Approachability ideal - on a question of Jakob and Poveda

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Abstract

We give a positive answer to a question of H. Jakob and A. Poveda by constructing a model in which for a singular cardinal λ of uncountable cofinality for every regular cardinal $\eta \in (\mathrm{cof}(\lambda), \lambda), \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+]$.

1 Introduction

H. Jakob and M. Levine [5] answered a long standing question of S. Shelah [8], p.439, by showing consistency of $\{\delta < \aleph_{\omega+1} \mid \mathrm{cof}(\delta) = \aleph_2\} \notin I[\aleph_{\omega+1}]$. Generalizations followed in [4] and [6]. In [6] the following question was asked:

Is it consistent to have a singular cardinal λ of uncountable cofinality for every regular cardinal $\eta \in (\mathrm{cof}(\lambda), \lambda), \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+]$.

The purpose of the present paper is to provide an affirmative answer.
We will follow lines of [5], [4] and [6]. The main issue will be to replace product of one element Prikry type forcings by a full support (Magidor) iteration without leaving gaps. This will require reworking some specific parts of arguments of [5], [4] and [6]. Also some new Prikry type forcing will be required at basic stages.

2 On Strong Prikry Condition

Let $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \tau, \beta < \tau\rangle$ be a full support (Magidor) iteration of Prikry-type forcing notions. The following lemma is similar to Lemma 6.2 from [2].

Lemma 2.1 *Let $p = \langle \dot{p}_\gamma \mid \gamma < \tau\rangle \in P_\tau$ and $D \subseteq P_\tau$ be dense open. Then there are:*

1. *$p^* \geq^* p$,*
2. *$P_\tau-$name $\dot{n}$ of a natural number,*
3. *$P_\tau-$names $\langle \dot{\gamma}_i \mid i < \dot{n}\rangle$ of ordinals below τ,*
4. *$P_\tau-$names $\langle \dot{D}(\gamma_i) \mid i < \dot{n}\rangle$ of dense open subsets of $\langle \dot{Q}_{\gamma_i} \mid i < \dot{n}\rangle$,*

such that for every $r \geq p^*$ which decides $\dot{n}$ and $\langle \dot{\gamma}_i \mid i < \dot{n} \rangle$,
if *$\langle \dot{q}_{\gamma_i} \mid i < n \rangle$ are such that for every $i' < i$, $r^\frown \langle \dot{q}_{\gamma_{i'}} \mid i' < i \rangle \Vdash \dot{q}_{\gamma_i} \geq \dot{r}_{\gamma_i}$ and $\dot{q}_{\gamma_i} \in \dot{D}(\gamma_i)$,*
then *$r^\frown \langle \dot{q}_{\gamma_i} \mid i < n \rangle \in D$.*

Proof. Deal first with a successor case. Let $\tau = \tau' + 1$. Let $G_{\tau'} \subseteq P_{\tau'}$ be a generic filter with $p \restriction \tau' \in G_{\tau'}$.
Set $D(\tau') = D[G_{\tau'}] = \{q_{\tau'} \in Q_{\tau'} \mid \exists q \in G_{\tau'} \quad q^\frown q_{\tau'} \in D\}$.
In V, the following set $D_{\tau'} \subseteq P_{\tau'}$ will be dense open above $p \restriction \tau'$:

$$D_{\tau'} = \{q \in P_{\tau'} \mid \exists \dot{q}_{\tau'} \quad q \Vdash \dot{q}_{\tau'} \in \dot{D}(\tau')\}.$$

Apply the lemma to $P_{\tau'}$.

Suppose now that τ is a limit ordinal. Assume that there is no direct extension of p which satisfies the conclusion of the lemma. We define by recursion on $\beta < \tau$:

$$p(\beta) = \langle p^*_\gamma \mid \gamma < \beta \rangle ^\frown \langle p_\gamma \mid \beta \leq \gamma < \tau \rangle \geq^* p,$$

so that $p(\beta) \restriction \beta \Vdash \neg\sigma_\beta$, where:

$$\sigma_\beta \equiv (\exists \dot{q} \in P_\tau \setminus \beta \, (\dot{q} \geq^* p \setminus \beta \text{ and } \dot{q} \Vdash \text{the conclusion of the lemma})).$$

Suppose that $\langle p(\gamma) \mid \gamma < \beta \rangle$ are defined and $\leq^*$-increasing. We define $p(\beta)$.

Case 1. $\beta = \beta' + 1$.
Force with $P_{\beta'} = P_\tau \restriction \beta'$. Let $G_{\beta'} \subseteq P_{\beta'}$ be generic with $p(\beta') \restriction \beta' \in G_{\beta'}$. By the Prikry property of $Q_{\beta'}$, there is $p^*_{\beta'} \geq^*_{\beta'} p_{\beta'}$ which decides σ_β.

Claim 1 $p^*_{\beta'} \Vdash_{Q_{\beta'}} \neg\sigma_\beta$.

Proof. Suppose otherwise. Then:

$$p^*_{\beta'} \Vdash_{Q_{\beta'}} \exists \dot{q} \in P_\tau \setminus \beta \, (\dot{q} \geq^* p \setminus \beta \text{ and } \dot{q} \Vdash_{P_\tau \setminus \beta} \text{the conclusion of the lemma }).$$

Fix such a name $\dot{q}$. Consider the following set:

$$D_{\beta'} = \{r \in Q_{\beta'} \mid r^\frown \dot{q} \Vdash_{P_\tau \setminus \beta'} \text{the conclusion of the lemma }\}.$$

Then $D_{\beta'}$ is dense open above $p^*_{\beta'}$. So, we can add β' to the finite set of coordinates. This implies that some $r \in G_{\beta'}$ forces $\sigma_{\beta'}$, which contradicts the choice of $p(\beta')$.
$\square$ of the claim.

Case 2. β is a limit ordinal.
We need to show that $p(\beta) = \langle p^*_\gamma \mid \gamma < \beta\rangle \frown \langle p_\gamma \mid \beta \leq \gamma < \tau\rangle$ is as desired, i.e., $p(\beta) \restriction \beta \Vdash \neg\sigma_\beta$.

Suppose otherwise. Then there is a condition $r = \langle \dot r_\gamma \mid \gamma < \beta\rangle \in P_\beta$ such that $r \geq p(\beta) \restriction \beta$ and $r \Vdash \sigma_\beta$. Extend it, if necessary, so that for some $\dot q$:

$$r \Vdash \dot q \geq^* p \setminus \beta \text{ and } \dot q \text{ forces the conclusion of the lemma.}$$

By the definition of the order on P_β, there is $\beta^* < \beta$ such that for every γ with $\beta^* \leq \gamma < \beta$, $r \restriction \gamma \Vdash \dot r_\gamma \geq^* \dot p^*_\gamma$.

