ID of the student:

03.08.2017, moed B

Tel-Aviv University

Engineering Faculty

Final exam on "Partial Differential Equations"

Lecturer: Prof. Yakov Yakubov

Prescriptions:

1. The duration of the exam is 3 hours.
2. The use of any material is forbidden except the plane calculator and two personal lists (4 pages) of formulas prepared by the student. The size of the lists is the standard A4 format.
3. Do not use any methods which have not been studied in the classes.

The structure of the final exam:

1. There are 5 questions in the exam. You should answer to **only** 4 questions.
2. The grade of each question is 25 points.
3. Indicate on the first page of the exam which questions should be checked.
4. In the case you solve all 5 questions and you do not indicate which questions should be checked, first 4 questions will be checked.

Good luck!

Question 1 **(a) (13 points)** Solve the following wave (string) problem

$$
\begin{aligned}
&u_{tt} = 4u_{xx}\ , \quad x > 0, t > 0 \\
&u(x,0) = x^2 \sin x, \ \ x \ge 0 \\
&u_t(x,0) = x^3, \ \ x \ge 0 \\
&u(0,t) = 0, \ \ t \ge 0
\end{aligned}
$$

(b) (12 points) Solve the Cauchy problem for the heat equation:

$$
\begin{cases}
u_t = a^2 u_{xx}, & -\infty < x < \infty, \ \ t > 0, \\
u(x,0) = f(x) = \begin{cases} x, & -1 \le x \le 0 \\ 0, & else \end{cases}
\end{cases}
$$

Question 2 **(a) (15 points)** Classify the equation

$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0, \ \ x > 0$, bring it to the canonical form and find its general solution.

(b) (10 points) Using the energy integral method, prove uniqueness of the solution of the problem:

$$
\begin{cases}
u_t - 9u_{xx} = \sin(e^{xt}), & 0 < x < \pi/4, t > 0, \\
u(0,t) = \sin t, \ \ u(\pi/4, t) = \cos t, & t \ge 0, \\
u(x,0) = \tan x, & 0 \le x \le \pi/4
\end{cases}
$$

Hint: choose the energy $E(t) = \frac{1}{2} \int_0^{\pi/4} w^2(x,t)dx$.

Question 3 **(a) (10 points)** Assume that $u(x,t)$ is a solution of the problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = x - \arctan x, & x \ge 0, \\ u_t(x,0) = x, & x \ge 0, \\ u_x(0,t) = 0, & t \ge 0 \end{cases}$$

Find $u(1,2)$.

(b) (15 points) Find the maximum of the solution $u(x,t)$ of the following problem in the domain $\frac{1}{4} \le x \le \frac{1}{2}, t \ge 0$:

$$\begin{cases} u_t - u_{xx} = -e^{-t}\sin(\pi x), & \frac{1}{4} < x < \frac{1}{2}, t > 0, \\ u(\frac{1}{4},t) = u(\frac{1}{2},t) = 0, & t \ge 0, \\ u(x,0) = \sin(2\pi x), & \frac{1}{4} \le x \le \frac{1}{2} \end{cases}$$

Can this maximum be attained by $u(x,t)$ at some inside point (x_0,t_0), i.e., $\frac{1}{4} < x_0 < \frac{1}{2}, t_0 > 0$?

Question 4 **(a) (20 points)** Solve the Laplace problem in the rectangle

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \frac{\pi}{2}, \ 0 < y < \frac{\pi}{2} \\ u(0,y) = 0, \quad u(\frac{\pi}{2},y) = 0, & 0 \le y \le \frac{\pi}{2} \\ u_y(x,0) = x, \quad u_y(x,\frac{\pi}{2}) = \sin(6x), & 0 \le x \le \frac{\pi}{2} \end{cases}$$

(b) (5 points) What does it say the min/max principle for the solution of the above problem?

Question 5 **(25 points)** Solve the following non-homogeneous wave problem

$$\begin{aligned} &u_{tt} = u_{xx} + \pi\cos(\pi x), \quad 0 < x < 1, \ t > 0, \\ &u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \ge 0, \\ &u(x,0) = 1 - \cos(\pi x), \quad u_t(x,0) = 2 + \cos(2\pi x), \quad 0 \le x \le 1. \end{aligned}$$