Name and ID of the student:

20.07.2016, moed B

Tel-Aviv University

Engineering Faculty

Final exam on "Partial Differential Equations"

Lecturer: Prof. Yakov Yakubov

Prescriptions:

1. The duration of the exam is 3 hours.
2. The use of any material is forbidden except the plane calculator and two personal lists (4 pages) of formulas prepared by the student. The size of the lists is the standard A4 format.
3. Do not use any methods which have not been studied in the classes.

The structure of the final exam:

1. There are 5 questions in the exam. You should answer to **only** 4 questions.
2. The grade of each question is 25 points.
3. Indicate on the first page of the exam which questions should be checked.
4. In the case you solve all 5 questions and you do not indicate which questions should be checked, first 4 questions will be checked.

Good luck!

Question 1 **(a) (15 points)** Solve (by separation of variables method) the following wave (string) problem

$$\begin{aligned}
&u_{tt} - u_{xx} = 0, && 0 < x < 1, \quad 0 < t, \\
&u(x,0) = 2\sin^2(\pi x),\ \ u_t(x,0) = 4\cos^2(\pi x), && 0 \le x \le 1, \\
&u_x(0,t) = u_x(1,t) = 0, && 0 \le t.
\end{aligned}$$

(b) (10 points) Using the energy integral $E(t) = \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)dx$, prove the uniqueness of the solution.

Question 2 **(a) (18 points)** Given the ring $D = \{(r,\theta) \ : \ 1 < r < 2, 0 \le \theta \le 2\pi\}$.

Find the solution of the following boundary value problem

$$\begin{aligned}
&\Delta u = 0, && (r,\theta) \in D, \\
&\frac{\partial u(1,\theta)}{\partial r} = 0,\ \ u(2,\theta) = \sin^2\theta, && 0 \le \theta \le 2\pi.
\end{aligned}$$

Remark. One can keep the solution in the polar coordinates.

(b) (7 points) Calculate the maximum of the solution $u(r,\theta)$ from (a) in the closed ring, i.e., in the ring including its boundary.

Question 3 **(a) (13 points)** Classify the equation

$u_{xx} + 4u_{xy} + 13u_{yy} + 3u_x + 24u_y - 9u + 9(x+y) = 0$ and bring it to the canonical form.

(b) (12 points) Solve the Cauchy problem for the heat equation:

$$\begin{aligned}
&u_t = a^2 u_{xx} \quad , \quad t > 0 \quad , \quad -\infty < x < \infty \\
&u(x,0) = f(x) = x^2 \quad , \quad -\infty < x < \infty.
\end{aligned}$$

Question 4 **(a) (15 points)** Solve the Laplace problem in the rectangle

$$\begin{aligned}
&\Delta u = 0, && 0 < x < 2, \quad 0 < y < 3, \\
&u(x,0) = 2\sin(6\pi x), \quad u(x,3) = 0, && 0 \le x \le 2, \\
&u(0,y) = u(2,y) = 0, && 0 \le y \le 3.
\end{aligned}$$

(b) (10 points) Given a bounded domain $\Omega \subset R^2$ and $\gamma \le 0$. Using one of the Green's formulas, prove the uniqueness of the solution of the problem

$$\begin{aligned}
&\Delta u(x,y) + \gamma u(x,y) = F(x,y), && (x,y) \in \Omega, \\
&u(x,y) = f(x,y), && (x,y) \in \partial\Omega,
\end{aligned}$$

where F and f are given continuous functions and $\partial\Omega$ denotes the boundary of the domain.

Question 5 **(a) (18 points)** Solve the following non-homogeneous heat problem

$$\begin{aligned}
&u_t - u_{xx} = -e^{-t}\sin(\pi x), && 0 < x < 1, \quad t > 0, \\
&u(0,t) = 1, \quad u(1,t) = 0, && t \ge 0, \\
&u(x,0) = 1 - x + \sin(2\pi x), && 0 \le x \le 1.
\end{aligned}$$

(b) (7 points) Show that the solution $u(x,t) \le 2$ in the domain $t \ge 0$ $, 0 \le x \le 1$.