Consider a P_{β^*}-name:

$$\dot q' = \langle \dot r_\gamma \mid \beta^* \leq \gamma < \beta\rangle \frown \dot q.$$

Then:

$$r \restriction \beta^* \Vdash (\dot q' \geq^* p \setminus \beta^* \text{ and } \dot q' \Vdash_{P_\tau \setminus \beta^*} \text{the conclusion of the lemma}).$$

Apply the lemma for the shorter iteration in the interval $[\beta^*, \beta)$ for the restriction of D to this interval. This will give a contradiction to the assumption that $p(\beta^*) \restriction \beta^* \Vdash \neg\sigma_{\beta^*}$.

This completes the construction. Consider $p(\tau) = \langle p^*_\gamma \mid \gamma < \tau\rangle$. We obtain a contradiction as in Case 2. This completes the proof of the lemma.
□

A stronger statement can be proved, assuming that we have enough closure of components of the iteration.

Lemma 2.2 *Suppose that for every $\beta < \tau$, $\langle \dot Q_\beta, \leq^*_\beta\rangle$ is $\leq \tau-$closed forcing. Then there are:*

1. *$p^* \geq^* p$,*
2. *$n < \omega$,*
3. *an increasing sequence $\langle \gamma_i \mid i < n\rangle$ of ordinals below τ,*
4. *$P_\tau-$names $\langle \dot D(\gamma_i) \mid i < n\rangle$ of dense open subsets of $\langle \dot Q_{\gamma_i} \mid i < n\rangle$,*

such that:
if *$\langle \dot q_{\gamma_i} \mid i < n\rangle$ are such that for every $i' < i$, $p^* \frown \langle \dot q_{\gamma_{i'}} \mid i' < i\rangle \Vdash \dot q_{\gamma_i} \geq \dot p^*_{\gamma_i}$ and $\dot q_{\gamma_i} \in \dot D(\gamma_i)$,*
then *$p^* \frown \langle \dot q_{\gamma_i} \mid i < n\rangle \in D$.*

Proof. The proof repeats those of 2.1, only at each stage we decide the relevant finite sequence of coordinates by taking a direct extension. This is possible since the required degree of closure implies that no new subsets are added to τ.
□

Remark 2.3 Note that if Q_{γ_i} satisfies strong Prikry condition, then we can pick $D(\gamma_i)$ accordingly.

3 Forcing Conditions

We define here particular forcing notion which will be used.

Assume GCH. Let τ be a regular cardinal and $\langle \rho_\xi \mid \xi < \tau \rangle$ be an increasing sequence of regular cardinals such that:

1. $\tau < \rho_0$,
2. if ξ, $0 < \xi < \tau$, is a limit ordinal, then $\rho_\xi > \left(\bigcup_{\xi' < \xi} \rho_{\xi'}\right)^+$.

We consider the following full (Magidor) support form of iteration

$$\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \tau, \beta < \tau \rangle$$

of Prikry type forcing notions such that for every $\beta < \tau$ and $\alpha \leq \tau$ the following hold:

1. $0_{P_\beta} \Vdash \langle \dot{Q}_\beta, \leq_{\dot{Q}_\beta}, \leq^*_{\dot{Q}_\beta} \rangle = \langle Q_\beta, \leq, \leq^* \rangle$, for some one-element Prikry forcing $\langle Q_\beta, \leq, \leq^* \rangle$ over ρ_β in V,
2. $\leq^*_\beta$ is $< \rho_{\beta-1}$-closed (or strategically closed) if β is a successor ordinal, and $\leq \bigcup_{\xi<\beta} \rho_\xi$-closed if β is a limit ordinal,
3. P_α consists of α-sequences $q = \langle \dot{q}_\xi \mid \xi < \alpha \rangle$ such that:
   (a) for all but finitely many ξ's, $\dot{q}_\xi$ is a positive set in the filter $\mathcal{U}_\xi \in V$; outside of this set, $\dot{q}_\xi$ is an ordinal $< \rho_\xi$,
   (b) for every limit $\xi < \alpha$, $q_\xi \in V$,
   (c) for every successor $\xi < \alpha$, $\dot{q}_\xi$ is a P_ξ-name of the following type:

$$\{\langle \check{\nu}, \dot{A}^\nu \rangle \mid \nu < \rho_{\xi-1},\ \nu \in q_{\xi-1} \text{ if it is not an ordinal,}$$

$$A^\nu \text{ a positive set with respect to } \mathcal{U}_\xi\},$$

   i.e., the decision about a positive set $\dot{q}_\xi$ depends only on one element Prikry sequence over $\rho_{\xi-1}$.

We define the forcing order $\leq$ and the direct extension order $\leq^*$ on P_α's in the usual fashion according to the coordinates.

The next lemma follows from the choice of $\langle \rho_\alpha \mid \alpha < \tau \rangle$ and the definition of the forcing.

Lemma 3.1 *$\langle P_\tau, \leq^* \rangle$ is ρ_0-closed. Moreover, for every $\alpha, 0 < \alpha < \tau$, $\langle P_\tau \setminus P_{\alpha+1}, \leq^* \rangle$ is $\leq \rho_\alpha$-closed if α is a limit ordinal and $< \rho_{\alpha-1}$-closed if α is a successor ordinal.*

3.1 Prikry Condition

Lemma 3.2 *For every $\alpha \leq \tau$, $\langle P_\alpha, \leq, \leq^* \rangle$ satisfies the Prikry condition, i.e., for every $q \in P_\alpha$ and every statement σ of the forcing language, there is $p \leq^* q$ which decides σ.*

Proof. We prove the lemma by induction on α. Assume that for every $\alpha' < \alpha$, $\langle P_{\alpha'}, \leq, \leq^* \rangle$ satisfies the Prikry condition. We need to show that then $\langle P_\alpha, \leq, \leq^* \rangle$ satisfies it as well.

Suppose otherwise. Then for some $q \in P_\alpha$ and a statement σ there is no $p \leq^* q$ deciding σ. Assume for simplicity that q does not have one-element Prikry sequences.

