Multi-valued Calculi for Logics Based on Non-determinism

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Abstract

Non-deterministic matrices (Nmatrices) are multiple-valued structures in which the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. We consider two different types of semantics which are based on Nmatrices: the dynamic one and the static one (the latter is new here). We use the Rasiowa-Sikorski (R-S) decomposition methodology to get sound and complete proof systems employing finite sets of mv-signed formulas for all propositional logics based on such structures with either of the above types of semantics. Later we demonstrate how these systems can be converted into cut-free ordinary Gentzen calculi which are also sound and complete for the corresponding non-deterministic semantics. As a by-product, we get new semantic characterizations for some well-known logics (like the logic **CAR** from [18, 28]).

Keywords: nondeterministic matrices, deduction systems, tableaux systems, R-S systems, n-sequents

1 Introduction

A non-deterministic matrix (Nmatrices) for a propositional language $\mathcal{L}$ is a multiple-valued structure where the operations corresponding to the logical connectives of $\mathcal{L}$ may provide more than one option for the truth-value to be assigned to a complex formula ψ, given the truth-values assigned to the immediate subformulas of ψ. Nmatrices were introduced in ([2, 3, 4]), and in [4] it was shown that logics which have a finite characteristic Nmatrix have the main good properties enjoyed by logics which have an ordinary (deterministic) finite characteristic matrix. In particular: they are decidable and finitary (i.e.: they satisfy the compactness theorem). Many concrete applications of finite Nmatrices were then given in [4, 5, 6, 7], where it was also shown that for these applications the use of Nmatrices is indeed *necessary*: no finite ordinary matrix can be used instead.

However, all the papers cited above concentrated on the problem of providing non-deterministic semantics for various proof systems. In other words: their aim was to find characteristic Nmatrices for logics which had been originally introduced through some proof system (either a Hilbert-type type one or a Gentzen-type one). Then the the resulting Nmatrices were used to show some interesting properties of these logics and interesting relations between them. In fact, [5, 6, 7] were devoted to developing general methods for doing this, and so they provided non-deterministic semantics for thousands of logics which were (and historically had first been) introduced via some formal proof system.

The present paper is the first work which goes in the opposite direction.[1] Its goal is to develop proof systems for logics which are defined by Nmatrices. So here the Nmatrices come first, and we describe general methods of developing proof systems for them, demonstrating their usefulness with some concrete examples. Another innovation of this paper is the distinction made between two different types of semantics which are based on Nmatrices: the dynamic one and the static one. So far, all the previous papers on the subject employed the dynamic semantics only. This is the first to introduce (and apply) also the static one.

The structure of the paper is as follows. In Sect. 2 we describe the two types of semantics for logical systems induced by Nmatrices (the dynamic semantics and the static semantics). Several running examples of two-valued, three-valued, and four-valued Nmatrices are given in the second part of that section. In Sect. 3 we introduce general proof systems for logics having finite-valued non-deterministic semantics, together with principles that allow us to simplify these systems in the individual cases. Our systems are based on a weakened version of the Rasiowa-Sikorski (R-S) decomposition methodology ([25, 30]), which can be easily translated to the well-known n-sequent formalism ([13]). The soundness and completeness of our general proof systems are then proved in Sect. 4, where some general applications are also described. In Sect. 5, the general method is illustrated on the various running examples introduced in Sect. 2.2. It turns out then that most of them correspond to well-known logics. We also show there how our proof systems can be used for obtaining cut-free ordinary (two-sided) Gentzen-type systems for these well-known logics.

2 Non-deterministic Matrices

2.1 Concept

In what follows, $\mathcal{L}$ is a propositional language, O_n ($n \geq 0$) is the set of its n-ary connectives, $\mathcal{W}$ is its set of wffs, p, q, r denote propositional variables, $\varphi, \psi, \phi, \tau$ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote finite sets of formulas.

Definition 2.1 *A* non-deterministic matrix *(*Nmatrix*) for $\mathcal{L}$ is a triple $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (containing its designated values), and $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond \in O_n$.*

Definition 2.2 *Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix.*

1. *A* dynamic valuation *in $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ such that for each n-ary connective $\diamond \in O_n$, the following holds for all $\psi_1, \ldots, \psi_n \in \mathcal{W}$:*

$$\text{(SLC)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

2. *A* static valuation *in $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ which satisfies Condition (SLC) together with the following compositionality principle: for each $\diamond \in O_n$ and for*

[1] Actually, some partial work in this direction was done already in [2, 4], where a connection was established between canonical Gentzen type systems and 2-valued Nmatrices. It was shown there how to construct a characteristic 2-valued Nmatrix for any given canonical system, *and* how to develop a canonical system for any given 2-valued Nmatrix. However, this was only done for the very special case of 2-valued Nmatrices (with dynamic semantics), while the present paper effects this for any finite-valued Nmatrix.

every $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n \in \mathcal{W}$,

$$\text{(CMP)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\varphi_1, \ldots, \varphi_n)) \text{ if } v(\psi_i) = v(\varphi_i) \ (i = 1 \ldots n)$$

Note 2.3 The well-known ordinary (deterministic) matrices correspond to the case when each $\widetilde{\diamond}$ is a function taking singleton values only. Then it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$; thus there is no difference between static and dynamic valuations, and we have full determinism.

Note 2.4 Like in usual multi-valued semantics, the principle here is that each formula has a definite logical value. This is why we exclude $\emptyset$ as a value of $\widetilde{\diamond}$. However, the absence of any logical value for a formula can still be simulated in our formalism by introducing a special logical value $\perp$ representing exactly this case (which is a well-known procedure in the framework of partial logics).

As one can see from the above definitions, the dynamic semantics corresponds to selecting the value of $v(\diamond(\psi_1, \ldots, \psi_n))$ out of the whole range of the allowed values in $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ separately and independently for each tuple $\langle v(\psi_1), \ldots, v(\psi_n) \rangle$. Thus the choice of one of the possible values is made at the lowest possible (local) level of computation, or on-line, and $v(\psi_1), \ldots, v(\psi_n)$ do not uniquely determine $v(\diamond(\psi_1, \ldots, \psi_n))$. This semantics corresponds to the highest level of non-determinism possible in the context of the definition of an Nmatrix.

On the other hand, with the static semantics this choice is made globally, system-wide. Indeed, Condition (CMP) says that the value of $v(\diamond(\psi_1, \ldots, \psi_n))$ is now uniquely determined by $v(\psi_1), \ldots, v(\psi_n)$ — which means that the interpretation of $\diamond$ is a function. The function is a "determinisation" of the non-deterministic interpretation $\widetilde{\diamond}$ to be applied in computing the value of any formula under the given valuation. This limits non-determinism, but we still have the freedom of choosing the above function among those compatible with the non-deterministic interpretation $\widetilde{\diamond}$ of $\diamond$. The selection is performed before any computation begins. So, in the static semantics one selects a function $f^v_\diamond : \mathcal{V}^n \to \mathcal{V}$ such that, for any $(t_1, \ldots, t_n) \in \mathcal{V}$, and any $\psi_1, \ldots, \psi_n \in \mathcal{W}$:

$$f^v_\diamond(t_1, \ldots, t_n) \in \widetilde{\diamond}(t_1, \ldots, t_n), \qquad v(\diamond(\psi_1, \ldots, \psi_n)) = f^v_\diamond(v(\psi_1) \ldots v(\psi_n))$$

Definition 2.5 *A valuation v in $\mathcal{M}$ satisfies a formula ψ ($v \models \psi$) if $v(\psi) \in \mathcal{D}$, and is a model of Γ ($v \models \Gamma$) if it satisfies every formula in Γ.*

Definition 2.6 *We say that ψ is dynamically (statically) valid in $\mathcal{M}$, in symbols $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$), if $v \models \psi$ for each dynamic (static) valuation v in $\mathcal{M}$.*

We say that Δ dynamically (statically) follows from Γ in $\mathcal{M}$, in symbols $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$), if for every dynamic (static) model v of Γ in $\mathcal{M}$ we have $v \models \phi$ for some $\phi \in \Delta$.

The relation $\vdash^d_{\mathcal{M}}$ ($\vdash^s_{\mathcal{M}}$) is called the dynamic (static) consequence relation induced by $\mathcal{M}$.

Note 2.7 Obviously, the static consequence relation includes the dynamic one, i.e. $\vdash^s_{\mathcal{M}} \supseteq \vdash^d_{\mathcal{M}}$. For ordinary matrices $\vdash^s_{\mathcal{M}} = \vdash^d_{\mathcal{M}}$, so we shall just write $\vdash_{\mathcal{M}}$ in this case.

2.2 Exemplary Nmatrices

Below we shall denote $\mathcal{N} = \mathcal{V} \setminus \mathcal{D}$, and shall usually identify singleton values of connectives with the truth values themselves.

Example 2.8 Consider $\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$, and assume $\mathcal{L}$ has binary connectives $\vee, \wedge, \supset$ interpreted classically, and a unary connective $\neg$, which is interpreted either classically or paraconsistently. This leads to the Nmatrix $\mathcal{M}_2 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ for $\mathcal{L}$, where:

$\widetilde{\vee}$	**f**	**t**
f	**f**	**t**
t	**t**	**t**

$\widetilde{\wedge}$	**f**	**t**
f	**f**	**f**
t	**f**	**t**

$\widetilde{\supset}$	**f**	**t**
f	**t**	**t**
t	**f**	**t**

$\widetilde{\neg}$	**f**	**t**
	t	$\{\mathbf{f}, \mathbf{t}\}$

Obviously, the dynamic semantics satisfies $\neg\varphi \vee \varphi$ but not $\varphi \supset \neg\neg\varphi$ (take $v(\varphi) = t, v(\neg\varphi) = t, v(\neg\neg\varphi) = f$). On the other hand, the static semantics does satisfy $\varphi \supset \neg\neg\varphi$ (it allows just two interpretations for $\neg$: the classical one and $\lambda x.t$, and $\varphi \supset \neg\neg\varphi$ is valid for both). However, it does not satisfy $\neg\varphi \wedge \varphi \supset \psi$ (take the interpretation of $\neg$ to be $\lambda x.t$, and let $v(\varphi) = t$, $v(\psi) = f$). It follows that the logics induced by the dynamic semantics and by the static semantics are different, although both are paraconsistent. We shall see in Section 5 that they both correspond to well-know logics.

