again, $\widetilde{D}_1(z) = 1$, and $\widetilde{D}_2(z) = -\lambda z^2 + z(\lambda + S_1\mu_1 + \mu_2) - qS_1\mu_1$, such that

$$\widetilde{D}_2(0) = -qS_1\mu_1 < 0, \quad \widetilde{D}_2(1) = pS_1\mu_1 + \mu_2 > 0, \quad \text{and} \quad \widetilde{D}_2(\infty) < 0.$$

It readily follows that the quadratic function $\widetilde{D}_2(z)$ has a single root in the open interval $(0,1)$, and a single root in $(1,\infty)$.

Thus, the theorem is true for $\widetilde{D}_1(z)$ and $\widetilde{D}_2(z)$.

We now assume that the theorem holds for all values up to $n-1$, and prove its validity for $n>2$.

For the inductive step, we need several propositions. Using the induction assumption, the $n-2$ distinct roots of $\widetilde{D}_{n-1}(z)$ in $(0,1)$ are denoted by

$$0 < z_1(n-1) < z_2(n-1) < \ldots < z_{n-2}(n-1) < 1.$$

Proposition 5.1. *Between any two roots of $\widetilde{D}_{n-1}(z)$ in $(0,1)$ there is a root of $\widetilde{D}_n(z)$.*

Proof: Let $z_i(n-1)$ and $z_{i+1}(n-1)$ be two consecutive roots of $\widetilde{D}_{n-1}(z)$ in the interval $(0,1)$. Then, from (5.9), for $j=i,\, i+1$,

$$D_n\big(z_j(n-1)\big) = f_{n-1}\big(z_j(n-1)\big)D_{n-1}\big(z_j(n-1)\big) - g_{n-2}\big(z_j(n-1)\big)D_{n-2}\big(z_j(n-1)\big) \tag{5.11}$$

Since $z_j(n-1)$ is a root of $\widetilde{D}_{n-1}(z)$, it is also a root of $D_{n-1}(z)$, so that $D_{n-1}\big(z_j(n-1)\big)=0$ for $j=i,\;\; i+1$.

Hence,

$$D_n\big(z_j(n-1)\big) = -g_{n-2}\big(z_j(n-1)\big)D_{n-2}\big(z_j(n-1)\big) \tag{5.12}$$

Since $g_n(z)>0$ for all $z>0$ and all n, the signs of $D_{n-2}\big(z_j(n-1)\big)$ and $D_n\big(z_j(n-1)\big)$ are opposite (both values are non-zero by the remark), and so are the signs of $\widetilde{D}_{n-2}\big(z_j(n-1)\big)$ and $\widetilde{D}_n\big(z_j(n-1)\big)$ for $j=i,\, i+1$. By the induction assumption, $\widetilde{D}_{n-2}(z)$ has a unique root between $z_i(n-1)$ and $z_{i+1}(n-1)$, and therefore changes its sign between the two roots. That is, $\widetilde{D}_{n-2}\big(z_i(n-1)\big)\cdot\widetilde{D}_{n-2}\big(z_{i+1}(n-1)\big)<0$. Therefore, by (5.12), $\widetilde{D}_n(z)$ also changes its sign between $z_i(n-1)$ and $z_{i+1}(n-1)$. Thus, $\widetilde{D}_n(z)$ has an odd number of roots (at least one) between $z_i(n-1)$ and $z_{i+1}(n-1)$.

Proposition 5.2. *Proposition 5.1 holds true, word for word, for the interval $(1,\infty)$, where the distinct $n-2$ roots are denoted by*
$1 < z_{n-1}(n-1) < z_n(n-1) < \cdots < z_{2n-4}(n-1) < \;\infty$.

