On covering vs pp below $o^{\P}$

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Abstract

It is shown that $\mathrm{cov}(\lambda,\lambda,\aleph_1,2) =^+ \mathrm{pp}(\lambda)$, $\mathrm{cof}(\lambda)=\aleph_0$, below $o^{\P}$, provided $2^{\aleph_1}<\lambda$.

1 Introduction

We deal here with a problem of Saharon Shelah [4],
Analytical Guide 14.7(β),p.459:

Is $cov(\lambda,\lambda,\aleph_1,2) =^+ pp(\lambda)$, when $cof(\lambda)=\aleph_0$?

It is shown that this is true, at least below $o^{\P}$, provided $2^{\aleph_1}<\lambda$.

The proof is based on results of S. Shelah [4], ch. IX, sec. 1 and of W. Mitchell and the author [2].

Let us state some basic definitions due to S. Shelah.

Definition 1. *$cov(\lambda,\mu,\theta,\sigma)=\min\{|\mathcal{P}| \mid \mathcal{P}$ is a family of subsets of λ each of cardinality $<\mu$ such that: for every $a\subseteq\lambda, |a|<\theta$ for some $\alpha<\sigma$ and $A_i\in\mathcal{P}, i<\alpha$ we have $a\subseteq\bigcup_{i<\alpha}A_i\}$.*

Definition 2. *Let λ be a singular cardinal. $pp(\lambda)=\sup\{tcf(\prod a/J) \mid a$ is a set of $cof(\lambda)$ regular cardinals , unbounded in λ, J an ideal on A including J^{bd}_a and $\prod a/J$ has true cofinality$\}$.*

$\eta =^+ pp(\lambda)$ means more than equality; the supremum in the right hand side is obtained if it is regular.

2 Main result

Our aim is to prove the following:

Theorem 1 ($\neg o^{\P}$)**.** *Let λ be a singular cardinal of cofinality ω and $2^{\aleph_1}<\lambda$. Then:*

$$pp(\lambda) =^+ cov(\lambda,\lambda,\aleph_1,2)$$

Proof. Suppose otherwise. We follow S. Shelah [4], chapter IX, Section 1. Using Claim 1.6 (p. 365), find a sequence $\bar{\lambda}=\langle\lambda_i \mid i<\alpha^*<\omega_1\rangle$ which satisfies $(*)^{12}_{\lambda}$. It consists of limit cardinals of cofinality $\aleph_1$.

If for every $i < \alpha^*$, $\mathrm{pp}(\lambda_i) = \lambda_i^+$, then the result follows by Claim 1.7 (p. 367) without any anti-large cardinal assumptions.

Suppose that for some i's, $\mathrm{pp}(\lambda_i) > \lambda_i^+$. By the Mitchell Covering Lemma [3], there is no μ such that $\lambda_i < \mu \leq \mathrm{pp}(\lambda_i)$ which is a measurable cardinal in the core model $\mathcal{K}$.

Indiscernibles analysis implies then that there is an increasing mod J_λ^{bd} sequence of functions $\langle p_\xi \mid \xi < \lambda^+ \rangle$ such that for every sequence

$$\vec{e} = \langle e_i \mid i < \alpha^* \rangle \in \prod_{i<\alpha^*} (\lambda_i, \mathrm{pp}(\lambda_i)] \cap Reg,$$

such that every e_i is of cardinality at most $\aleph_1$, and for every $g \in \prod_{i<\alpha^*} e_i$, there is $\xi < \lambda^+$ such that:

$$g < p_\xi \pmod{J_\lambda^{\mathrm{bd}}}$$

Just take a covering model N with $g \in N$ and then use its Skolem function h^N.

2.1 Simpler case

Suppose first that $\mathrm{pp}(\lambda_i) = \lambda_i^{+\eta} > \lambda_i^+$, for every $i < \omega_1$, for some fixed $\eta, 1 < \eta < \aleph_1$.

Fix $i < \omega_1$. Apply [2]. Due to a smallness of η, we can find disjoint sets $\langle a_\mu^i \mid \mu \in Reg \cap (\lambda_i, \mathrm{pp}(\lambda_i))] \rangle$ and find increasing $\pmod{J_{\lambda_i}^{\mathrm{bd}}}$ sequences $\langle f_\xi^{i,\mu} \mid \xi < \mu \rangle$ in $\prod a_\mu^i$, for every regular cardinal μ $(\lambda_i < \mu \leq \mathrm{pp}(\lambda_i))$ which witnesses the cofinality μ.

Consider $a^i = (\lambda_i \setminus \bigcup_\mu a_\mu^i) \cap Reg$.

Let $C_i = \langle c_i^j \mid j < \omega_1 \rangle$ be the maximal ω_1-sequence consisting of principal indiscernibles for λ_i, i.e., basically a canonical Magidor sequence for λ_i. The model $\mathcal{K}[C_i]$ still satisfies GCH, but $\mathrm{cof}(\lambda_i) = \omega_1$ in $\mathcal{K}[C_i]$, as witnessed by C_i.