Example 2.9 Assume now $\mathcal{V} = \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$, and $\mathcal{L}$ has a unary connective $\neg$ and a binary connective $\vee$. Consider the Nmatrix $\mathcal{M}_{MK} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, with $\mathcal{O} = \{\widetilde{\neg}, \widetilde{\vee}\}$, where:

$\widetilde{\neg}$	**f**	**e**	**t**
	t	**e**	**f**

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	$\{\mathbf{e}, \mathbf{t}\}$
t	**t**	**t**	**t**

One can easily see that with the static semantics the above Nmatrix corresponds to the intersection of two well-known, three-valued logics: namely, 3-valued Kleene and McCarthy [26] logics [2]. Indeed: if in the static semantics we take $f^K_\vee(\mathbf{e}, \mathbf{t}) = \mathbf{t}$, we shall obtain Kleene logic, while by choosing $f^M_\vee(\mathbf{e}, \mathbf{t}) = \mathbf{e}$ we shall get McCarthy logic. Therefore using our procedures for developing proof systems one can obtain proof systems for both these logics, which coincide with the result of modifying the proof system for $\vdash^s_{\mathcal{M}_{MK}}$ (as developed in Section 5.3) by strengthening in each case exactly one of its disjunction rules.

Example 2.10 Consider the following two 3-valued Nmatrices $\mathcal{M}^3_L, \mathcal{M}^3_S$. In both we have $\mathcal{V} = \{\mathbf{f}, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}$. Also the interpretations of disjunction, conjunction and implication are the same in both of them, corresponding to those in positive classical logic:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{N} & \text{if } a, b \in \mathcal{N}, \end{cases}$$

[2] The standard language of these logics includes the connective $\wedge$ as well, but in both of them $\wedge$ is definable in terms of $\neg, \vee$ by De Morgan laws.

$$a\tilde{\wedge}b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D}, \\ \mathcal{N} & \text{if either } a \in \mathcal{N} \text{ or } b \in \mathcal{N}, \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{N} \text{ or } b \in \mathcal{D}, \\ \mathcal{N} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{N}. \end{cases}$$

However, negation is interpreted differently: more liberally in $\mathcal{M}^3_L$, and more strictly in $\mathcal{M}^3_S$:

$$\mathcal{M}^3_L : \quad \begin{array}{c|ccc} \tilde{\neg} & \mathbf{f} & \top & \mathbf{t} \\ \hline & \mathbf{t} & \mathcal{V} & \mathbf{f} \end{array} \qquad\qquad \mathcal{M}^3_S : \quad \begin{array}{c|ccc} \tilde{\neg} & \mathbf{f} & \top & \mathbf{t} \\ \hline & \mathbf{t} & \mathcal{D} & \mathbf{f} \end{array}$$

Later we shall show, using our general method and theorems, that the dynamic semantics for $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ induce the same logic (i.e., consequence relation). However, the proof mechanisms developed here give a deeper insight into the matter. Indeed, we show that the sets of *3-sequents* which are derivable in the respective proof systems we develop for $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ in Section 5.4 (and so, by completeness, also the sets of 3-sequents dynamically valid in $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$, respectively) do differ from each other.

Example 2.11 After considering 2-valued Nmatrices and 3-valued Nmatrices, our last example is a 4-valued Nmatrix. This is the Nmatrix $\mathcal{M}_4 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V} = \{\mathbf{f}, \perp, \top, \mathbf{t}\}$, $\mathcal{D} = \{\top, \mathbf{t}\}$, $\wedge, \vee, \supset$ are are defined by the general rules given in Example 2.10 (applied, however, to the sets $\mathcal{D}$ and $\mathcal{N} = \mathcal{V} \setminus \mathcal{D}$ appearing in the current exmaple), while $\neg$ is the negation of the bilattice $FOUR$ ([11, 22, 21, 1]):

$$\begin{array}{c|cccc} \tilde{\neg} & \mathbf{f} & \perp & \top & \mathbf{t} \\ \hline & \mathbf{t} & \perp & \top & \mathbf{f} \end{array}$$

3 Proof Theory

In this section we present proof systems for propositional logics based on finite Nmatrices (for both versions of the semantics: the dynamic and the static one). They will consist of rules operating on finite sets of *signed formulas*, and axioms being sets of such formulas. The deduction formalism used here is similar to the so-called Rasiowa-Sikorski (R-S) systems ([30], [25]), known also as dual tableaux ([23]). However, in contrast to the former, the deduction rules in our formalism are not invertible.

Definition 3.1 *Let $\mathcal{V}$ be a finite set (of truth-values), let $\mathcal{L}$ be a propositional language with a set $\mathcal{W}$ of wffs, and let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$ with $\mathcal{V}$ as its set of truth-values.*

- *A signed formula over $\mathcal{L}$ and $\mathcal{V}$ is an expression of the form $a : \psi$, where $a \in \mathcal{V}, \psi \in \mathcal{W}$.*
- *A valuation v in $\mathcal{M}$ satisfies a signed formula $a : \psi$, in symbols $v \models a : \psi$, if $v(\psi) = a$.*

Signed formulas will be denoted by $\alpha, \beta, \ldots$, and sets of signed formulas — by Ω, Σ, Φ.

In terms of satisfaction by a valuation and validity, sets of signed formulas will be interpreted *disjunctively*:

Definition 3.2

- *A valuation v in $\mathcal{M}$ satisfies a set of signed formulas Ω iff it satisfies* some *signed formula $\alpha \in \Omega$.*
- *A set of signed formulas Ω is said to be dynamically (statically) valid in an Nmatrix $\mathcal{M}$, in symbols $\models^d_{\mathcal{M}} \Omega$ ($\models^s_{\mathcal{M}} \Omega$), if $v \models \Omega$ for every dynamic (static) valuation v in $\mathcal{M}$.*

As before, let $N = \mathcal{V} \setminus \mathcal{D}$ denote the set of non-designated logical values. Further, for any set of logical values $A \subseteq \mathcal{V}$ and any set of formulas $F \subseteq \mathcal{W}$, denote $A : F = \{a : \psi \mid a \in A, \psi \in F\}$. The following straightforward observation is the key for using proof systems based on sets of signed formulas to characterize the logics induced by Nmatrices.

Proposition 3.3 *For any Nmatrix $\mathcal{M}$ over $\mathcal{V}$, and any finite sets of formulas $\Gamma, \Delta \subset \mathcal{W}$, $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$) holds iff the set of signed formulas $(\mathcal{N} : \Gamma) \cup (\mathcal{D} : \Delta)$ is dynamically (statically) valid in $\mathcal{M}$. In particular, a formula φ is dynamically (statically) valid in $\mathcal{M}$ iff the set $\mathcal{D} : \{\varphi\}$ is dynamically (statically) valid in $\mathcal{M}$.*

Definition 3.4 *Let $\mathcal{M}$ be an Nmatrix, and SF — a deduction system based on finite sets of signed formulas over the language of $\mathcal{M}$ and the set of truth-values of $\mathcal{M}$. We say that:*

- *SF is dynamically (statically) complete for $\mathcal{M}$ if for all finite sets of formulas $\Gamma, \Delta \subset \mathcal{W}$: $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$) iff $\vdash_{SF} (\mathcal{N} : \Gamma) \cup (\mathcal{D} : \Delta)$.*
- *SF is weakly dynamically (statically) complete for $\mathcal{M}$ if for all formulas $\varphi \in \mathcal{W}$: $\models^d_{\mathcal{M}} \varphi$ ($\models^s_{\mathcal{M}} \varphi$) iff $\vdash_{SF} (\mathcal{D} : \{\varphi\})$.*
- *SF is fully dynamically (statically) complete for $\mathcal{M}$ if for any set of signed formulas Ω :$\models^d_{\mathcal{M}} \Omega$ ($\vdash^s_{\mathcal{M}} \Omega$) iff $\vdash_{SF} \Omega$.*

By Prop. 3.3, full completeness implies completeness, which in turn implies weak completeness.

3.1 Dynamic Semantics

The deduction system $SF^d_{\mathcal{M}}$ for the dynamic semantics of an n-valued Nmatrix $\mathcal{M}$ contains:

- **Axioms**: Each set of signed formulas containing $\{a : \varphi \mid a \in \mathcal{V}\}$, where φ is any formula in $\mathcal{W}$;
- **Inference rules:** For every m-ary connective $\diamond \in \mathcal{O}$ and any logical values $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$, the rule:

$$(\diamond\text{-D}) \qquad \frac{\Omega, a_1 : \varphi_1 \quad \ldots \quad \Omega, a_m : \varphi_m}{\Omega, b_1 : \diamond(\varphi_1, \ldots, \varphi_m), \ldots, b_k : \diamond(\varphi_1, \ldots, \varphi_m)}$$

3.2 Static Semantics

The proof system $SF^s_{\mathcal{M}}$ for the static semantics of $\mathcal{M}$ is obtained out of the system $SF^d_{\mathcal{M}}$ for the dynamic semantics by adding, for any $\diamond \in O_m$ and any $a_1, \ldots, a_m, b \in \mathcal{V}$ such that $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$, the rule:

$$(\diamond\text{-S}) \quad \frac{\Omega, a_1 : \varphi_1 \quad \ldots \quad \Omega, a_m : \varphi_m \quad \Omega, a_1 : \psi_1 \quad \ldots \quad \Omega, a_m : \psi_m \quad \Omega, b : \diamond(\psi_1, \ldots, \psi_m)}{\Omega, b : \diamond(\varphi_1, \ldots, \varphi_m)}$$

Obviously, these $(2m+1)$-premise inference rules are not very convenient — and, more importantly, they are not analytic. They can, however, be simplified at the price of extending the language with constants corresponding to all truth values. Moreover: in that case we can resign from repeating the inference rules from the dynamic semantics, adding instead equivalent axioms for the constants.