Proposition 5.3. *The smallest root of $\widetilde{D}_n(z)$, denoted $z_1(n)$, lies to the left of the smallest root of $\widetilde{D}_{n-1}(z)$. That is, $z_1(n) < z_1(n-1)$.*

Proof: As before, by (5.12), the signs of $\widetilde{D}_n(z_1(n-1))$ and $\widetilde{D}_{n-2}\big(z_1(n-1)\big)$ are opposite. However, $\widetilde{D}_n(0)$ and $\widetilde{D}_{n-2}(0)$ have the same sign. This follows since $\widetilde{D}_n(0)$, $\widetilde{D}_{n-1}(0)$ and $\widetilde{D}_{n-2}(0)$, being the zero coefficients of the corresponding polynomials, satisfy (by Theorem 5.1), $\widetilde{D}_n(0)\cdot\widetilde{D}_{n-1}(0) < 0$ and $\widetilde{D}_{n-1}(0)\cdot\widetilde{D}_{n-2}(0) < 0$, so that $\widetilde{D}_n(0)\cdot\widetilde{D}_{n-2}(0) > 0$. By the induction assumption, $\widetilde{D}_{n-2}(z)$ has no root in $(0, z_1(n-1))$, and therefore does not change its sign in $(0, z_1(n-1))$. Thus, $\widetilde{D}_n(z)$ *does* change its sign in $(0, z_1(n-1))$, which implies that it has an odd number of roots (at least one) there. Therefore, $z_1(n) < z_1(n-1)$.

Proposition 5.4. *For every n, $\widetilde{D}_n(1) > 0$.*

Proof: Since $\widetilde{D}_1(1) = 1 > 0$, it is enough to show that

$$\frac{\widetilde{D}_n(1)}{\widetilde{D}_{n-1}(1)} > pS_1\mu_1 > 0 \quad (n > 1) \tag{5.13}$$

Clearly, $\frac{\widetilde{D}_2(1)}{\widetilde{D}_1(1)} = \frac{pS_1\mu_1+\mu_2}{1} > pS_1\mu_1$.

We now assume that (5.13) holds up to $n-1$, and prove its validity for $n > 2$.

Since $z = 1$ is a root of $p(z)$, and $D_n(1) = \widetilde{D}_n(1)$, we have, from (5.6), (5.7) and (5.9),

$$\frac{\widetilde{D}_n(1)}{\widetilde{D}_{n-1}(1)} = K_{n-1}\mu_2 + pS_1\mu_1 - pS_1\mu_1K_{n-2}\mu_2\frac{\widetilde{D}_{n-2}(1)}{\widetilde{D}_{n-1}(1)} \tag{5.14}$$

Using the induction assumption for $\frac{\widetilde{D}_{n-2}(1)}{\widetilde{D}_{n-1}(1)}$, we readily have

$$\begin{aligned}\frac{\widetilde{D}_n(1)}{\widetilde{D}_{n-1}(1)} &> K_{n-1}\mu_2 + pS_1\mu_1 - pS_1\mu_1\,K_{n-2}\mu_2\cdot\frac{1}{pS_1\mu_1}\\ &= pS_1\mu_1 + \mu_2(K_{n-1}-K_{n-2}) \ge pS_1\mu_1\end{aligned} \tag{5.15}$$

Proposition 5.5. *Let $\widehat{z}(n)$ denote the largest root of $\widetilde{D}_n(z)$ in $(0,1)$. Then, $\widehat{z}(n) > z_{n-2}(n-1)$, i.e., $\widehat{z}(n)$ is greater than the largest root of $\widetilde{D}_{n-1}(z)$ in that interval.*

Proof: Again, by (5.12), $\widetilde{D}_n\big(z_{n-2}(n-1)\big)$ and $\widetilde{D}_{n-2}\big(z_{n-2}(n-1)\big)$ differ in sign. On the other hand, by Proposition 5.4, $\widetilde{D}_n(1)$ and $\widetilde{D}_{n-2}(1)$ have the same (positive) sign.

However, by the induction assumption, $\widetilde{D}_{n-2}(z)$ has no roots in $\big(z_{n-2}(n-1), 1\big)$, and therefore does not change sign in that interval. Thus, $\widetilde{D}_n(z)$ must change its sign in $\big(z_{n-2}(n-1), 1\big)$, and hence has an odd number of roots there. That is, its largest root in $(0,1)$, $\widehat{z}(n)$, is greater than $z_{n-2}(n-1)$.

Proposition 5.6. *Let* $\underline{z}(n)$ *denote the smallest root of* $\widetilde{D}_n(z)$ *in* $(1,\infty)$. *Then* $\underline{z}(n) < z_{n-1}(n-1)$.