Let $q = \langle \dot{q}_\xi \mid \xi < \alpha \rangle$. Pick the least $\nu \in q_0$. Consider

$$q^\frown \nu = \langle \nu, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_\xi, \cdots \mid 0 < \xi < \alpha \rangle.$$

We ask whether there is $p \geq^* q^\frown \nu$ which decides σ. If there is no such p, then set $q^1 = q$. Otherwise, pick such p and set

$$q^1 = \langle q_0, p_1{}^\frown q_1, p_2, \ldots, p_\xi, \cdots \mid 1 < \xi < \alpha \rangle.$$

Continue, pick $\nu_1 = \min(q_0 \setminus \{\nu\})$. Consider

$$q^1{}^\frown \nu_1 = \langle \nu_1, q^1_1, \dot{q}^1_2, \ldots, \dot{q}^1_\xi, \cdots \mid 0 < \xi < \alpha \rangle.$$

If there is no $p \geq^* q^1{}^\frown \nu_1$ which decides σ, then let $q^2 = q^1$. Otherwise, we pick $p \geq^* q^1{}^\frown \nu_1$ which decides σ and set

$$q^2 = \langle q_0, p_1{}^\frown q^1_1, p_2, \ldots, p_\xi, \cdots \mid 1 < \xi < \alpha \rangle.$$

Continue further in a similar fashion. At limit stages take intersections of corresponding positive sets. Note that we have enough closure in order to do this.

Finally, q^{ρ_0} will be constructed. Using the induction assumption, we can find $q^*_0 \geq^* q_0$ such that q^*_0 decides (in P_1) whether $q^{\rho_0} \setminus 1$ decides σ and if it decides, which way.

By the assumption we made about P_α, q^*_0 must force that $q^{\rho_0} \setminus 1$ does not decide σ. Let $q(1) = (q^{\rho_0} \setminus 1)^\frown q^*_0$.

Now we repeat the process above only going through all possibilities over the first two coordinates of $q(1)$, i.e., through all the conditions in P_2 which extend $\langle q(1)_0, \dot{q}(1)_1 \rangle$. This way $q(1)^{\rho_2}$ and then $q(2)$ are constructed.

Continue the process through all coordinates. Finally $q(\alpha)$ will be obtained. Pick some $p \geq q(\alpha)$ which decides σ. By the definition of the order $\leq$, a non-direct extension is used only at finitely many coordinates. This easily gives the desired contradiction. □

The next lemma is similar.

Lemma 3.3 *For every $\alpha \leq \tau$, $\langle P_\alpha, \leq, \leq^* \rangle$ satisfies the strong Prikry condition. I.e., for every $q \in P_\alpha$ and a dense open subset D of P_α there are $p \geq^* q$ and a set $b \subseteq \alpha$ of finitely many coordinates such that any extension of p at these coordinates will be in D.*

4 Blocks with direct order extension

Let $\langle \rho_\xi \mid \xi < \tau \rangle$ and $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \tau, \beta < \tau \rangle$ be as in the previous section and let $\eta, \tau < \eta < \rho_0$, be a regular cardinal.
Denote P_τ by P.

Consider $P * \mathrm{Col}(\eta, \rho_\tau^+)$. Further we will refer to such forcings as *basic blocks for* $\tau, \eta, \langle \rho_\xi \mid \xi < \tau \rangle$ and denote by $BB(\tau, \eta, \langle \rho_\xi \mid \xi < \tau \rangle)$.

Let $\langle p, \dot{c} \rangle \in P * \mathrm{Col}(\eta, \rho_\tau^+)$.

We call $\langle p, \dot{c} \rangle$ a *nice condition* iff:

there is $\nu < \eta$ such that $p \Vdash \mathrm{dom}(\dot{c}) = \check{\nu}$ such that
for every $\gamma < \nu$ (ν is as in (1)), there are $n^\gamma < \omega$, $i_1^\gamma < \cdots < i_{n^\gamma}^\gamma < \tau$, $\{\delta^\gamma(\vec{\mu}) < \rho_\tau^+ \mid \vec{\mu} \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}\}$ and $\dot{A}_1^\gamma, \ldots, \dot{A}_{n^\gamma}^\gamma$ such that:

(a) $p \Vdash \dot{A}_k^\gamma$ is a maximal antichain above p_γ in $\langle \dot{Q}_\gamma, \leq_\gamma \rangle$, for every k ($1 \leq k \leq n^\gamma$),

(b) for every $\vec{\mu} \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}$ with $p \Vdash \check{\vec{\mu}} \in \dot{A}_k^\gamma$, $p \frown \vec{\mu} \Vdash \dot{c}(\gamma) = \delta^\gamma(\vec{\mu})$.

Given a nice condition $\langle p, \dot{c} \rangle$, let us further replace a name $\dot{c}$, which can be in general rather complicated, by the canonical name determined by $n^\gamma < \omega$, $i_1^\gamma < \cdots < i_{n^\gamma}^\gamma < \tau$, $\{\delta^\gamma(\vec{\mu}) < \rho_\tau^+ \mid \vec{\mu} \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}\}$ and $\dot{A}_1^\gamma, \ldots, \dot{A}_{n^\gamma}^\gamma$, as above.

We argue that nice conditions are dense in $\langle P * \mathrm{Col}(\eta, \rho_\tau^+), \leq^* \rangle$.

Lemma 4.1 *Let $\langle q, \dot{e} \rangle \in P$, then there is a nice condition $\langle p, \dot{c} \rangle \geq^* \langle q, \dot{e} \rangle$.*

Proof. First find $\langle q, \dot{d} \rangle \geq^* \langle q, \dot{e} \rangle$ and a name $\dot{\nu}$ of an ordinal $< \eta$ such that $q \Vdash \mathrm{dom}(\dot{d}) = \dot{\nu}$.

Second, using the completeness of the order $\leq^*$, find $r \geq^* q$ and $\nu < \eta$ such that $r \Vdash \dot{\nu} = \check{\nu}$.

Then we proceed by induction and construct a $\leq^*$-increasing sequence of conditions $\langle r_\gamma \mid \gamma < \nu \rangle$ in P such that there are $n^\gamma < \omega$, $i_1^\gamma < \cdots < i_{n^\gamma}^\gamma < \tau$, $\{\delta^\gamma(\vec{\mu}) < \rho_\tau^+ \mid \vec{\mu} \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}\}$ and $\dot{A}_1^\gamma, \ldots, \dot{A}_{n^\gamma}^\gamma$ such that for every $\langle \mu_1, \ldots, \mu_{n^\gamma} \rangle \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}$, if $r_\gamma \Vdash \langle \mu_1, \ldots, \mu_{n^\gamma} \rangle \in \prod_{1 \leq k \leq n^\gamma} \dot{A}_k^\gamma$, then

$$r_\gamma \frown \langle \mu_1, \ldots, \mu_{n^\gamma} \rangle \Vdash \dot{d}(\gamma) = \delta^\gamma(\langle \mu_1, \ldots, \mu_{n^\gamma} \rangle).$$

Finally, let $p \geq^* r_\gamma$ for every $\gamma < \nu$, and $\dot{c} = \dot{d}$. Then $\langle p, \dot{c} \rangle$ is as desired.
□

Denote by D the set of all nice conditions of P. By the previous lemma, $\langle P, \leq^* \rangle$ is equivalent to $\langle D, \leq^* \rangle$.