Let us denote the constant corresponding to $a \in \mathcal{V}$ by $\underline{a}$. Then the proof system $SF^{sc}_{\mathcal{M}}$ for the static semantics of the language featuring constants consists of:

- **Axioms**: Each set of signed formulas containing either:
  1. $\{a : \varphi \mid a \in \mathcal{V}\}$, where φ is any formula in $\mathcal{W}$; or
  2. $\{a : \underline{a}\}$, for any $a \in \mathcal{V}$; or
  3. $\{b_1 : \diamond(\underline{a_1}, \ldots, \underline{a_m}), \ldots, b_k : \diamond(\underline{a_1}, \ldots, \underline{a_m})\}$ for any $\diamond \in O_m$ and any $a_1, \ldots, a_m$, $b_1, \ldots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$.
- **Inference rules:** For any $\diamond \in O_m$ and any $a_1, \ldots, a_m, b \in \mathcal{V}$ such that $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$, the rule

$$(\diamond\text{-SC}) \qquad \frac{\Omega, a_1 : \varphi_1 \quad \ldots \quad \Omega, a_m : \varphi_m \quad \Omega, b : \diamond(\underline{a_1}, \ldots, \underline{a_m})}{\Omega, b : \diamond(\varphi_1, \ldots, \varphi_m)}$$

Clearly, both the systems presented above are sound. In the sequel we shall denote by $\vdash^{sc}_{\mathcal{M}}$ the consequence relation induced by the static semantics of an Nmatrix $\mathcal{M}$ on the language $\mathcal{L}$ extended with the constants representing the truth values of $\mathcal{M}$.

Note 3.5 Examining the generic deduction systems given above, we can easily observe that the inference rules of the static semantics really differ from those of the dynamic semantics only in case of truly non-deterministic values of the connectives. Indeed, if the value of the connective is a singleton, i.e. $\widetilde{\diamond}(\underline{a_1}, \ldots, \underline{a_m}) = \{b\}$, the rule ($\diamond$-S) is just a weaker version of ($\diamond$-D), and so need not be included in $SF^s_{\mathcal{M}}$. As for $SF^{sc}_{\mathcal{M}}$, the last premise of rule ($\diamond$-SC) is derivable in the system by virtue of the singleton set $\{b : \ \diamond(\underline{a_1}, \ldots, \underline{a_m})\}$ being an axiom — hence it can be skipped (the formal justification is provided by Principle 2 in the next section). As the other premises of the "static" and "dynamic" rules coincide, and so do the conclusions in such a "singleton" case, the rules can be considered identical. Moreover: in this case the "static" Axiom 3 corresponding to such a singleton value of the connective can be deleted too, since it is derivable from rule ($\diamond$-D) and the basic axioms for the constants ("static" Axiom 2).

Note 3.6 It can easily be proved that the weakening rule (from Ω infer Ω' in case $\Omega \subset \Omega'$) is admissible in our systems. This is the reason why it has not been necessary to officially include it among our rules, although we shall henceforth treat it as one of these rules in practice.

3.3 Streamlining of the Deduction Systems

As in other similar cases, the instances of the generic deduction systems we have given in Sect. 3.1, 3.2 corresponding to individual Nmatrices are hardly the optimal deduction systems for these matrices. Hence we now give some principles for streamlining such systems. Later we shall illustrate their application on the deduction systems for our exemplary Nmatrices.

Denote the system under consideration by $\mathcal{R}$. Our streamlining principles consist in: deleting a derivable rule (Princ. 1), replacing a rule by one with weaker premises (Princ. 2), and combining two rules with the same conclusion (Princ. 3):

Principle 1 If a rule in $\mathcal{R}$ is derivable from other rules, it can be deleted.

Principle 2 If $\dfrac{S}{\Sigma}$ (where S is a set of premises) is a rule in $\mathcal{R}$, S' is a subset of S and $\dfrac{S'}{\Sigma}$ is derivable in $\mathcal{R}$, then $\dfrac{S}{\Sigma}$ can be replaced with $\dfrac{S'}{\Sigma}$. In particular: if $\dfrac{S}{\Sigma}$ is a rule in $\mathcal{R}$, $\pi \in S$ and π is derivable from $S \setminus \{\pi\}$ in $\mathcal{R}$, then $\dfrac{S}{\Sigma}$ can be replaced with $\dfrac{S \setminus \{\pi\}}{\Sigma}$.

Principle 3 Rules $\dfrac{\Omega_1 \quad \dots \quad \Omega_k}{\Sigma}$ and $\dfrac{\Omega'_1 \quad \dots \quad \Omega'_l}{\Sigma}$ can be replaced with the rule $\dfrac{\{\Omega_i \cup \Omega'_j\}_{1\leq i\leq k, 1\leq j\leq l}}{\Sigma}$

The validity of Principles 1 and 2 is obvious, while the validity of Principle 3 follows from the use of contexts in our rules: if $\dfrac{\Phi_1 \quad \dots \quad \Phi_k}{\Sigma}$ is a valid application of rule R, and Σ' is a set of signed formulas, then $\dfrac{\Phi_1 \cup \Sigma' \quad \dots \quad \Phi_k \cup \Sigma'}{\Sigma \cup \Sigma'}$ is also a valid application of rule R. Hence for each $1 \leq i \leq k$, $\Omega_i \cup \Sigma$ follows from $\{\Omega_i \cup \Omega'_j\} \mid 1 \leq j \leq l\}$ using the second rule, and then Σ follows from these k sets using the first rule.

4 Completeness and Its Applications

4.1 Completeness Proofs

In this section we shall prove completeness of the calculi presented in Sect. 3.

Theorem 4.1 *The calculus $SF^d_{\mathcal{M}}$ is fully dynamically complete for $\mathcal{M}$.*

Proof: It is straightforward to show that $\vdash_{SF^d_{\mathcal{M}}} \Omega$ implies $\models^d_{\mathcal{M}} \Omega$. It remains to show that a set Ω of signed formulas not provable in $SF^d_{\mathcal{M}}$ is not valid, i.e., there is a valuation v in $\mathcal{M}$ which refutes all formulas in Ω.

Define a set Ω of signed formulas to be *saturated* if the following holds: if $\widetilde{\diamond}(a_1, \dots, a_m) = \{b_1, \dots, b_k\}$, and $b_1 : \diamond(\varphi_1, \dots, \varphi_m), \dots, b_k : \diamond(\varphi_1, \dots, \varphi_m)$ are all in Ω, then $a_i : \varphi_i \in \Omega$ for some $1 \leq i \leq m$. Let Ω be not provable in $SF^d_{\mathcal{M}}$, and suppose it is not saturated. Now assume for contradiction that $\Omega \cup \{a_i : \varphi\}$ is provable for $1 \leq i \leq m$. Then from rule $(\diamond - D)$ we get that $\Omega \cup \{b_1 : \diamond(\varphi_1, \ldots, \varphi_m), \ldots, b_k : \diamond(\varphi_1, \ldots, \varphi_m)\}$ is provable - but the latter set is identical with Ω. Hence there is an i such that $\Omega \cup (a_i : \varphi_i)$ is not provable. By repeating this procedure, we can extend Ω to a saturated, non-provable set in a finite number of steps (since each time we add only subformulas of the formulas already in Ω).

Thus it suffices to show that if Ω is saturated and not provable in $SF^d_{\mathcal{M}}$ then it has a countermodel v. We define such a v by induction on the complexity of formulas. According to our goal, v is defined so that $v(\varphi) \neq i$ for any $(i : \varphi) \in \Omega$.

First, let p be any propositional variable. As the unprovable Ω cannot contain any axiom of $SF^{\mathcal{M}}_d$, $(i_0 : p) \notin \Omega$ for some i_0. We put $v(p) = i_0$, which meets the goal condition. Suppose now we have defined v for formulas of complexity up to l, and that $\psi = \diamond(\varphi_1, \ldots, \varphi_m)$, where each φ_i is of complexity at most l. Hence $v(\varphi_i)$ is already defined for each i. Assume $a_i = v(\varphi_i)$, and $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$. Then there must be an i_0 such that $b_{i_0} : \diamond(\varphi_1, \ldots, \varphi_m) \notin \Omega$. Indeed: as Ω is saturated, otherwise there would be j with $a_j : \varphi_j \in \Omega$, contradicting the induction hypothesis on φ_j. Now take $v(\psi) = b_{i_0}$.