Proof: Similar to Proposition 5.5, $\widetilde{D}_{n-2}(\cdot)$ and $\widetilde{D}_n(\cdot)$ have opposite signs at $z_{n-1}(n-1)$, and the same sign at $z=1$. Since, by the induction assumption, $\widetilde{D}_{n-2}(z)$ has no roots in $\big(1, z_{n-1}(n-1)\big)$, it follows that $\widetilde{D}_n(z)$ has an odd number of roots in that interval, which implies that the smallest one, $\underline{z}(n)$, lies to the left of $z_{n-1}(n-1)$.

Proposition 5.7. *Let* $\overline{z}(n)$ *denote the largest root of* $\widetilde{D}_n(z)$ *in* $(1,\infty)$. *Then* $\overline{z}(n) > z_{2n-4}(n-1)$.

Proof: As before, $\widetilde{D}_n\big(z_{2n-4}(n-1)\big)\cdot\widetilde{D}_{n-2}\big(z_{2n-4}(n-1)\big) < 0$. We wish to show that $\widetilde{D}_n(\infty)\cdot\widetilde{D}_{n-2}(\infty) > 0$. In order to do this it is enough to show that $\widetilde{D}_n(\infty)\cdot\widetilde{D}_{n-1}(\infty) < 0$. As (by Theorem 5.1) $\widetilde{D}_n(z)$ and $\widetilde{D}_{n-1}(z)$ are (alternating signs) polynomials of even degrees, the signs of the leading power and the zero element are the same. Also, as the sign of $d_0(n)$ is opposite to that of $d_0(n-1)$, it follows that the sign of the leading power of $\widetilde{D}_n(z)$ differs from that of $\widetilde{D}_{n-1}(z)$. Clearly, the sign of the leading power determines the sign of $\widetilde{D}_n(\infty)$.

Since $\widetilde{D}_{n-2}(z)$ does not change sign in $\big(z_{2n-4}(n-1), \infty\big)$, necessarily $\widetilde{D}_n(z)$ does, and therefore has an odd number of roots there, which implies that $\overline{z}(n) > z_{2n-4}(n-1)$.

To complete the proof of the Interlacing Theorem, it remains only to show that $\widetilde{D}_n(z)$ possesses exactly $n-1$ roots in $(0,1)$ and $n-1$ roots in $(1,\infty)$.

By the induction hypothesis, $\widetilde{D}_{n-1}(z)$ has exactly $n-2$ roots in $(0,1)$. We have shown (Proposition 5.1) that between any two consecutive roots of $\widetilde{D}_{n-1}(z)$ there is a set of roots (at least one) of $\widetilde{D}_n(z)$, so that there are at least $n-3$ such roots. In addition, there is a non-empty set of roots of $\widetilde{D}_n(z)$ in each of the intervals $\big(0, z_1(n-1)\big)$ and $\big(z_{n-2}(n-1), 1\big)$. Hence, $\widetilde{D}_n(z)$ has at least $n-1$ roots in $(0,1)$. The same situation occurs in $(1,\infty)$, so that $\widetilde{D}_n(z)$ has at least $2n-2$ roots in $(0,\infty)$. But the degree of the polynomial $\widetilde{D}_n(z)$ is $2n-2$, which implies that each of the above sets consists of exactly one root of $\widetilde{D}_n(z)$. Thus, $\widetilde{D}_n(z)$ has exactly $n-1$ roots in $(0,1)$ and the same number of roots in $(1,\infty)$, with all roots being *distinct*. Q.E.D.

To summarize, it has been shown that the determinant of the Hessenberg matrix $|H(z)| = B_{M+1}(z)$ has $M+1$ real roots in $(0,1]$, since $D_{M+1}(z)$ possesses M real roots in $(0,1)$ and $p(z)$ has $z=1$ as its only root in $(0,1]$.

In the Appendix we present a few numerical results regarding the calculation of the roots of $B_{M+1}(z)$ for various values of $(\lambda, \mu_1,\ \mu_2,\ S_1,\ S_2,\ p)$. Observe that $B_n(z)$, $1 \leq n \leq M+1$, can be considered as a homogenous polynomial in λ, μ_2 and $S_1\mu_1$, as is easily seen from the recursion formula (5.4) and the expressions for $B_1(z)$ and $B_2(z)$. Thus, multiplying λ, μ_2 and $S_1\mu_1$ by the *same* factor, leaving the other parameters unchanged, results in an equation having the same roots.