Lemma 4.2 $|D| = \rho_\tau^+$.

Proof. We have $|P| = \rho_\tau^+$. Let us calculate the number of $\dot{c}$'s such that $\langle p, \dot{c} \rangle \in D$ for some p. $\dot{c}$ is completely determined by $\nu < \eta$, $\langle n^\gamma \mid \gamma < \nu \rangle$, $\langle \vec{i^\gamma} \mid \gamma < \nu \rangle$, where $\vec{i^\gamma} = \langle i_1^\gamma, \ldots, i_{n^\gamma}^\gamma \rangle$

is a finite sequence of ordinals $< \tau$, sequences $\langle \vec{\mu}^\gamma \mid \gamma < \nu \rangle$ where $\vec{\mu}^\gamma = \langle \mu_1^\gamma, \ldots, \mu_{n^\gamma}^\gamma \rangle \in \prod_{1 \leq k \leq n^\gamma} \rho_{i_k^\gamma}$ and sequences $\langle \vec{\xi}^\gamma \mid \gamma < \nu \rangle$ where $\vec{\xi}^{\gamma, \vec{\mu}^\gamma} = \langle \xi_1^{\gamma, \vec{\mu}^\gamma}, \ldots, \xi_{n^\gamma}^{\gamma, \vec{\mu}^\gamma} \rangle$ of ordinals $< \rho_\tau^+$.

The total number of possibilities does not exceed ρ_τ^+.

□

Lemma 4.3 *$\langle P * Col(\eta, \rho_\tau^+), \leq^* \rangle$ is equivalent to $Col(\eta, \rho_\tau^+)$.*

Proof. $\langle P * \mathrm{Col}(\eta, \rho_\tau^+), \leq^* \rangle$ is an η-closed forcing of cardinality ρ_τ^+. It is enough to show that it collapses ρ_τ^+ to η.

For every $\alpha < \rho_\tau^+$, set

$$D_\alpha = \left\{ \langle p, \dot{c} \rangle \in P * \mathrm{Col}(\eta, \rho_\tau^+) \;\middle|\; \exists \nu < \eta \text{ such that } p \Vdash \langle \nu, \alpha \rangle \in \dot{c} \right\}.$$

Then D_α is dense.

Clearly, $\langle D_\alpha \mid \alpha < \rho_\tau^+ \rangle$ ensures that ρ_τ^+ falls to η.

□

Suppose now that we force with Easton support iterations of blocks $P * \mathrm{Col}(\eta, \rho_\tau^+)$ unboundedly often below a Mahlo cardinal κ. Then such an iteration is equivalent to $\mathrm{Col}(\eta, < \kappa)$.

5 The Crucial Lemma – Adapted

S. Shelah [7], starting from a supercompact cardinal κ, showed non-triviality of $I[\aleph_{\omega+1}]$. Using supercompacts models he found a stationary subset S of $\kappa^{+\omega+1}$ not in $I[\kappa^{+\omega+1}]$, which concentrates on ordinals of some fixed cofinality $\rho^{+\omega+1}$ for some regular cardinal $\rho < \kappa$. Next step was to collapse down below $\aleph_\omega$ and to turn $\kappa^{+\omega+1}$ into $\aleph_{\omega+1}$ preserving stationarity of S and $S \notin I[\kappa^{+\omega+1}]$. It was important for him to preserve $\rho^{+\omega+1}$ as a cardinal. So $Col(\omega, \rho^{+\omega})$ was used followed by $Col(\omega_2, \kappa)$. This way $\rho^{+\omega+1}$ is preserved, turns into ω_1, and so S concentrates on ordinals of cofinality ω_1. The question was whether, for example, it is possible to have cofinality ω_2 instead.

The crucial idea of H. Jakob and M. Levine was that, under certain circumstances, it is possible to collapse such $\rho^{+\omega+1}$ to any $\aleph_n, 0 < n < \omega$ still preserving non-approachability. Namely, they observed that if a countable subset without a cover in V of cardinality less than $\rho^{+\omega}$ was added then collapses will preserve non-approachability. The crucial point was to show that after such collapse any increasing and unbounded function $F : \aleph_n \to (\rho^{+\omega+1})^V$ will have an initial segment whithout a small cover in V.

We will need here to make a small adoption of H. Jakob and M. Levine observationto the setting of specific form of iteration used here.

Let $\tau < \eta$ be regular cardinals. Suppose that $\langle \rho_\xi \mid \xi < \tau \rangle$ is an increasing sequence of regular cardinals with $\rho_0 > \eta$. Assume that $\dot{Q}_\xi$ is based on ρ_ξ. Let $\rho_\tau = \bigcup_{\xi < \tau} \rho_\xi$. Let

$P * Col(\eta, (\rho_\tau^+)^V)$ be a $BB(\tau, \eta, \langle \rho_\xi \mid \xi < \tau \rangle)$ iteration. Let $G * H \subseteq P * \mathrm{Col}(\eta, (\rho_\tau^+)^V)$ be generic.

Lemma 5.1 *Let $F : \eta \longrightarrow (\rho_\tau^+)^V$ be an increasing unbounded function in $V[G * H]$. Then there is $\nu < \eta$ such that $\mathrm{rng}(F \restriction \nu)$ cannot be covered by a set in V of cardinality less than ρ_τ.*

Proof. Fix some condition $(p, \dot{c})$ forcing that $\dot{F} : \eta \longrightarrow (\rho_\tau^+)^V$ is increasing and unbounded.