By construction, the valuation v defined above refutes Ω. Moreover, for any m-ary connective $\diamond$ and any formulas $\varphi_1, \ldots, \varphi_m$ of $\mathcal{L}$ we have $v(\diamond(\varphi_1, \ldots, \varphi_m)) \in \widetilde{\diamond}(\varphi_1, \ldots, \varphi_m)$, so v is a well-defined valuation compliant with the dynamic semantics of the given Nmatrix. $\Box$

Corollary 4.2 *The calculus $SF^d_{\mathcal{M}}$ is (weakly) dynamically complete with respect to $\mathcal{M}$.*

Theorem 4.3 *The calculus $SF^{sc}_{\mathcal{M}}$ is fully statically complete for $\mathcal{M}$.*

Proof: Again, the soundness of $SF^{sc}_{\mathcal{M}}$ is an obvious consequence of the definition of the static semantics. The proof of completeness is analogous to the dynamic case, but now we define a set Ω of signed formulas to be *saturated* if whenever $b : \diamond(\varphi_1, \ldots, \varphi_m)$ is in Ω, and $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$, then either $a_i : \varphi_i \in \Omega$ for some $1 \leq i \leq m$ or $b : \diamond(\underline{a_1}, \ldots, \underline{a_m})$ is in Ω. The definition of a counter-valuation v for a given unprovable, saturated Ω starts this time by taking $v(\underline{a}) = a$ for any constant $a \in \mathcal{V}$. The case where φ is a propositional variable is handled like in the dynamic case. Next, if φ is of the form $\diamond(\underline{a_1}, \ldots, \underline{a_m})$ then there should be some $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$ such that $b : \diamond(\underline{a_1}, \ldots, \underline{a_m})$ is not in Ω (otherwise Ω would be provable by Axiom 3 of $SF^{sc}_{\mathcal{M}}$). We let $v(\varphi) = b$ in this case. Finally, for other complex formulas we define inductively $v(\diamond(\varphi_1, \ldots, \varphi_m)) = v(\diamond(\underline{a_1}, \ldots, \underline{a_m}))$, where $a_i = v(\varphi_i)$ $(i = 1, \ldots, m)$. $\Box$

Corollary 4.4 *The calculus $SF^{sc}_{\mathcal{M}}$ is (weakly) statically complete with respect to $\mathcal{M}$.*

Note 4.5 It can be directly seen that the completeness of $SF^{sc}_{\mathcal{M}}$ implies completeness of the system $SF^{s+c}_{\mathcal{M}}$ for the language with the constants, obtained out of the system $SF^{s}_{\mathcal{M}}$ by adding the basic axioms for constants $\{a : \underline{a}\}$. Indeed, it is easy to see that out of these axioms plus rule ($\diamond$-D) of $SF^{s+c}_{\mathcal{M}}$ we can infer the one remaining Axiom 3. and the rule ($\diamond$-SC) of $SF^{sc}_{\mathcal{M}}$. Hence $SF^{s+c}_{\mathcal{M}}$ is at least as strong as $SF^{sc}_{\mathcal{M}}$, which means it must be complete like $SF^{sc}_{\mathcal{M}}$.

As to the system $SF^s_{\mathcal{M}}$, its completeness can be proved directly, using a method analogous to the case of $SF^d_{\mathcal{M}}$, but we skip it, since we will not make any use of the completeness of $SF^s_{\mathcal{M}}$ here.

4.2 Applications

4.2.1 Elimination of Generalized Cuts

The analogue of the well known cut rule for ordinary sequents is the following *generalized cut* rule for sets of signed formulas:

$$\frac{\Omega \cup \{i : \varphi \mid i \in I\} \qquad \Omega \cup \{j : \varphi \mid j \in J\}}{\Omega} \qquad \text{for } I, J \subseteq \mathcal{V}, I \cap J = \emptyset$$

Theorem 4.6 *The generalized cut rule is admissible in $SF^d_{\mathcal{M}}$ and in $SF^{sc}_{\mathcal{M}}$.*

Proof: The rule is obviously sound for both the dynamic semantics and the static one. Hence its admissibility follows from the completeness theorems.

4.2.2 Proof Search: R-S Systems

The proof systems developed here operate on sets of multi-valued (mv) signed formulas. As we have mentioned in the introduction, they follow the pattern of mv Rasiowa-Sikorski (R-S) systems ([25, 30]). Such systems are commonly used for proof search by applying the inference rules "backwards". Besides R-S systems, another, much better known type of system used for this purpose are mv tableaux (R-S systems are sometimes called "dual tableaux" — see [23]). From a syntactical point of view, the two types of systems are quite similar. They are based on rules which decompose formulas into simpler ones, and use those rules to generate labeled decomposition trees for formulas, which are termed "proofs" if they satisfy certain closure properties. The main difference between R-S systems and tableaux is in the *semantic* interpretation of what is done in them: Tableaux interpret sets of signed formulas conjunctively, while R-S systems interpret them disjunctively. Accordingly, an open branch of a tableau yields a valuation which *satisfies* all the signed formulas on that branch, while an open branch of an R-S tree yields a valuation which *refutes* all the signed formulas on that branch.

In the two-valued case, both mechanisms are equivalent from the proof-theoretic viewpoint, differing just in presentation. However, in the multi-valued case, the difference is much more significant. To prove $\Gamma \vdash \Delta$, the R-S systems use a single decomposition tree for the set $(\mathcal{N} : \Gamma) \cup (\mathcal{D} : \Delta)$ – and this is also the case for the systems developed here. On the other hand, the tableaux formalism uses a separate tableau for each assignment of labels in $\mathcal{D}$ to the formulas in Γ, and labels in $\mathcal{N}$ to the formulas in Δ. Thus to show the validity of a formula φ one has to show that $\{a : \varphi\}$ has a closed tableau for each $a \in \mathcal{N}$.[3]

[3]In [23] this problem is dealt with by allowing *sets of truth values* to function as signs in signed formulas. This is not necessary in case of R-S systems, but the use of this method might possibly improve their efficiency too. This will be a topic of future research.

4.2.3 Obtaining Cut-free, Ordinary Two-sided Gentzen-type Systems

In [9] we describe a general procedure which allows us to obtain in many important cases an *ordinary* (two-sided), cut-free Gentzen-type system from an R-S style mv-calculus. In the next section we present examples of systems obtained with this method from the mv-calculi developed here.

5 Examples

In this section we show what R-S deduction systems result from applying the general schemes of axioms and inference rules given in Sect. 3.1, 3.2 to the concrete cases of our exemplary Nmatrices. We further show how these systems can be reduced with help of the reduction principles given in Sect. 3.3, and what are the equivalent complete and cut-free systems of ordinary, two-sided sequents obtained from these R-S systems using the general methods of [9].

It is useful to employ in what follows another notational variant for our proof systems, presenting them as calculi of n-sequents (also termed *many-sided* sequents, or *many-placed* sequents) [4]. Without loss of generality we may assume that the set of truth values of the language $\mathcal{L}$ is $\mathcal{V} = \{0, 1, \ldots, n-1\}$ (where $n \geq 2$), with the set of designated values being $\mathcal{D} = \{d, \ldots, n-1\}$ (where $d \geq 1$)[5].

Definition 5.1

- *By an n-sequent over the language $\mathcal{L}$ we mean an expression Σ of the form $\Gamma_0|\Gamma_1|\ldots|\Gamma_{n-1}$, where for each i $\Gamma_i \subseteq \mathcal{W}$ is a finite set of formulas of $\mathcal{L}$.*
- *A valuation v in an Nmatrix $\mathcal{M}$ satisfies the sequent $\Sigma = \Gamma_0|\Gamma_1|\ldots|\Gamma_{n-1}$, written $v \models \Sigma$, if there exists an $i, 0 \leq i \leq n-1$, and $\varphi_i \in \Gamma_i$ such that $v(\varphi_i) = i$.*
- *A sequent Σ is said to be dynamically (statically) valid in an Nmatrix $\mathcal{M}$, in symbols $\models^d_{\mathcal{M}} \Sigma$ ($\models^s_{\mathcal{M}} \Sigma$) if $v \models \Sigma$ for every dynamic (static) valuation v in $\mathcal{M}$.*

Clearly, in view of the above, a set S of signed formulas over the language L is equivalent to the sequent $\Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{n-1}$, where $\Gamma_i = \{\psi \mid (i : \psi) \in S\}$. This gives a simple method for translating our proof systems (based on sets of signed formulas) into sequent calculi, which we will use in this section.

To make the presentation more intuitive, the sequent bar $|$ which separates the non-designated values from the designated ones will be replaced with the symbol $\Rightarrow$ used in ordinary sequents.

5.1 Dynamic Semantics of $\mathcal{M}_2$: the Logic CLuN

Although cut-free canonical Gentzen-type systems were already developed for all dynamic 2-valued Nmatrices in [2, 4], it should be illuminating to see how the methods

[4] Proof systems based on n-sequents were invented and reinvented several times in the past. See e.g. [31, 29, 14]. See also [32, 23, 13] for further details and references. Now formulating our *general* methods and calculi in this format is extremely cumbersome and difficult to read. However, when the number n of truth-values is not big (as is the case in our examples), the use of n-sequents makes it easier to grasp what is going on, and especially the process of converting the resulting calculi to ordinary, 2-sided Gentzen-type systems becomes smoother and more transparent.

[5] If $\mathcal{V} \neq \{0, 1, \ldots, n-1\}$, we shall assume that the order of sequent positions corresponding to the individual truth values coincides with the order of those values in the truth tables.

of this paper work in this simple case (it should also be instructive to compare it with the corresponding static case, which has not been dealt with before). So consider the dynamic semantics of the 2-valued Nmatrix $\mathcal{M}_2$ from Example 2.8. We describe the $\{\vee, \neg\}$ fragments of the corresponding systems (implication and conjunction can be handled similarly).

The basic rules for $\vee, \neg$ we get from the procedure of Sect. 3.1 are:

$$(1)\ \frac{\Omega, \mathbf{f} : \varphi \quad \Omega, \mathbf{f} : \psi}{\Omega, \mathbf{f} : \varphi \vee \psi} \qquad (2)\ \frac{\Omega, \mathbf{f} : \varphi \quad \Omega, \mathbf{t} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi} \qquad (3)\ \frac{\Omega, \mathbf{t} : \varphi \quad \Omega, \mathbf{f} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

$$(4)\ \frac{\Omega, \mathbf{t} : \varphi \quad \Omega, \mathbf{t} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi} \qquad (5)\ \frac{\Omega, \mathbf{f} : \varphi}{\Omega, \mathbf{t} : \neg\varphi} \qquad (6)\ \left\{\frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{f} : \neg\varphi, \mathbf{t} : \neg\varphi}\right\}$$

This system is hardly an optimal one, so we next apply our reduction principles 1–3 of Sect. 3.3 to bring it to a more compact and pleasing form.

