ID of the student:

20.06.2018, moed A

Tel-Aviv University

Engineering Faculty

Final exam on "Partial Differential Equations"

Lecturer: Prof. Yakov Yakubov

Prescriptions:

1. The duration of the exam is 3 hours.
2. The use of any material is forbidden except the plane calculator and two personal lists (4 pages) of formulas prepared by the student. The size of the lists is the standard A4 format.
3. Do not use any methods which have not been studied in the classes.

The structure of the final exam:

1. There are 5 questions in the exam. You should answer to **only** 4 questions.
2. The grade of each question is 25 points.
3. Indicate on the first page of the exam which questions should be checked.
4. In the case you solve all 5 questions and you do not indicate which questions should be checked, first 4 questions will be checked.

Good luck!

Question 1 **(a) (13 points)** Solve the following wave problem

$$
\begin{aligned}
&u_{tt} = 9u_{xx}\ , \quad x>0, t>0\\
&u(x,0) = x - x^2,\ \ x \ge 0\\
&u_t(x,0) = \cos x,\ \ x \ge 0\\
&u_x(0,t) = 0,\ \ t \ge 0
\end{aligned}
$$

(b) (12 points) Solve the following Dirichlet problem for the Laplace equation in polar coordinates

$$
\begin{aligned}
&\Delta u = 0, \quad 0 \le r < 1,\ \ -\pi < \theta \le \pi,\\
&u(1,\theta) = f(\theta) = \begin{cases} -1, & -\pi < \theta \le 0,\\ 3, & 0 < \theta \le \pi \end{cases}
\end{aligned}
$$

Question 2 **(a) (15 points)** Classify the equation $9u_{xx} - 6u_{xy} + u_{yy} = x^2 y$, bring it to the canonical form and find its general solution.

(b) (10 points) Assume that $u \in C^2(D) \cap C(\overline{D})$ is a solution of the problem

$$
\begin{aligned}
&\Delta u = -(x^2 + y^2), \quad (x,y) \in D,\\
&u(x,y) = 0, \quad (x,y) \in \partial D,
\end{aligned}
$$

in the square $D = (-1,1)\times(-1,1)$ (here $\overline{D} = [-1,1]\times[-1,1]$). Prove that

$\dfrac{1}{12} \le u(0,0) \le \dfrac{1}{6}$.

Hint: one can use the auxiliary function $v(x,y) = u(x,y) + \dfrac{1}{12}(x^4 + y^4)$.

Question 3 **(a) (15 points)** Solve the Cauchy problem ($-\infty < x < \infty, t > 0$)

$$
\begin{cases}
u_t = a^2 u_{xx},\\
u(x,0) = \begin{cases} T_1, & x < x_1,\\ T_2, & x_1 \le x \le x_2,\\ T_3, & x > x_2 \end{cases}
\end{cases}
$$

where T_1, T_2, and T_3 are some constants.

(b) (10 points) Using the energy integral method prove uniqueness of the solution of the Dirichlet problem for the following heat equation (k is a positive constant)

$$\begin{cases} u_t - ku_{xx} = F(x,t), & 0 < x < L, t > 0, \\ u(0,t) = a(t), u(L,t) = b(t), & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq L \end{cases}$$

Hint: Show that the energy $E(t) = \frac{1}{2}\int_0^L w^2(x,t)dx$ satisfies $E'(t) \leq 0$.

<u>Question 4</u> **(a) (17 points)** Solve the following problem

$$\begin{cases} \Delta u = 0, \ \ 1 < x^2 + y^2 < 4, \\ \dfrac{\partial u}{\partial n} = x + y, \ \ x^2 + y^2 = 1, \\ u = 0, \quad x^2 + y^2 = 4. \end{cases}$$

(b) (8 points) Prove that the solution of the above problem is unique.

<u>Question 5</u> **(25 points)** Solve the following non-homogeneous heat problem

$$\begin{cases} u_t - a^2 u_{xx} = 0 & 0 < x < 1, t > 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \\ u(0,t) = 0, \ \ u_x(1,t) = 2t, & t \geq 0. \end{cases}$$