Claim 2 *Let $\xi < \tau$ be such that p_ξ is a set of positive measure. Then there are $\langle p^*(\xi), \dot{c}^*(\xi) \rangle \geq^* \langle p, \dot{c} \rangle$, $\nu(\xi) < \eta$, and disjoint sets $\langle A^\xi_\gamma \mid \exists s \geq p^*(\xi) \restriction \xi \quad s \Vdash_{P_\xi} \gamma \in \dot{p}(\xi)^*_\xi \rangle$ in V of cardinality at most ρ_τ such that:*

$$\langle p^*(\xi), \dot{c}^*(\xi) \rangle \Vdash \forall \gamma \in \dot{p}_\xi(\xi) \quad \dot{F}(\nu(\xi)) \in A^\xi_\gamma.$$

Proof. Let $G_\xi \subseteq P_\xi$ be generic. Work in $V[G_\xi]$. Proceed by induction on $\gamma \in p_\xi$. So, suppose for each $\gamma' < \gamma$, the construction is done. Set $B = \bigcup_{\gamma' < \gamma} A^\xi_{\gamma'}$. Then $|B| \leq \rho_\tau$. So, there are $\nu^\gamma < \eta$ and $\langle r, \dot{c}' \rangle \geq^* \langle p \setminus \xi, \dot{c} \rangle$ which forces $\dot{F}(\nu^\gamma) \notin B$.

Now, using Strong Prikry Condition and $|P_\xi| < \rho_\tau$, we define A^ξ_γ in V of cardinality $\leq \rho_\tau$. Note that ν^γ depends on G_ξ. Assume that it is picked to be the least possible. Let $\dot{\nu}^\gamma$ be its P_ξ-name.

In $V[G_\xi]$, by shrinking, we can assume that ν^γ does not depend on γ. Then, instead of $\dot{\nu}^\gamma$, we are left with ν. Take a direct extension of $p \restriction \xi$ which decides ν. This will be our $\nu(\xi)$.
$\square$ of the claim.

The rest of the argument is as in [5] or [4] or [6].
Proceed by induction on $\xi < \tau$ and construct a $\leq^*$ $-$increasing sequence $\langle p(\xi), \dot{c}(\xi) \rangle$ with members satisfying the conclusion of the claim. Let $\langle p^*, \dot{c}^* \rangle \geq^* \langle p(\xi), \dot{c}(\xi) \rangle$, for every $\xi < \tau$. Set $\nu = \sup_{\xi < \tau} \nu(\xi) + 1$. We claim that

$$\langle p^*, \dot{c}^* \rangle \Vdash \mathrm{rng}(\dot{F} \restriction \nu) \text{ cannot be covered by a set in } V \text{ of cardinality less than } \rho_\tau.$$

Otherwise, there will be $\langle r, \dot{e} \rangle \geq \langle p^*, \dot{c}^* \rangle$ and $X \subseteq \rho_\tau$ in V and of cardinality less than ρ_τ there, such that

$$\langle r, \dot{e} \rangle \Vdash \mathrm{rng}(\dot{F} \restriction \nu) \subseteq X.$$

Pick $\xi < \tau$ such that $\rho_\xi > |X|$ and $\dot{r}_\xi$ is a positive set. Find some $\gamma < \rho_\xi$ and $s \geq r \restriction \xi \quad s \Vdash_{P_\xi} \gamma \in \dot{r}(\xi)_\xi$ such that $X \cap A^\xi_\gamma = \emptyset$.
This provides the desired contradiction, since $r' {}^\frown r \setminus \xi \geq r \geq p^*$, and so

$$\langle r' {}^\frown \gamma {}^\frown r \setminus \xi + 1, \dot{e} \rangle \Vdash \mathrm{rng}(\dot{F} \restriction \nu) \subseteq X.$$

On the other hand, by the claim,

$$\langle p^*(\xi), \dot{c}^*(\xi)\rangle \Vdash \forall \gamma' \in \dot{p}_\xi(\xi) \quad \dot{F}(\nu(\xi)) \in A^\xi_{\gamma'}.$$

□

6 Nice Filters

Assume GCH. Let κ be a supercompact cardinal, and let $\tau < \eta$ be regular cardinals. Let $P \subseteq V_\kappa$ be a κ-c.c., η-closed forcing and let $\dot{R} \in V^P$ be a $\leq \kappa$-directed closed forcing. Suppose that P itself has the following form: For a club $C \subseteq \kappa$, $\alpha \in C \implies$

$$P \restriction \alpha = P_\alpha * \dot{l}(\alpha) * \mathrm{Col}(\eta, 2^{|\dot{l}(\alpha)|}),$$

where $P_\alpha \subseteq V_\alpha$ has α-c.c., $\dot{l}(\alpha) \in V_\kappa$ is a P_α-name of a $\leq \alpha$-directed closed forcing, and l is a Laver function. Let $\lambda > \kappa$ be a regular large enough cardinal. Take a normal ultrafilter W over $\mathcal{P}_\kappa(\lambda)$, and let $j : V \longrightarrow M$ be its ultrapower embedding. We would like to extend the generic embedding to an embedding $j^* : V^{P*\dot{R}} \longrightarrow M^{j(P*\dot{R})}$.

We have $\kappa \in j(C)$. Arrange $j(l)(\kappa) = R$. Then,

$$j(P) \restriction \kappa = P * R * \mathrm{Col}(\eta, 2^{|R|}).$$

Let $G(P) \subseteq P$ and $G(R) \subseteq R$ be generic over V and over $V[G(P)]$ respectively.

Over M, we will use $j(P)/(G(P) * G(R))$. Let r^* be a $j(P)/(G(P) * G(R))$ name of an element of $j(R)$ which is stronger than every element of $G(R)$.

Define a filter W^* extending W in $V[G(R) * G(P)]$ as follows:

$$X \in W^* \iff \text{there is } (p, \dot{r}) \in G(P) * G(R) \text{ such that, in } M,$$

$$(p, \dot{r}) \frown r^* \Vdash j''\lambda \in j(\dot{X}).$$

Lemma 6.1 *W^* is an η-complete filter over $(\mathcal{P}_\kappa(\lambda))^V$.*

Proof. Follows from the η-closure of $P * R$.
□

Consider $(W^*)^+$ the set of all positive sets of W^*. Let $t \in j(P) * j(R)$ be such that:

(a) $t \restriction P * R \in G(P) * G(R)$,

(b) $t \geq r^*$.

Fix a function $f_t : \mathcal{P}_\kappa(\lambda) \longrightarrow P * R$ which represents t. Set:

$$A_t = \{x \in \mathcal{P}_\kappa(\lambda) \mid f_t(x) \in G(P) * G(R)\}.$$

Lemma 6.2 $A_t \in (W^*)^+$.