Negation: The rule for negation enclosed in brackets has a tautological conclusion, and therefore it can be deleted by Principle 1. This leaves us with the single negation rule (5), which in the n-sequent notation mentioned above takes the form:

$$(5)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

Note that as $n = 2$ in this example, we obtain the standard sequents here. It should also be noted that by adding a rule dual to (5) we obtain a proof system for classical logic.

Disjunction: Rules 3,4 have a common conclusion, so they can be grouped together under Princ. 3 in Sect. 3.3, yielding a rule with four premises. Two of them are subsumed by the first one, so we can eliminate them by Principle 2 to get the rule

$$\frac{\Omega, \mathbf{t} : \varphi \quad \Omega, \mathbf{f} : \psi, \mathbf{t} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

However, the second premise in this rule is tautological, so we can delete it too, getting

$$\frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

Using Princ. 3 for combining the latter rule with Rule 2, and deleting again a tautological premise, we finally get

$$\frac{\Omega, \mathbf{t} : \varphi, \mathbf{t} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

Together with Rule 1, we thus get the following reduced set of disjunction rules:

$$\frac{\Omega, \mathbf{f} : \varphi \quad \Omega, \mathbf{f} : \psi}{\Omega, \mathbf{f} : \varphi \vee \psi} \qquad \frac{\Omega, \mathbf{t} : \varphi, \mathbf{t} : \psi}{\Omega, \mathbf{t} : \varphi \vee \psi}$$

Again, in the alternative notational variant of the sequent calculus, the above rules can be rewritten to the standard set of disjunction rules in the sequent calculus:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Let $SF^{d}_{\mathcal{M}_2}$ be the proof system for the dynamic semantics presented above. It is already an ordinary Gentzen-type system, and it is cut-free by Theorem 4.6. As its single rule for negation obviously translates to $\neg\varphi \vee \varphi$, a sound and complete Hilbert-type axiomatization for $\vdash^{d}_{\mathcal{M}_2}$ is obtained by adding this axiom schema to a standard Hilbert-type formulation of positive classical logic.

Note 5.2 *The Gentzen-type and Hilbert-type systems we have just derived are in fact the standard ones used in the literature for Batens' logic* **CLuN** *from [10]. This fact implies that* $\vdash^{d}_{\mathcal{M}_2} = \vdash_{\mathbf{CLuN}}$.[6] *(Officially, this result was first shown in [2, 3, 4], but it should be noted that already in [10],* $\mathcal{M}_2$ *and its the dynamic semantics had implicitly been introduced, together with the sound and complete Hilbert-type system we have derived for it here).*

5.2 Static Semantics of $\mathcal{M}_2$: the Logic CAR

Now let us turn to the static semantics of the Nmatrix $\mathcal{M}_2$. We know from Note 3.5 in Sect. 3.2 that static axioms corresponding to deterministic values of connectives can be deleted. The only non-deterministic value of a connective is $\neg\mathbf{t} = \{\mathbf{f}, \mathbf{t}\}$, and the corresponding axiom is $\{\mathbf{f} : \neg\underline{\mathbf{t}}, \mathbf{t} : \neg\underline{\mathbf{t}}\}$ — which can also be deleted as a special case of the general axiom for the dynamic semantics. Thus we only have two additional axioms for the constants: $\{\mathbf{f} : \underline{\mathbf{f}}\}, \{\mathbf{t} : \underline{\mathbf{t}}\}$. In the sequent notation, these axioms take the form $\underline{\mathbf{f}} \Rightarrow$, $\Rightarrow \underline{\mathbf{t}}$, respectively, while the general axiom inherited from the dynamic semantics becomes $\varphi \Rightarrow \varphi$, which is the well-known basic axiom of the standard sequent calculus.

As to the inference rules, we know again from Note 3.5 that the "static" rules for deterministic values of connectives coincide with the corresponding "dynamic" rules. Hence in this example it is enough to give the two static rules corresponding to $\neg\mathbf{t} = \{\mathbf{f}, \mathbf{t}\}$:

$$\frac{\Omega, \mathbf{t} : \varphi \quad \Omega, \mathbf{f} : \neg\underline{\mathbf{t}}}{\Omega, \mathbf{f} : \neg\varphi} \qquad \frac{\Omega, \mathbf{t} : \varphi \quad \Omega, \mathbf{t} : \neg\underline{\mathbf{t}}}{\Omega, \mathbf{t} : \neg\varphi}$$

which in the sequent notation become

$$\text{(I)} \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma, \neg\underline{\mathbf{t}} \Rightarrow \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \qquad \text{(II)} \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\underline{\mathbf{t}}}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

Accordingly, denote (following section 3.2) by $SF^{sc}_{\mathcal{M}_2}$ the deduction system obtained by adding to the system $SF^{d}_{\mathcal{M}_2}$ above the basic axioms $\underline{\mathbf{f}} \Rightarrow$ and $\Rightarrow \underline{\mathbf{t}}$, as well as rules (I), (II). We know from Sect. 3 that $SF^{sc}_{\mathcal{M}_2}$ is complete for $\vdash^{s}_{\mathcal{M}_2}$, and the cut rule is admissible in it. Our next goal is to provide for $\vdash^{s}_{\mathcal{M}_2}$ a system without constants having the same properties.

[6] The original language of **CLuN** includes also the propositional constant $\perp$, which can easily be added here too.

As a starting point we take the system $SF^s_{\mathcal{M}_2}$ (recall that it is obtained out of $SF^d_{\mathcal{M}_2}$ by adding the rule schema $(\diamond - S)$ of Sect. 3.2). This system has two additional rules for negation:

$$\text{(I-s)}\quad \frac{\{\Gamma \Rightarrow \Delta, \varphi\}\quad \Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg\varphi} \qquad \text{(II-s)}\quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \{\Gamma \Rightarrow \Delta, \psi\}\ \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}$$

It can be easily seen that rules (I-s), (II-s) remain semantically valid if we drop the premises included in brackets, so in what follows we will use these weaker versions of them (this is not essential, but it makes further reasoning easier). Obviously, the aforementioned rules do not have the subformula property — so our immediate goal is to transform them to valid analytic rules with the same power. Let us begin with rule (II-s). Its problematic premise, which makes the rule non-analytic, is $\Gamma, \neg\psi \Rightarrow \Delta$. We take a special instance of (II-s), where this premise is an axiom, and together with weakening obtain the following derivation:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi, \neg\psi} \qquad \Gamma, \neg\psi \Rightarrow \Delta, \neg\psi}{\Gamma, \neg\varphi \Rightarrow \Delta, \neg\psi}$$

leading to the simpler analytic rule

$$\text{(IV)}\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta, \neg\psi}$$

It can easily be checked that a similar procedure applied to rule (I-s) yields the same rule.

Definition 5.3 *Let $G^s_{\mathcal{M}_2}$ be the system obtained from $SF^s_{\mathcal{M}_2}$ by replacing rules (I-s) and (II-s) with rule (IV) above, and let $G^{sc}_{\mathcal{M}_2}$ be obtained from $SF^{sc}_{\mathcal{M}_2}$ by replacing rules (I) and (II) with the same rule (IV).*

Theorem 5.4

1. *$G^s_{\mathcal{M}_2}$ is sound and complete for $\vdash^s_{\mathcal{M}_2}$, and the cut rule is admissible in it.*
2. *$G^{sc}_{\mathcal{M}_2}$ is sound and complete for $\vdash^{sc}_{\mathcal{M}_2}$, and the cut rule is admissible in it.*

Proof. The first part of the theorem is an immediate corollary of the second, because all the rules of $G^{sc}_{\mathcal{M}_2}$ have the subformula property (and so the axioms for the constants cannot be used in a cut-free proof of a sequent in the language of $G^s_{\mathcal{M}_2}$).

For the second part, note first that the soundness of $G^{sc}_{\mathcal{M}_2}$ follows from that of $SF^{sc}_{\mathcal{M}_2}$, because rule (IV) is derivable in the latter. To complete the proof it suffices (by the completeness of $SF^{sc}_{\mathcal{M}_2}$) to show that rules (I) and (II) are admissible in $G^{sc}_{\mathcal{M}_2}$.

To show admissibility of Rule I in $G^{sc}_{\mathcal{M}_2}$, assume we have (cut-free) proofs of the sequents (A) $\Gamma \Rightarrow \Delta, \varphi$ and (B) $\Gamma, \neg\mathbf{t} \Rightarrow \Delta$ in $G^{sc}_{\mathcal{M}_2}$. Change in the proof of (B) every ancestor of the constant $\mathbf{t}$ in the indicated occurrence of $\neg\mathbf{t}$ on the left hand side of (B) to φ, and add the context $\Gamma \Rightarrow \Delta$ to all the sequents in the proof. Neither of this influences the validity of the applications of rules — and the left hand side of the sequent (B) changes to the desired form. The axioms without an ancestor of the considered occurrence of $\mathbf{t}$ are translated to axioms followed by weakening. The axioms that might contain such an ancestor are $\Rightarrow \mathbf{t}$ and $\neg\mathbf{t} \Rightarrow \neg\mathbf{t}$. The first translates to $\Gamma \Rightarrow \Delta, \varphi$, which is just the provable sequent (A). The second yields $\Gamma, \neg\varphi \Rightarrow \Delta, \neg\mathbf{t}$. However, the latter sequent is derivable from (A) using Rule (IV). As a result, we get a (cut-free) proof of $\Gamma, \neg\varphi \Rightarrow \Delta$.