For example, the roots of the equation $|H(z)| = 0$ with parameters $(1,2,2,2,1,1/2)$, as appearing in the first row of Table 1 in the Appendix, are equal to the roots of that equation with parameters $(1.5,3,3,2,1,1/2)$, which appear in the tenth row of Table 1.

ACKNOWLEDGEMENT

We wish to thank Prof. N. Dyn for suggesting the interlacing approach, and for very valuable advice.

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APPENDIX

For various values of the parameters $(\lambda, \mu_1, \mu_2, S_1, S_2, p)$, we present a few numerical examples of the actual values of the roots $\{z_k\}$, obtained from the solution of the equation $|H(z)| = 0$.

As was proved in Section 5, this equation has $2M+2$ roots for every M. $M+1$ roots are in $(0,1]$ and the other $M+1$ are in $(1,\infty)$. As was shown, for every M, $z=1$ is a root, and $z = \frac{S_1\mu_1}{\lambda}$ is another root in $(1,\infty)$. (It is also observed that the root $z = \frac{S_1\mu_1}{\lambda}$ is always the next one after $z=1$.)

Table 1: Values of the Roots of $|H(z)|$ for $M=2$.

λ	μ_1	μ_2	S_1	S_2	p	Values of roots
1	2	2	2	1	1/2	(0.17, 0.53, 1, 4, 5.82, 7.46)
1	2	2	2	1	2/3	(0.09, 0.43, 1, 4, 5.81, 7.66)
1	2	3	2	1	2/3	(0.07, 0.41, 1, 4, 6.70, 8.80)
1	2	3	2	1	1/2	(0.13, 0.49, 1, 4, 6.76, 8.12)
1	3	3	2	1	1/2	(0.17, 0.55, 1, 6, 8.63, 10.68)
1	3	3	2	1	2/3	(0.09, 0.45, 1, 6, 8.58, 10.86)
2	3	3	2	1	2/3	(0.08, 0.41, 1, 3, 4.45, 6.03)
2	3	3	2	1	3/4	(0.05, 0.36, 1, 3, 4.46, 6.12)
1.5	3	3	2	1	2/3	(0.05, 0.37, 1, 4, 5.81, 7.75)
1.5	3	3	2	1	1/2	(0.17, 0.53, 1, 4, 5.82, 7.46)
1.5	3	2	2	1	2/3	(0.11, 0.43, 1, 4, 5.25, 6.85)
1.5	3	2	2	1	1/2	(0.20, 0.55, 1, 4, 5.23, 6.66)

Table 2: Values of the Roots of $|H(z)|$ for $M=3$.

λ	μ_1	μ_2	S_1	S_2	p	Values of roots
2	2	2	2	1	1/2	(0.12, 0.26, 0.62, 1, 2, 2.79, 3.73, 4.44)
2	2	2	2	1	3/4	(0.03, 0.12, 0.48, 1, 2, 2.78, 3.87, 4.69)
3	2	2	2	1	3/4	(0.03, 0.11, 0.41, 1, 1.33, 2.02, 2.88, 3.52)
3	2	2	2	1	1/2	(0.11, 0.24, 0.56, 1, 1.33, 2, 2.75, 3.31)
3	4	3	2	1	1/2	(0.15, 0.30, 0.66, 1, 2.66, 3.38, 4.36, 5.12)
3	4	3	2	1	2/3	(0.07, 0.19, 0.56, 1, 2.66, 3.37, 4.46, 5.31)
4	4	3	2	1	2/3	(0.07, 0.18, 0.52, 1, 2, 2.62, 3.56, 4.27)
4	4	3	2	1	1/2	(0.14, 0.28, 0.63, 1, 2, 2.60, 3.46, 4.11)
5	4	5	2	1	1/2	(0.10, 0.23, 0.58, 1, 1.6, 2.47, 3.36, 4.03)
5	4	5	2	1	2/3	(0.05, 0.15, 0.51, 1, 1.6, 2.44, 3.44, 4.18)
5	3	4	2	1	2/3	(0.04, 0.13, 0.45, 1, 1.2, 2, 2.86, 3.49)
5	3	4	2	1	3/4	(0.03, 0.10, 0.41, 1, 1.2, 2.01, 2.89, 3.55)