Proof. Otherwise, for some $(p, \dot{r}) \in G(P) * G(R)$,

$$(p, \dot{r}) \frown r^* \Vdash j''\lambda \notin j(\dot{A}_t).$$

Then, by the definition of A_t,

$$(p, \dot{r}) \frown r^* \Vdash t \notin G(j(P) * j(R)).$$

However, by (a) and (b) above, t is compatible with $(p, \dot{r}) \frown r^*$, which yields a contradiction.
□

Lemma 6.3 *The set of A_t with $t \in j(P) * j(R)$ satisfying (a) and (b) is dense in $(W^*)^+$ (mod W).*

Proof. Let $A \in (W^*)^+$. Then, for some $s \in (j(P) * j(R))/(G(P) * G(R))$ with $s \geq r^*$, we have:

$$s \Vdash j''\lambda \in j(\dot{A}).$$

Pick some $(p, \dot{r}) \in G(P) * G(R)$ such that:

$$(p, \dot{r}) \Vdash (s \Vdash j''\lambda \in j(\dot{A})).$$

Set $t = (p, \dot{r}) \frown s$. Then, $t \Vdash j''\lambda \in j(\dot{A})$ and t satisfies conditions (a) and (b).

We claim that $A_t \subseteq A \pmod{W}$. Thus, by elementarity,

$$B = \{x \in \mathcal{P}_\kappa(\lambda) \mid f_t(x) \Vdash x \in A\} \in W.$$

Let $x \in B \cap A_t$, then $f_t(x) \Vdash x \in A$ and $f_t(x) \in G(P) * G(R)$. Hence, $x \in A$ and we are done.
□

The above lemma implies the following:

Lemma 6.4 *$\langle (W^*)^+, \subseteq \rangle$ has a dense subset which is η-closed.*

7 Direct Systems

Let κ be a λ-supercompact cardinal for a Mahlo cardinal λ.

Let $\mathcal{U}$ be a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$. For every inaccessible μ where $\kappa < \mu < \lambda$, let:

$$\mathcal{U}_\mu = \mathcal{U} \restriction \mu = \{X \restriction \mu \mid X \in \mathcal{U}\}$$

where $X \restriction \mu = \{x \cap \mu \mid x \in X\}$.

Then $\mathcal{U}_\mu$ is a normal ultrafilter over $\mathcal{P}_\kappa(\mu)$ for every μ such that $\kappa < \mu < \lambda$.

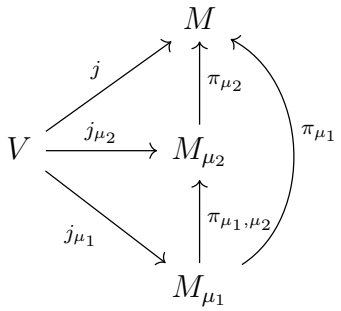
Let $j_\mu : V \longrightarrow M_\mu \simeq V^{\mathcal{P}_\kappa(\mu)}/\mathcal{U}_\mu$ be the corresponding ultrapower embedding.

Consider projections $\pi_{\mu_1,\mu_2} : \mathcal{U}_{\mu_2} \longrightarrow \mathcal{U}_{\mu_1}$ for every μ_1, μ_2 with $\kappa < \mu_1 \leq \mu_2 \leq \lambda$, defined by mapping $x \in \mathcal{P}_\kappa(\mu_2)$ to $x \cap \mu_1$. Further, let us identify π_{μ_1,μ_2} with the corresponding embeddings between the ultrapowers.

Consider a directed system:

$$\langle M_\mu, \pi_{\mu_1,\mu_2} \mid \kappa < \mu < \lambda,\ \mu_1 \leq \mu_2 < \lambda \rangle$$

Let M be its limit and denote by π_μ's the corresponding embeddings from M_μ to M. We will have the following commutative diagram:



Note that:

1. ${}^\lambda M \subseteq M$
2. $j(\kappa) \geq \lambda$
3. $|j(\kappa)| = \lambda$
4. $j''\lambda$ is unbounded in $j(\lambda)$

Suppose now that we have a forcing $R * \dot{T}$, where R is like $\mathrm{Col}(\kappa, < \lambda)$ and $\dot{T}$ is a λ-closed forcing in V^R. Let $G * H$ be a generic filter. We shall argue that j extends to:

$$j^* : V[G * H] \longrightarrow M[G^* * H^*]$$

Deal first with $j(R)$. Use (1) to construct an M-generic subset G' of $j(R)$. So, for every $\beta < j(\lambda)$, we have $f_\beta : j(\kappa) \longrightarrow \beta$, the collapse of β from G'. For every $\beta < j(\lambda)$ of the form $j(\alpha)$, replace $f_\beta \restriction \kappa$ by the α-th function from G. Note that for any given $\beta < j(\lambda)$, $|\beta \cap j''\lambda| < \lambda$.

So, $\beta \cap j''\lambda \in M$. Hence, such changes leave us inside $j(R)$. Denote the result by G^*. By the $j(\lambda)$-c.c. of $j(R)$, G^* will be $j(R)$-generic over M. In addition, now $j''G \subseteq G^*$.

Turn to the second forcing $\dot{T}$. We claim that $j''H$ generates a generic over $M[G^*]$ for $j(\dot{T})$. This follows from the λ-closure of $\dot{T}$ by the standard Kunen-Paris arguments.

Note that the forcing $\dot{T}$ may destroy the λ-compactness of κ. This is the reason for using the $\mathcal{U}_\mu$'s ($\mu < \lambda$).

8 Nice Filters Revisited

Assume GCH. Let κ be a λ-supercompact cardinal, and let $\tau < \eta$ be regular cardinals. Let $P \subseteq V_\kappa$ be a κ-c.c., η-closed forcing (typically $\mathrm{Col}(\eta, < \kappa)$), $\dot{R}$ a P-name of a forcing like $\mathrm{Col}(\kappa, < \lambda)$, and $\dot{T}$ a $P * \dot{R}$-name of a λ-closed forcing.

We proceed similarly to Section 4, but replace positive sets for the filter W^* over $\mathcal{P}_\kappa(\lambda)$ by positive sets for the direct limit embedding of Section 2. The point is that $\dot{T}$ may destroy the λ-supercompactness of κ (or what remains of it after $P * \dot{R}$), and so we deal with the direct limit instead.

So, for every inaccessible μ where $\kappa \leq \mu < \lambda$, we have a normal ultrafilter $\mathcal{U}_\mu$ on $\mathcal{P}_\kappa(\mu)$.

Define $\mathcal{U}^*_\mu$ in $V[G(P), G(R), G(T)]$, where $G(P) \subseteq P$, $G(R) \subseteq R$, and $G(T) \subseteq T$ are generic, as follows:

$$X \in \mathcal{U}^*_\mu \iff \text{there is } (p, \dot{\nu}, \dot{t}) \in G(P) * G(R) * G(T) \text{ such that, in } M,$$

$$(p, \dot{\nu} \frown \nu', \dot{t}) \Vdash j''\mu \in j(\dot{X})$$

where ν' is from the fixed master condition sequence for $j(R)/G(P) * G(R)$.