In case of Rule II, we assume we have (cut-free) proofs of its premises in $G^{sc}_{\mathcal{M}_2}$. Change in the proof of $\Gamma \Rightarrow \Delta, \neg\mathbf{t}$ each ancestor of the indicated occurrence of $\mathbf{t}$ to φ. As a result, the axiom $\mathbf{t} \Rightarrow \mathbf{t}$ is replaced by the sequent $\varphi \Rightarrow \mathbf{t}$ derivable from the axiom $\Rightarrow \mathbf{t}$ by weakening, and the axiom $\neg\mathbf{t} \Rightarrow \neg\mathbf{t}$ — by the sequent $\neg\mathbf{t} \Rightarrow \neg\varphi$, derivable from $\Rightarrow \mathbf{t}$ using Rule (IV). Since these are the only axioms giving rise to the occurrences of $\mathbf{t}$ we are replacing, and the applications of rules are preserved under this transformation, we eventually obtain a cut-free proof of $\Gamma \Rightarrow \Delta, \neg\varphi$. Hence Rule II is admissible too. □

Note 5.5 An examination of the derivation above of Rule II reveals that it is sufficient to have a proof of the second premise of that rule in order to get a proof of its conclusion. This indicates that Rule II should remain valid if its first premise is deleted. A direct check proves that it is indeed so.

Note 5.6 The two negation rules of $G^{sc}_{\mathcal{M}_2}$ translate to $\neg\varphi \vee \varphi$ and $\varphi \supset (\neg\varphi \supset \neg\psi)$. Hence a sound and complete Hilbert-type axiomatization for $\vdash^s_{\mathcal{M}_2}$ is obtained by adding these two axiom schemes to a standard Hilbert-type formulation of positive classical logic.

Note 5.7 The logic **CAR** from [18, 28] is the minimal extension of **CLuN** for which substitution of equivalents is a valid rule. In [18] a corresponding sound and complete Hilbert type system was presented[7]. That system differs from the one we have just derived for $\vdash^s_{\mathcal{M}_2}$ only by having $(\psi \supset \varphi) \supset (\neg\varphi \supset \neg\psi)$ as an axiom instead of $\varphi \supset (\neg\varphi \supset \neg\psi)$. Since the two systems can easily be seen to be equivalent, it follows that **CAR** is in fact the logic corresponding to the static semantics of $\mathcal{M}_2$:

Corollary 5.8 CAR *is sound and complete with respect to the static semantics of* $\mathcal{M}_2$*, i.e.* $\vdash^s_{\mathcal{M}_2} = \vdash_{CAR}$.

5.3 $\mathcal{M}_{MK}$ *and the Logics of McCarthy and Kleene*

Next, let us see what systems we get for the 3-valued Nmatrix $\mathcal{M}_{MK}$ from Example 2.9.

Dynamic semantics:

Negation:

$$\frac{\Omega, \mathbf{f} : \varphi}{\Omega, \mathbf{t} : \neg\varphi} \qquad \frac{\Omega, \mathbf{e} : \varphi}{\Omega, \mathbf{e} : \neg\varphi} \qquad \frac{\Omega, \mathbf{t} : \varphi}{\Omega, \mathbf{f} : \neg\varphi}$$

[7] Proof systems for this logic, though not its semantics, were already considered by Curry in [20].

Disjunction: We get the following nine rules:

$$(\mathbf{f}\vee\mathbf{f})\quad \frac{\Omega,\mathbf{f}:\varphi\quad \Omega,\mathbf{f}:\psi}{\Omega,\mathbf{f}:\varphi\vee\psi}\qquad\qquad (\mathbf{f}\vee\mathbf{e})\quad \frac{\Omega,\mathbf{f}:\varphi\quad \Omega,\mathbf{e}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi}$$

$$(\mathbf{f}\vee\mathbf{t})\quad \frac{\Omega,\mathbf{f}:\varphi\quad \Omega,\mathbf{t}:\psi}{\Omega,\mathbf{t}:\varphi\vee\psi}\qquad\qquad (\mathbf{e}\vee\mathbf{f})\quad \frac{\Omega,\mathbf{e}:\varphi\quad \Omega,\mathbf{f}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi}$$

$$(\mathbf{e}\vee\mathbf{e})\quad \frac{\Omega,\mathbf{e}:\varphi\quad \Omega,\mathbf{e}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi}\qquad\qquad (\mathbf{e}\vee\mathbf{t})\quad \frac{\Omega,\mathbf{e}:\varphi\quad \Omega,\mathbf{t}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi,\mathbf{t}:\varphi\vee\psi}$$

$$(\mathbf{t}\vee\mathbf{f})\quad \frac{\Omega,\mathbf{t}:\varphi\quad \Omega,\mathbf{f}:\psi}{\Omega,\mathbf{t}:\varphi\vee\psi}\qquad\qquad (\mathbf{t}\vee\mathbf{e})\quad \frac{\Omega,\mathbf{t}:\varphi\quad \Omega,\mathbf{e}:\psi}{\Omega,\mathbf{t}:\varphi\vee\psi}$$

$$(\mathbf{t}\vee\mathbf{t})\quad \frac{\Omega,\mathbf{t}:\varphi\quad \Omega,\mathbf{t}:\psi}{\Omega,\mathbf{t}:\varphi\vee\psi}$$

The four rules with the conclusion $\Omega,\mathbf{t}:\varphi\vee\psi$ can be combined using our reduction principles to just one rule. Likewise, the rules $(\mathbf{e}\vee\mathbf{f})$ and $(\mathbf{e}\vee\mathbf{e})$ can be combined too. Hence the set of nine rules given above can be reduced to the following five:

$$\frac{\Omega,\mathbf{f}:\varphi\quad \Omega,\mathbf{f}:\psi}{\Omega,\mathbf{f}:\varphi\vee\psi}\qquad\qquad \frac{\Omega,\mathbf{f}:\varphi\quad \Omega,\mathbf{e}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi}$$

$$\frac{\Omega,\mathbf{f}:\varphi,\mathbf{t}:\varphi\quad \Omega,\mathbf{t}:\varphi,\mathbf{t}:\psi}{\Omega,\mathbf{t}:\varphi\vee\psi}\qquad\qquad \frac{\Omega,\mathbf{e}:\varphi\quad \Omega,\mathbf{f}:\psi,\mathbf{e}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi}$$

$$\frac{\Omega,\mathbf{e}:\varphi\quad \Omega,\mathbf{t}:\psi}{\Omega,\mathbf{e}:\varphi\vee\psi,\mathbf{t}:\varphi\vee\psi}$$

which in the sequent notation can be written as:

$$\frac{\Gamma_{\mathbf{f}},\varphi|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}},\psi|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}},\varphi\vee\psi|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}}}\qquad\qquad \frac{\Gamma_{\mathbf{f}},\varphi|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\psi\Rightarrow\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\vee\psi\Rightarrow\Gamma_{\mathbf{t}}}$$

$$\frac{\Gamma_{\mathbf{f}},\varphi|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\varphi\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\varphi,\psi}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\varphi\vee\psi}\qquad\qquad \frac{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}},\psi|\Gamma_{\mathbf{e}},\psi\Rightarrow\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\vee\psi\Rightarrow\Gamma_{\mathbf{t}}}$$

$$\frac{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\psi}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\vee\psi\Rightarrow\Gamma_{\mathbf{t}},\varphi\vee\psi}$$

Static semantics: The only non-deterministic value of a connective we have here is $\mathbf{e}\vee\mathbf{t}=\{\mathbf{e},\mathbf{t}\}$. Hence together with the basic axioms for constants we obtain the following additional axioms for the static semantics: $\{\mathbf{f}:\underline{\mathbf{f}}\}$, $\{\mathbf{e}:\underline{\mathbf{e}}\}$, $\{\mathbf{t}:\underline{\mathbf{t}}\}$, $\{\mathbf{e}:\underline{\mathbf{e}}\vee\underline{\mathbf{t}},\ \mathbf{t}:\underline{\mathbf{e}}\vee\underline{\mathbf{t}}\}$, which in the sequent notation take the form: $\underline{\mathbf{f}}|\ \Rightarrow\ $, $|\underline{\mathbf{e}}\Rightarrow\ $, $|\ \Rightarrow\underline{\mathbf{t}}$, $|\underline{\mathbf{e}}\vee\underline{\mathbf{t}}\Rightarrow\underline{\mathbf{e}}\vee\underline{\mathbf{t}}$. As to the inference rules, the only ones differing from the rules for the dynamic semantics are those corresponding to $\mathbf{e}\vee\mathbf{t}=\{\mathbf{e},\mathbf{t}\}$:

$$\frac{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\psi\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\underline{\mathbf{e}}\vee\underline{\mathbf{t}}\Rightarrow\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\vee\psi\Rightarrow\Gamma_{\mathbf{t}}}$$

$$\frac{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}},\varphi\Rightarrow\Gamma_{\mathbf{t}}\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\psi\quad \Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\underline{\mathbf{e}}\vee\underline{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}}\Rightarrow\Gamma_{\mathbf{t}},\varphi\vee\psi}$$

