ID of the student:

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Tel-Aviv University

Engineering Faculty

Final exam on "Partial Differential Equations"

Lecturer: Prof. Yakov Yakubov

<u>Prescriptions:</u>

1. The duration of the exam is 3 hours.
2. The use of any material is forbidden except the plane calculator and two personal lists (4 pages) of formulas prepared by the student. The size of the lists is the standard A4 format.
3. Do not use any methods which have not been studied in the classes.

<u>The structure of the final exam:</u>

1. There are 5 questions in the exam. You should answer to **<u>only</u>** 4 questions.
2. The grade of each question is 25 points.
3. Indicate on the first page of the exam which questions should be checked.
4. In the case you solve all 5 questions and you do not indicate which questions should be checked, first 4 questions will be checked.

Good luck!

<u>Question 1</u> **(a)** **(16 points)** Solve the following wave problem

$$
\begin{aligned}
&u_{tt} = 4u_{xx}\ , \quad x>0, t>0\\
&u(x,0) = x^2, \ \ x \ge 0\\
&u_t(x,0) = \sin x, \ \ x \ge 0\\
&u(0,t) = 0, \ \ t \ge 0
\end{aligned}
$$

and calculate $u(\pi, \frac{3\pi}{2})$.

(b) (9 points) Prove that a solution of the following problem is unique in $x^2 + y^2 \le 2$

$$
\begin{aligned}
&2\Delta u - u = \sin x, \quad x^2 + y^2 < 2,\\
&\frac{\partial u}{\partial n} = y, \quad x^2 + y^2 = 2
\end{aligned}
$$

<u>Question 2</u> **(a) (15 points)** Classify the equation $9u_{xx} - 6u_{xy} + u_{yy} = x^2 y$, bring it to the canonical form and find its general solution.

(b) (10 points) Assume that $u \in C^2(D) \cap C(\overline{D})$ is a solution of the problem

$$
\begin{aligned}
&\Delta u = -(x^2 + y^2), \quad (x,y) \in D,\\
&u(x,y) = 0, \quad (x,y) \in \partial D,
\end{aligned}
$$

in the square $D = (-1,1) \times (-1,1)$ (here $\overline{D} = [-1,1] \times [-1,1]$). Prove that

$\frac{1}{12} \le u(0,0) \le \frac{1}{6}$.

Hint: one can use the auxiliary function $v(x,y) = u(x,y) + \frac{1}{12}(x^4 + y^4)$.

<u>Question 3</u> **(a) (15 points)** Solve the Cauchy problem for the heat equation:

$$
\begin{aligned}
&u_t = a^2 u_{xx} \quad , \quad t>0 \quad , \quad -\infty < x < \infty\\
&u(x,0) = f(x) = e^x \quad , \quad -\infty < x < \infty.
\end{aligned}
$$

(b) (10 points) Given the Cauchy problem for the wave equation

$$
\begin{aligned}
&u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty, \ t>0,\\
&u(x,0) = f(x), \ \ u_t(x,0) = g(x), \ \ -\infty < x < \infty
\end{aligned}
$$

where $f(x), g(x)$ are even functions. Prove that also the solution $u(x,t)$ of the problem is even with respect to x.

Question 4 **(a) (19 points)** Given a constant $\alpha < 1$. Find the solution to the following problem in polar coordinates

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\alpha}{r^2}u = 0, \quad 0 < \theta < \pi, 0 < r < 1, \\ u(r,0) = u(r,\pi) = 0, \quad 0 \le r \le 1, \\ u_r(1,\theta) = \theta^2 - \pi\theta, \quad 0 \le \theta \le \pi \end{cases}$$

(b) (6 points) Given a harmonic function $u(x, y)$ in the disk with radius 1 centered at the origin. The value of the function on the boundary of the disk is equal to $x^2 + b$. Given that $u(0,0) = 0$. Find b.

Question 5 **(25 points)** Solve the following non-homogeneous heat problem

$$\begin{cases} u_t - u_{xx} = 3t^2 + e^{-\frac{25}{4}t}\sin(\frac{5}{2}x), & 0 < x < \pi, t > 0, \\ u(x,0) = x + \sin(\frac{x}{2}), & 0 \le x \le \pi, \\ u(0,t) = t^3, \ \ u_x(\pi,t) = 1, & t \ge 0. \end{cases}$$

Hint: look for the "correction function" in the form $w(x,t) = a(t) + b(t)x$.