Now, instead of a single filter, we will have a directed sequence:

$$\langle \mathcal{U}^*_\mu, \pi^*_{\mu_1,\mu_2} \mid \kappa \leq \mu < \lambda,\ \kappa \leq \mu_1 \leq \mu_2 \leq \lambda \rangle$$

which extends naturally the original sequence:

$$\langle \mathcal{U}_\mu, \pi_{\mu_1,\mu_2} \mid \kappa \leq \mu < \lambda,\ \kappa \leq \mu_1 \leq \mu_2 < \lambda \rangle$$

We would like to use positive sets from filters of the sequence in order to define a one-element Prikry-type forcing Q.

Set $Q_0 = \bigcup\{(\mathcal{U}^*_\mu)^+ \mid \kappa \leq \mu < \lambda\}$ and $Q_1 = \kappa$.

Define the order $\leq_1$ on Q_1 by setting "$\leq_1$" $\equiv$ "$=$", i.e., it is just atomic.

Define the order $\leq_0$ on Q_0 as follows: $X_1 \leq_0 X_2$ iff there are μ_1, μ_2 with $\kappa \leq \mu_1 \leq \mu_2 < \lambda$ such that $X_1 \in (\mathcal{U}^*_{\mu_1})^+$, $X_2 \in (\mathcal{U}^*_{\mu_2})^+$, and $(\pi^*_{\mu_1,\mu_2})''X_2 \subseteq X_1$.

Set $Q = Q_0 \cup Q_1$. Define $\leq$ and $\leq^*$ orders on Q as follows:

Let $\leq^* = \leq_0 \cup \leq_1$, and if $q \in Q_1, X \in Q_0$, then set $X \leq q$ iff there is $x \in X$ such that $x \cap \kappa = q$.

The next lemma follows directly from the definition of $\langle Q, \leq, \leq^* \rangle$.

Lemma 8.1 *$\langle Q, \leq, \leq^* \rangle$ is a Prikry-type forcing.*

Proof. Let σ be a statement of the forcing language and $q \in Q$. If $q \in Q_1$, then $q \Vdash \sigma$ or $q \Vdash \neg\sigma$, since the only extension of q in this case is q itself.
□

Suppose that $q \in Q_0$. Then, for some μ where $\kappa \leq \mu < \lambda$, $q \in (\mathcal{U}^*_\mu)^+$.

Define $f : q \to 2$ by setting:

$$f(x) = \begin{cases} 0 & \text{if } x \cap \kappa \Vdash \sigma \\ 1 & \text{if } x \cap \kappa \Vdash \neg\sigma \end{cases}$$

In M, find a condition s which decides the value of $j^*(f)(j''\mu)$. Say the decided value is 0. Take a function h_s which represents s in the ultrapower by some $\mathcal{U}_{\mu'}$ with $\mu' \geq \mu$.

Set:

$$A_s = \{x \in \mathcal{P}_\kappa(\mu') \mid f(x) \in G(P) * G(R) * G(T),\ f_s(x) = 0,\ x \cap \mu \in q\}$$

Then $A_s \in (\mathcal{U}^*_{\mu'})^+$, and so $A_s \geq q$. Now, for every $x \in A_s$, $f_s(x) = 0$. Then $x \cap \kappa \Vdash \sigma$. Hence, $A_s \Vdash \sigma$ and we are done.

□

Lemma 8.2 *$\langle Q_0, \leq_0\rangle$ is equivalent to an η-closed forcing notion.*

Proof. Note that $\langle Q_0, \leq_0\rangle$ is the forcing with positive sets of filters $\mathcal{U}^*_\mu$'s, so it is equivalent to $j(P) * j(R) * j(\dot{T})$. The last forcing is η-closed.

□

9 Main result

Here we will apply the previous techniques to answer affirmatively a question of H. Jakob and A. Poveda [6].

Theorem 9.1 *Assume GCH. Let τ be a regular cardinal. Suppose that $\langle \kappa_\gamma \mid \gamma < \tau$ is an increasing sequence of supercompact cardinals. Let $\kappa_\tau = \bigcup_{\gamma<\tau} \kappa_\gamma$. Then there is a generic extension such that:*
for every regular cardinal $\eta, \tau < \eta < \kappa^+_\tau$,

$$\{\nu < \kappa^+_\tau \mid \mathrm{cof}(\nu) = \eta\} \notin I[\kappa^+_\tau].$$

Proof. For every $\gamma < \tau$ let $\ell_\gamma : \kappa_\gamma \to V_{\kappa_\gamma}$ be a Laver function of the supercompact cardinal κ_γ.

Define by induction Easton support iteration of Prikry type forcing notions $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa_\tau, \beta < \kappa_\tau\rangle$.

If $\alpha < \tau^+$, then P_α is a trivial forcing. Suppose that $\alpha \geq \tau^+$. Define Q_α.

Let $\tau < \alpha < \kappa_\tau$ and P_α is defined. If α is not an inaccessible or if $|P_\alpha| > \alpha$, then let Q_α be trivial. Suppose that this is not the case.

Assume first that $\tau < \alpha < \kappa_0$.

Consider $l_0(\alpha)$. We set Q_α to be trivial, unless $l_0(\alpha)$ is a pair consisting of a $P_\alpha-$name

of $\alpha-$closed forcing $\dot{R}$ and $P_\alpha * \dot{R}-$name of a basic block (see Section 4) of the form $BB(\tau, \tau^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$. If this is the case, then set $Q_\alpha = \dot{R} * BB(\tau, \tau^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$.

Assume that $\kappa_\xi \leq \alpha < \kappa_{\xi+1}$.

Consider $l_{\xi+1}(\alpha)$. We set Q_α to be trivial, unless $l_{\xi+1}(\alpha)$ is a pair consisting of a $P_\alpha-$name of $\alpha-$closed forcing $\dot{R}$ [1] and $P_\alpha * \dot{R}-$name of a basic block of the form $BB(\tau, \kappa_\xi, \langle \rho_\gamma \mid \gamma < \tau \rangle)$. If this is the case, then set $Q_\alpha = \dot{R} * BB(\tau, \kappa_\xi, \langle \rho_\gamma \mid \gamma < \tau \rangle)$.

Finally assume that for some limit $\xi < \tau$, $(\bigcup_{\zeta<\xi} \kappa_\zeta)^+ \leq \alpha < \kappa_\xi$.