We shall now translate the above systems of 3-sequent inference rules and axioms to ordinary, 2-sided sequent calculi[8]. The translation is based on the observation that, for any formula φ and any valuation v in $\mathcal{M}_{MK}$, the following relationships hold:

$$\begin{array}{lll} v(\varphi) = \mathbf{t} & \text{iff} & v(\varphi) \in \mathcal{D} \\ v(\varphi) = \mathbf{e} & \text{iff} & v(\varphi) \in \mathcal{N} \ \& \ v(\neg\varphi) \in \mathcal{N} \\ v(\varphi) = \mathbf{f} & \text{iff} & v(\varphi) \in \mathcal{N} \ \& \ v(\neg\varphi) \in \mathcal{D} \end{array} \tag{5.1}$$

where $\mathcal{D} = \{\mathbf{t}\}, \mathcal{N} = \{\mathbf{f}, \mathbf{e}\}$ are the sets of designated and non-designated values of $\mathcal{M}_{MK}$, respectively. Now, for any valuation v, v satisfies an ordinary sequent $\Gamma \Rightarrow \Delta$ if either $v(\varphi) \in \mathcal{N}$ for some $\varphi \in \Gamma$, or $v(\psi) \in \mathcal{D}$ for some $\psi \in \Gamma$. It easily follows therefore from (5.1) above that v satisfies $\Gamma_{\mathbf{f}}|\Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}$ *if and only if* it satisfies all the ordinary sequents of the form $\Gamma'_{\mathbf{f}}, \Gamma'_{\mathbf{e}}, \neg\Gamma''_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}, \neg\Gamma''_{\mathbf{f}}$, where $\Gamma_l = \Gamma'_l \uplus \Gamma''_l$ for $l \in \{\mathbf{f}, \mathbf{e}\}$. Based on this, we translate the 3-sequent inference rules as follows. Given a 3-sequent rule $R = \dfrac{S}{\Sigma}$, we translate each 3-sequent in the premises S and in the conclusion Σ to the equivalent set of ordinary sequents defined above, obtaining the sets S' and Σ' of ordinary sequents, respectively. Then a rule R is replaced by the equivalent set of 2-sequent rules which allow us to derive each sequent in Σ' out of the sequents in S'.

By way of example, consider the rule

$$\frac{\Gamma_{\mathbf{f}}, \varphi|\Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}}, \psi|\Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}, \varphi \vee \psi|\Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}}$$

Translating the premises and the conclusion to sets of ordinary sequents, we get as a result the following two rules of the ordinary sequent calculus:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

It can be directly checked that these four-premise rules can be optimized by deleting part of their premises while preserving their semantic validity, which yields:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

Proceeding in a similar way with all rules and axioms of the 3-sequent calculi presented above, we get the ordinary sequent calculi given below.

Dynamic semantics: The system GMK^d is defined as follows:

Axioms: $\varphi \Rightarrow \varphi$

Inference rules:

[8] As noted above, the translation presented here is a particular instance of a general method described in [9].

(1) $\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$ (2) $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}$ (3) $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$

(4) $\dfrac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$ (5) $\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$ (6) $\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$

(7) $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$ (8) $\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$

Static semantics: The system GMK^{sc} is obtained from GMK^d by augmenting it with:

Additional axioms: $\underline{\mathbf{f}} \Rightarrow,\ \underline{\mathbf{e}} \Rightarrow,\ \neg\underline{\mathbf{e}} \Rightarrow,\ \Rightarrow \underline{\mathbf{t}}$

Additional inference rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \neg\varphi \Rightarrow \Delta \qquad \Gamma, \underline{\mathbf{e}} \vee \underline{\mathbf{t}} \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \underline{\mathbf{e}} \vee \underline{\mathbf{t}}}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Theorem 5.9 *The two-sided calculi GMK^d, GMK^{sc} are sound and complete for the dynamic and the static semantic of $\mathcal{M}_{MK}$, respectively, and the cut rule is admissible in them.*

Proof. Soundness of the above systems follows immediately from soundness of the original 3-sequent calculi out of which they have been obtained, since our translation preserves validity of rules. To show completeness and admissibility of cuts assume e.g. that $\Gamma \Rightarrow \Delta$ is valid according to the dynamic semantics. Then $\Gamma \vdash^d_{\mathcal{M}_{MK}} \Delta$. It follows by Proposition 3.3 that the 3-sequent $\Gamma \mid \Gamma \Rightarrow \Delta$ is valid, and so provable in $SF^d_{\mathcal{M}_{MK}}$. Any such proof of $\Gamma \mid \Gamma \Rightarrow \Delta$ can be directly translated by our procedure to a cut-free proof in GMK^d of any of its 2-sequent translations. In particular, $\Gamma \Rightarrow \Delta$ has such a proof.

∎

Using similar procedures, we can obtain sequent calculi for Kleene and McCarthy logics. As we have mentioned when introducing $\mathcal{M}_{MK}$, the resulting systems correspond to GMK^d with one disjunction rule strengthened in each case. Namely, to obtain McCarthy logic, we replace rule (4) above with

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \neg\varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

while for Kleene logic we delete the first premise in rule (7), obtaining the rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Finally, it should be noted that except (4), the two-sided rules developed above either coincide with the rules developed for McCarthy logic in [24], or can be shown to be mutually derivable with them. For example, rule (7) can be "split" using the converse of Principle 3 into the following rules from [24]:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

5.4 Dynamic Semantics of $\mathcal{M}_L^3$ and $\mathcal{M}_S^3$: the Logic $\mathbf{C}_{\min}$

From now on, we will only use the sequent notation in the proof systems for our exemplary logics. Let us turn to the dynamic semantics of the Nmatrices $\mathcal{M}_L^3, \mathcal{M}_S^3$ from Example 2.10. In case of disjunction, the basic 3-sequent rules of Sect. 3.1 are as follows:

$$(\mathbf{f} \vee \mathbf{f}) \quad \frac{\Gamma_{\mathbf{f}}, \varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}}, \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}, \varphi \vee \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}} \qquad (\mathbf{f} \vee \top) \quad \frac{\Gamma_{\mathbf{f}}, \varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \psi | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

$$(\mathbf{f} \vee \mathbf{t}) \quad \frac{\Gamma_{\mathbf{f}}, \varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi} \qquad (\top \vee \mathbf{f}) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}}, \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

$$(\top \vee \top) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \psi | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi} \qquad (\top \vee \mathbf{t}) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

$$(\mathbf{t} \vee \mathbf{f}) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \varphi \quad \Gamma_{\mathbf{f}}, \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi} \qquad (\mathbf{t} \vee \top) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \varphi \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \psi | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

$$(\mathbf{t} \vee \mathbf{t}) \quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \varphi \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

Using again reduction principles 1–3 we can reduce them to just two rules, analogous to the standard ones:

$$\frac{\Gamma_{\mathbf{f}}, \varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}}, \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}, \varphi \vee \psi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi, \psi | \Gamma_{\mathbf{t}}, \varphi, \psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi \vee \psi | \Gamma_{\mathbf{t}}, \varphi \vee \psi}$$

The difference between the two Nmatrices $\mathcal{M}_L^3, \mathcal{M}_S^3$ concerns only negation. Accordingly, the corresponding systems differ only in the negation rules. Thus for the liberal case we get the following 3 rules:

$$\frac{\Gamma_{\mathbf{f}}, \varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \neg\varphi} \qquad \left\{ \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top, \varphi | \Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}, \neg\varphi \Rightarrow \Gamma_\top, \neg\varphi | \Gamma_{\mathbf{t}}, \neg\varphi} \right\} \qquad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}, \varphi}{\Gamma_{\mathbf{f}}, \neg\varphi \Rightarrow \Gamma_\top | \Gamma_{\mathbf{t}}}$$

However, the rule for the single non-deterministic value of the negation of $\mathcal{M}_L^3$ (the one enclosed in brackets) is trivial (since its conclusion is valid), and can be deleted. The final system $SF_{\mathcal{M}_L^3}$ for $\mathcal{M}_L^3$ we get is:

$$(\vee\Rightarrow)\quad \frac{\Gamma_{\mathbf{f}},\varphi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}} \quad \Gamma_{\mathbf{f}},\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}},\varphi\vee\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi,\psi|\Gamma_{\mathbf{t}},\varphi,\psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi\vee\psi|\Gamma_{\mathbf{t}},\varphi\vee\psi}\quad(\Rightarrow\vee)$$

$$(\wedge\Rightarrow)\quad \frac{\Gamma_{\mathbf{f}},\varphi,\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}},\varphi\wedge\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi|\Gamma_{\mathbf{t}},\varphi \quad \Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\psi|\Gamma_{\mathbf{t}},\psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi\wedge\psi|\Gamma_{\mathbf{t}},\varphi\wedge\psi}\quad(\Rightarrow\wedge)$$

$$(\supset\Rightarrow)\quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi|\Gamma_{\mathbf{t}},\varphi \quad \Gamma_{\mathbf{f}},\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}},\varphi\supset\psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}},\varphi \Rightarrow \Gamma_\top,\psi|\Gamma_{\mathbf{t}},\psi}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi\supset\psi|\Gamma_{\mathbf{t}},\varphi\supset\psi}\quad(\Rightarrow\supset)$$

$$(\neg\Rightarrow)\quad \frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}},\varphi}{\Gamma_{\mathbf{f}},\neg\varphi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}},\varphi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}},\neg\varphi}\quad(\Rightarrow\neg)$$

Note 5.10 It is easy to see, by just inspecting the rules of $SF_{\mathcal{M}^3_L}$, that if $\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}$ is provable in $SF_{\mathcal{M}^3_L}$ then $\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top \cup \Gamma_{\mathbf{t}}$ is provable in classical logic. The converse fails, of course: $\neg p, p \Rightarrow$ is classically valid, but $\neg p, p \Rightarrow\ |$ is not valid in $\mathcal{M}^3_L$, and so it is not provable in $SF_{\mathcal{M}^3_L}$.