Lemma 1. *For every $b \subseteq a_i, |b| = \aleph_1$ unbounded in λ_i and every $g \in \prod b$ there is $f \in \mathcal{K}[C_i]$ such that $f \restriction b > g$ mod $J_{\lambda_i}^{bd}$.*

Proof. Take a covering model N with b, g inside. For every $\kappa \in Reg \cap \lambda_i$, set $f(\kappa) = \sup(h^{N\prime\prime}(c_i^{j_\kappa} + 1) \cap \kappa)$, where $h^N \in \mathcal{K}$ is a Skolem function of N and $j_\kappa < \omega_1$ is the last j such that the $j-$th element c_i^j of C_i is below κ.

□

Now, using GCH and singularity of λ_i, it is possible to find in $\mathcal{K}[C_i]$, a sequence of functions in $\prod a_i$, $\langle f_\xi^i \mid \xi < \lambda_i^+ \rangle$ which is increasing $\pmod{J_{\lambda_i}^{\mathrm{bd}}}$ such that for every $b \subseteq a_i, |b| = \aleph_1$ unbounded in λ_i and every $g \in \prod b$, there is $\xi < \lambda_i^+$ such that:

$$g < f_\xi^i \pmod{J_{\lambda_i}^{\mathrm{bd}}}$$

Note that $\lambda_i^+ = (\lambda_i^+)^{\mathcal{K}}$ by the weak covering lemma, since $\neg o^{\P}$ is assumed.

We combine now the functions together. Let $\xi < \lambda^+$. For every $i < \alpha^*$, consider $p_\xi \restriction (\lambda_i, \lambda_i^{+\eta}] \cap Reg$. Then, for every regular $\mu \in (\lambda_i, \lambda_i^{+\eta}]$, we have a

function $f^{i,\mu}_{p_\xi(\mu)} \in \prod a^i_\mu$. If $\mu = \lambda_i^+$, then take the maximum between $f^{i,\mu}_{p_\xi(\mu)}$ and $f^i_{p_\xi(\mu)}$. Let F be the family of all such functions.

From this point one can proceed similar to Shelah's proof of Claim 1.7 ([4], p. 368–369). Let us provide the argument for completeness.

For every $i < \alpha^*$, fix a cofinal in λ_i sequence $\langle \lambda_{i,j} \mid j < \omega_1 \rangle \omega\omega$. Now, for every $j < \omega_1$, by the property of $\langle \lambda_i \mid i < \alpha^* \rangle$ applied to F and $\langle \lambda_{i,j} \mid i < \alpha^* \rangle$, for some $x_j \subseteq Reg \cap \bigcup_{i<\alpha^*} [\lambda_{i,j}, \lambda_i), |x_j| = \aleph_0$ and $g_j \in \prod x_j$,
there is no $f \in F$ which dominates g_j mod J^{bd}_λ.

Set $x^* = \bigcup_{j<\omega_1} x_j$ and let $g^* \in \prod x^*$ be defined as follows:

$$g^*(\kappa) = \sup\{g_j(\kappa) \mid j < \omega_1\}.$$

Fix $i < \alpha^*$. For every regular $\mu \in (\lambda_i, \lambda_i^{+\eta}]$, pick $\gamma^i_\mu < \mu$ such that $f^{i,\mu}_{\gamma^i_\mu} > g^* \restriction a^i_\mu$ mod $J^{bd}_{\lambda_i}$. Recall that $\mathrm{cof}(\lambda_i) = \omega_1$ and $\eta < \omega_1$, so there is $\beta^i < \lambda\lambda_i$ such that for every regular $\kappa, \beta^i < \kappa < \lambda_i$, $f^{i,\mu}_{\gamma^i_\mu}(\kappa) > g^*(\kappa)$, for every μ.

Let $s \in \prod_{i<\alpha^*} Reg \cap (\lambda_i, \lambda_i^{+\eta}]$ be a function defined by setting $s(\mu) = \gamma^i_\mu$, for every regular $\mu \in (\lambda_i, \lambda_i^{+\eta}]$.
Pick now $\xi < \lambda^+$ such that $p_\xi > s$ mod J^{bd}_λ. Assume for simplicity that p_ξ dominates s everywhere. Otherwise just work above the point from which the domination starts. Let $f \in F$ be the function which p_ξ defines. Then for every $i < \alpha^*$,

$$g^* \restriction \lambda_i < f$$

mod $J^{bd}_{\lambda_i}$. Hence, for some $\xi_i < \omega_1$,

$$g^* \restriction (\lambda_{i,\xi_i}, \lambda_i) < f.$$

Let $\xi^* = \bigcup\{\xi_i \mid i < \alpha^*\}$. But then, $g_{\xi^*} < f$, which is impossible by its choice. Contradiction.