Consider $l_\xi(\alpha)$. We set Q_α to be trivial, unless $l_\xi(\alpha)$ is a pair consisting of a $P_\alpha-$name of $\alpha-$closed forcing $\dot{R}$ and $P_\alpha * \dot{R}-$name of a basic block of the form $BB(\tau, (\bigcup_{\zeta<\xi} \kappa_\zeta)^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$. If this is the case, then set
$Q_\alpha = \dot{R} * BB(\tau, (\bigcup_{\zeta<\xi} \kappa_\zeta)^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$.

Let $P = P_{\kappa_\tau}$ and $G \subseteq P$ be a generic.

No cardinals $\geq \kappa_\tau$ or $\leq \tau$ are collapsed in $V[G]$. The regular cardinals in the interval (τ, κ_τ) will be τ^+, κ_ξ's , for $\xi < \tau$, and $(\bigcup_{\zeta<\xi} \kappa_\zeta)^+$, for limit $\xi < \tau$.

We will have in $V[G]$ that for every regular cardinal $\eta, \tau < \eta < \kappa_\tau^+$,

$$\{\nu < \kappa_\tau^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\kappa_\tau^+].$$

It follows from the construction using Lemma 5.1, as in [3] and [5] or [4] or [6].

□

10 Global results

By S. Shelah [7] existence of a proper class of supercompact cardinals implies that AP_λ fails for a proper class of singular cardinals λ. It was shown by O. Ben-Neria, C. Lambie-Hanson, S. Unger [1] and the author [3] that it is possible to have $\neg\mathrm{AP}_\lambda$ for a proper class of singular cardinals λ which is closed. In addition SCH breaks down at such λ's. H. Jakob and A. Poveda [6] constructed a model in which for ever singular cardinal λ, there are unboundedly many regular cardinals $\eta, \mathrm{cof}(\lambda) < \eta < \lambda$, with

$$\{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+].$$

In particular, AP_λ fails at **every** singular cardinal λ.[2]

Our aim here will be to extend the above result of H. Jakob and A. Poveda by constructing a model in which:

for ever singular cardinal λ, for every regular cardinal $\eta, \mathrm{cof}(\lambda) < \eta < \lambda$,

$$\{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+].$$

[1] Following Section 4, it is possible just to use iterations of Levy collapses here. The same for all further entrances of $\dot{R}$.

[2] SCH holds in their model. It is a major open problem whether $\neg\mathrm{AP}_{\aleph_\omega} + 2^{\aleph_\omega} > \aleph_{\omega+1}$ is consistent.

Assume GCH. Suppose that θ is the least inaccessible cardinal which is a limit of $\theta-$supercompact cardinals.
Let $\langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an increasing continuous, unbounded in θ sequence such that, for every $\alpha < \theta$, $\kappa_{\alpha+1}$ is a $\theta-$supercompact cardinal. We will show the following:

Theorem 10.1 *There is a generic extension V^* which preserves inaccessibility of θ such that, in V^*, for every singular cardinal $\lambda < \theta$, for every regular cardinal $\eta, \mathrm{cof}(\lambda) < \eta < \lambda$,*

$$\{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+].$$

Proof. Similar to 9.1, only iterate all the way to θ.

For every $\gamma < \theta$ let $\ell_\gamma : \kappa_{\gamma+1} \to V_{\kappa_{\gamma+1}}$ be a Laver function of the supercompact cardinal $\kappa_{\gamma+1}$.
Define by induction Easton support iteration of Prikry type forcing notions $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \theta, \beta < \theta \rangle$.
If $\alpha < \omega_1$, then P_α is a trivial forcing. Suppose that $\alpha \geq \omega_1$. Define Q_α.
Let $\omega < \alpha < \theta$ and P_α is defined. If α is not an inaccessible or if $|P_\alpha| > \alpha$, then let Q_α be trivial. Suppose that this is not the case.

Assume first that $\tau < \alpha < \kappa_1$.
Consider $l_0(\alpha)$. We set Q_α to be trivial, unless $l_0(\alpha)$ is a pair consisting of a $P_\alpha-$name of $\alpha-$closed forcing $\dot{R}$ and $P_\alpha * \dot{R}-$name of a basic block of the form $BB(\omega, \omega_1, \langle \rho_n \mid n < \omega \rangle)$. If this is the case, then set $Q_\alpha = \dot{R} * BB(\omega, \omega_1, \langle \rho_n \mid n < \omega \rangle)$.

Assume that $\xi < \theta$ is a successor ordinal and $\kappa_\xi \leq \alpha < \kappa_{\xi+1}$.
Consider $l_{\xi+1}(\alpha)$. We set Q_α to be trivial, unless $l_{\xi+1}(\alpha)$ is a pair consisting of a $P_\alpha-$name of $\alpha-$closed forcing $\dot{R}$ and $P_\alpha * \dot{R}-$name of a basic block of the form $BB(\tau, \kappa_\xi, \langle \rho_\gamma \mid \gamma < \tau \rangle)$. If this is the case, then set $Q_\alpha = \dot{R} * BB(\tau, \kappa_\xi, \langle \rho_\gamma \mid \gamma < \tau \rangle)$.

Finally assume that for some limit $\xi < \theta$, $(\kappa_\xi)^+ \leq \alpha < \kappa_{\xi+1}$.
Consider $l_\xi(\alpha)$. We set Q_α to be trivial, unless $l_\xi(\alpha)$ is a pair consisting of a $P_\alpha-$name of $\alpha-$closed forcing $\dot{R}$ and $P_\alpha * \dot{R}-$name of a basic block of the form $BB(\tau, \kappa_\xi^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$. If this is the case, then set $Q_\alpha = \dot{R} * BB(\tau, \kappa_\xi^+, \langle \rho_\gamma \mid \gamma < \tau \rangle)$.

Let $P = P_\theta$ and $G \subseteq P$ be a generic.
No cardinals $\geq \theta$ are collapsed in $V[G]$ and θ remains an inaccessible. The uncountable cardinals below θ will be ω_1 and κ_ξ's , for $\xi < \tau$.
We will have in $V[G]$ that for every singular cardinal $\lambda < \theta$, for every regular cardinal $\eta, \mathrm{cof}(\lambda) < \eta < \lambda$,

$$\{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+].$$

It follows from the construction using Lemma 5.1, as in [4] or [6].
□

By cutting the universe at θ, we obtain the following:

Corollary 10.2 $(V[G])_\theta \models ZFC+$ *for every singular cardinal* λ*, for every regular cardinal* $\eta, \mathrm{cof}(\lambda) < \eta < \lambda$*,*

$$\{\nu < \lambda^+ \mid \mathrm{cof}(\nu) = \eta\} \notin I[\lambda^+].$$

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