The system $SF_{\mathcal{M}^3_S}$ for $\mathcal{M}^3_S$ is obtained from $SF_{\mathcal{M}^3_L}$ by adding the rule

$$\frac{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\varphi|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}} \Rightarrow \Gamma_\top,\neg\varphi|\Gamma_{\mathbf{t}},\neg\varphi}$$

(corresponding to the single non-deterministic value of negation in $\mathcal{M}^3_S$).

We can now use the systems $SF_{\mathcal{M}^3_L}$ and $SF_{\mathcal{M}^3_S}$ we have just developed to provide an ordinary sequential system equivalent to both, showing in this way that the corresponding logics are in fact identical. This time the crucial observation is that, for any formula φ and any valuation v in $\mathcal{M}^3_S$, the following relationships hold:

$$\begin{array}{lll} v(\varphi)=\mathbf{t} & \text{iff} & v(\varphi)\in\mathcal{D}\ \&\ v(\neg\varphi)\in\mathcal{N} \\ v(\varphi)=\top & \text{iff} & v(\varphi)\in\mathcal{D}\ \&\ v(\neg\varphi)\in\mathcal{D} \\ v(\varphi)=\mathbf{f} & \text{iff} & v(\varphi)\in\mathcal{N} \end{array} \tag{5.2}$$

where now $\mathcal{D}=\{\mathbf{t},\top\}, \mathcal{N}=\{\mathbf{f}\}$. Like in the case of $\mathcal{M}_{MK}$, these relationships can be used (again with the method of [9] — see details there) to translate the system $SF_{\mathcal{M}^3_S}$ (though not $SF_{\mathcal{M}^3_L}$) to the following cut-free, ordinary sequent calculus:

Definition 5.11 *Let GC_{min} be the system obtained by augmenting the standard (cut-free) Gentzen-type system for positive classical logic with the following negation rules:*

$$\frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma,\neg\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow,\Delta,\neg\varphi}$$

Theorem 5.12 *GC_{min} is sound and complete for both $\vdash^d_{\mathcal{M}^3_L}$ and $\vdash^d_{\mathcal{M}^3_S}$, and the cut rule is admissible in it.*

Proof. That GC_{min} admits cut-elimination, and that it is sound and complete for $\vdash^d_{\mathcal{M}^3_S}$, follows from its being a translation of $SF_{\mathcal{M}^3_S}$ (the proof is similar to that of theorem 5.9, so we omit the details). It can be directly checked that GC_{min} is sound for $\vdash^d_{\mathcal{M}^3_L}$ too. Since $\vdash^d_{\mathcal{M}^3_S}$ is a refinement of $\vdash^d_{\mathcal{M}^3_L}$, we have $\vdash_{GC_{min}} \subseteq \vdash^d_{\mathcal{M}^3_L} \subseteq \vdash^d_{\mathcal{M}^3_S} \subseteq \vdash_{GC_{min}}$. Hence all the three logics are identical. ∎

Corollary 5.13 $\vdash^d_{\mathcal{M}^3_L} = \vdash^d_{\mathcal{M}^3_S}$.

Note 5.14 Despite the fact that $SF_{\mathcal{M}^3_L}$ and $SF_{\mathcal{M}^3_S}$ define the same logic (i.e. the same consequence relation), these systems are *not* equivalent. Thus $\Rightarrow \neg\varphi|\neg\varphi, \varphi$ is a theorem of $SF_{\mathcal{M}^3_S}$, but in general not of $SF_{\mathcal{M}^3_L}$.

Note 5.15 The two rules for negation in GC_{min} obviously translate to $\neg\varphi \vee \varphi$ and $\neg\neg\varphi \supset \varphi$. Hence GC_{min} is equivalent to the Hilbert-type system obtained by adding these two axiom schemes to a standard Hilbert-type formulation of positive classical logic. The result, usually denoted by $\mathbf{C_{min}}$, is one of the basic paraconsistent logics (see e.g. [16]). Thus both $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ provide sound and complete semantics for $\mathbf{C_{min}}$. Note that the goal of translating $SF_{\mathcal{M}^3_S}$ to an ordinary Gentzen-type system, together with the induction in the related proof, inevitably lead to GC_{min} and $\mathbf{C_{min}}$ (and that the admissibility of the cut rule in GC_{min} inevitably follows from the corresponding property of $SF_{\mathcal{M}^3_S}$).[9]

5.5 Dynamic Semantics of the Four-valued $\mathcal{M}_4$

Finally, consider the 4-valued Nmatrix defined in Example 2.11. The original deduction system for the dynamic semantics generated using our general method has 16 (4×4) rules for each binary connective, and 4 rules for negation. However, the rules for each binary connective can again be reduced to two. As a result, for the $\{\supset, \neg\}$-fragment we get:

$$(\supset)\quad \frac{\Gamma_{\mathbf{f}}, \varphi|\Gamma_\bot, \varphi \Rightarrow \Gamma_\top, \psi|\Gamma_{\mathbf{t}}, \psi}{\Gamma_{\mathbf{f}}|\Gamma_\bot \Rightarrow \Gamma_\top, \varphi \supset \psi|\Gamma_{\mathbf{t}}, \varphi \supset \psi} \qquad \frac{\Gamma_{\mathbf{f}}|\Gamma_\bot \Rightarrow \Gamma_\top, \varphi|\Gamma_{\mathbf{t}}, \varphi \quad \Gamma_{\mathbf{f}}, \psi|\Gamma_\bot, \psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}, \varphi \supset \psi|\Gamma_\bot, \varphi \supset \psi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}$$

$$(\neg)\quad \frac{\Gamma_{\mathbf{f}}, \varphi|\Gamma_\bot \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_\bot \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}, \neg\varphi} \qquad \frac{\Gamma_{\mathbf{f}}|\Gamma_\bot, \varphi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_\bot, \neg\varphi \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}$$

$$\frac{\Gamma_{\mathbf{f}}|\Gamma_\bot \Rightarrow \Gamma_\top, \varphi|\Gamma_{\mathbf{t}}}{\Gamma_{\mathbf{f}}|\Gamma_\bot \Rightarrow \Gamma_\top, \neg\varphi|\Gamma_{\mathbf{t}}} \qquad \frac{\Gamma_{\mathbf{f}}|\Gamma_\bot|\Gamma_\top \Rightarrow \Gamma_{\mathbf{t}}, \varphi}{\Gamma_{\mathbf{f}}, \neg\varphi|\Gamma_\bot \Rightarrow \Gamma_\top|\Gamma_{\mathbf{t}}}$$

Using methods similar to those employed in Section 5.3 (again see [9] for details), the above complete 4-sequent calculus for $\mathcal{M}_4$ can once more be translated into an equivalent ordinary sequent calculus. This time it is the standard Gentzen-type system for positive classical logic augmented with two rules for double negation:

[9] The fact that $\mathbf{C_{min}}$ is sound and complete for $\mathcal{M}^3_S$ was first proved in [3, 4]. In [5] $\mathcal{M}^3_S$ was even *constructed* from GC_{min} by a general procedure for deriving characteristic Nmatrices for a certain family of paraconsistent logics (As noted in the introduction, what is done here is exactly the converse). In contrast, the soundness and completeness of $\mathbf{C_{min}}$ with respect to $\mathcal{M}^3_L$ is new here.

Theorem 5.16 *Let GM_4 be the proof system obtained by augmenting the standard Gentzen-type system for positive classical logic with the following two rules for negation:*

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}, \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

Then GM_4 is sound and complete for the dynamic semantics of $\mathcal{M}_4$, and the cut rule is admissible in it.[10]

Note 5.17 As the two negation rules of GM_4 translate to $\varphi \supset \neg\neg\varphi$ and $\neg\neg\varphi \supset \varphi$, the logic of $\mathcal{M}_4$ can be axiomatized by adding these two axioms to a standard Hilbert-type formulation of positive classical logic.

6 Related Work

The concept of a non-deterministic matrix, together with the associated dynamic semantics, was first introduced and used in [2, 3, 4]. Two precursors that should be mentioned are the special two-valued Nmatrix $\mathcal{M}_2$ from Example 1, which was essentially used already in [10], and the particular 3-valued instance of the same idea (with a similar name) which has been used in [19].[11] As has been noted in the introduction, an extensive investigation of dynamic finite Nmatrices, with a lot of applications, was carried out in [5, 6, 7], with [5] discussing applications of 3-valued Nmatrices, [6] — mainly of 4-valued ones, and [7] — mainly of 5-valued ones. Some applications of *infinite*-valued Nmatrices (in which they cannot be replaced by any finite ones) were given in [8].

Two related semantic frameworks that also have some nondeterministic aspects are bivaluations (see e.g. [12, 17]), and Carnielli's possible-translations semantics (PTS), which has been used extensively by Carnielli, Marcos and others to provide semantics for various logics (see [15, 17, 27]) [12]. However, both frameworks, as they are currently defined and used, are far too general, being practically able to provide semantics for every propositional logic[13]. As a result, no general theorems concerning decidability and compactness, of the type proved for finite Nmatrices in [7], hold for them. Moreover: no general method for developing proof systems from these types of semantics (like the one we have provided here for Nmatrices) exists, or even seems possible without restricting them somehow. Still, it seems that, with appropriate restrictions, PTS might provide a powerful generalization of the use of Nmatrices. This should be an important subject for future research.

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[10] This theorem is one of the hundreds results of this sort which were proved in [6]. Again we emphasize that what is important and new in the present paper is not Theorem 5.16 itself, but the way it is derived here.

[11] The first was the direct inspiration of [2, 3, 4]. About the second we learned from J. Marcos only after [2] had been published.

[12] PTS was in fact originally called "non-deterministic semantics" (see [15]), but later its name has been changed to the present one.

[13] Thus any valuation v in an Nmatrix trivially defines a bivaluation v_b where $v_b(\psi) = t$ if $v(\psi)$ is designated, $v_b(\psi) = f$ otherwise.

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