2.2 General case

Let us deal with the general case now.
Fix $i < \omega_1$. The treatment of the set of regular cardinals below λ_i which are outside of indiscernibles exactly as those of a_i above. Note that every set in $[\mathrm{pp}(\lambda_i)]^{\omega_1}$ can be covered by a set in $\mathcal{K}$ of cardinality λ_i there. Let $H_i = (\mathcal{P}_{\lambda_i{}^+}(\mathrm{pp}(\lambda_i)))^{\mathcal{K}}$. It has cardinality $\leq \mathrm{pp}(\lambda_i)^+$ and is unbounded in $\mathcal{P}_{\lambda_i{}^+}(\mathrm{pp}(\lambda_i))$.

Now we use reconstruction games of [1] and [2]. Fix a winning strategy σ for Player II in such a game. Recall that for every $B \subseteq \mathrm{pp}(\lambda_i)$, $B \in \mathcal{K}$, $|B|^{\mathcal{K}} = \lambda_i$, we have $\sigma(B) = \vec{\nu} = \langle \vec{\nu}_\gamma \mid \gamma \in B \rangle$, a set of indiscernibles corresponding to measures of an extender over λ_i with indices in B.

Given $B \in H_i$. Define $h_B \in \prod \mathrm{Reg} \cap \lambda_i$ as follows:

$h_B(\kappa) = \sup(\sigma(B) \cap \kappa)$, if it is $< \kappa$ and 0 otherwise.

Let $a \subseteq \mathrm{Reg} \cap \lambda_i$, $|a| = \aleph_1$ be unbounded and $g \in \prod a$. Let $I(g) \subseteq [\lambda_i^+, \mathrm{pp}(\lambda_i)]$ be a set of indices which correspond to $\mathrm{rng}(g)$ and $B(g) \in H_i$ be a cover of $I(g)$.

Then,

$$h_{B(g)} > g \pmod{J^{\mathrm{bd}}_{\lambda_i}}.$$

Turn now to

$$\prod_{i<\alpha^*} [\lambda_i^+, \mathrm{pp}(\lambda_i)]/J^{\mathrm{bd}}_{\alpha^*}$$

By the anti-large cardinal assumption, there are no indiscernibles inside the intervals $[\lambda_i, \mathrm{pp}(\lambda_i)]$; otherwise, there would be measurables of $\mathcal{K}$ inside, which in turn would imply overlapping extenders there.
Take a covering model N with $\langle \lambda_i \mid i < \alpha^* \rangle$ inside.
Let $\vec{B} = \langle B_i \mid i < \alpha^* \rangle \in N \cap \prod_{i<\alpha^*} H_i$. For every $i < \alpha^*$, there are finitely many principle indiscernibles $\vec{\kappa}_i = \langle \kappa_{i1}, ..., \kappa_{i,n_i} \rangle$ and $B_i^* \in H_i, B_i^* \supseteq B_i$ such that $B_i^* = h^N(k_i, \vec{\kappa}_i)$, for some $k_i < \omega$, i.e. B_i^* depends on indiscernibles $\vec{\kappa}_i$ only. Note that members of the original sequence $\vec{B}$ may depend on non-principal indiscernibles. That is eliminated by passing to a bit larger sequence $\vec{B}^*$. Then, the sequence $\langle B_i^* \mid i < \alpha^* \rangle$ depends only on indiscernibles in the set $C = \{\kappa_{ij} \mid i < \alpha^*, 1 \leq j \leq n_i\}$. The last set forms a Prikry or Magidor sequence over $\mathcal{K}$, since it consists of countably many principal indiscernibles. In particular, $\mathcal{K}[C]$ satisfies $2^\lambda = \lambda^+$.

Then, there is a family $G \in \mathcal{K}[C]$ of cardinality λ^+ which consists of elements of $\prod_{i<\alpha^*} H_i$ such that for every $\langle B_i \mid i < \alpha^* \rangle$ there is $\langle B_i^* \mid i < \alpha^* \rangle \in G$ such that $B_i \subseteq B_i^*, i < \alpha^*$.

For every $\vec{B}^* = \langle B_i^* \mid i < \alpha^* \rangle \in G$ we combine the functions $h_{B_i^*}, i < \alpha^*$, into a single function $h_{\vec{B}^*}$. The total number of such functions is λ^+.

Now we proceed as in the end of the Shelah proof of Claim 1.7 ([4], p. 368–369) or as above in a simpler case, using the family of functions $\{h_{\vec{B}^*} \mid \vec{B}^* \in G\}$ and $\langle f_\xi^i \mid \xi < \lambda_i^+ \rangle, i < \alpha^*$ for λ_i^+'s as in 2.1.

□

References

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