

Holomorphic disks and tropical Lagrangians

2026 Tel Aviv Seminar

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Abstract: In continuation of a work with Sushmita Venugopalan, I will describe a calculus for counting disks bounding tropical Lagrangians, with applications to computing Fukaya categories of del Pezzos.

I Tropical Lagrangians

II A tropical formula for disk counts

III Fukaya categories for del Pezzos

CONFERENCE ADVERTISEMENTS

(a) Workshop on Lagrangian Floer theory and applications.
Dates: September 28 – October 2, 2026 Location: CMSA G10, 20 Garden St., Cambridge MA 02138. This workshop is part of the Lagrangian Floer theory and applications Program.
Organizers: Denis Auroux (Harvard), Jonny Evans (Lancaster), and Chris Woodward (Rutgers). To express interest in attending the workshop or program, and to apply for funding if available, please fill out the form on the *program page*.

(b) Rutgers Gauge Theory, Low-Dimensional Topology, and Geometric Analysis Conference 2026. Our five-day meeting is an official satellite event for the International Congress of Mathematicians, July 23-30, 2026, Philadelphia.

INTERNATIONAL TRAVEL AND FREE SPEECH

In-person attendance at both events is lower than expected due to mathematicians facing obstacles to travel because of, for example, political speech, which is completely understandable.

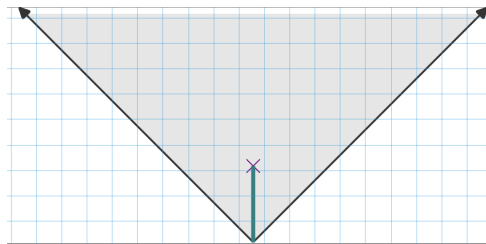
I, too, feel concerned crossing the border in the United States (due to their increasing examination of cell phones for political statements) as I would crossing the border in Russia, or Israel. (A group of people I am associated with, at American Friends Service Committee, has been banned from visiting Israel due to our political speech on the West Bank/Gaza situation.)

Hopefully the concerning trend towards blocking travel due to political speech will reverse in the near future. In the meantime, thank you to the organizers of online seminars.

TROPICAL LAGRANGIANS

Let $\Phi : X \rightarrow B$ be the moment map for a toric manifold X , or almost toric manifold. Call $\Phi(X)$ the *moment polytope*.

Example: $X = T^*S^2$, $\Phi : X \rightarrow B$ the standard integrable system for spherical pendulum given by energy and angular momentum, $\Phi(X) = \text{Cone}((1, 1), (-1, 1))$, $\text{Critval}(X) = \{(0, 0), (0, 1)\}$ stable and unstable equilibria. Unstable equilibrium is a focus-focus point (fiber with focus-focus point is a nodal torus.)

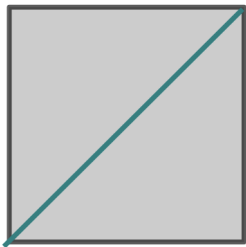


A *tropical Lagrangian* for a tropical graph in B is a family of Lagrangians $L \subset X$ “approaching the inverse image of Λ ” in some good sense. (Details omitted.)

Thm (Mikhalkin, Matessi, Hicks): For suitable tropical graphs, there exists a corresponding tropical lagrangian.

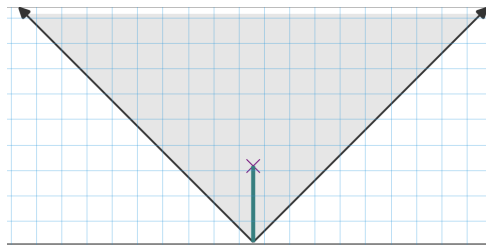
Suitable means univalent vertices can be at codimension two faces $F \subset \Phi(B)$, in which case they lie in the plane which “bisects” the moment image at that face, or at focus-focus values.

Example: The diagonal in $X = (S^2)^- \times S^2$ (that is, the graph of the identity as a symplectomorphism) is a tropical Lagrangian with graph the diagonal in $\Phi(X) = [-1, 1]^2$.

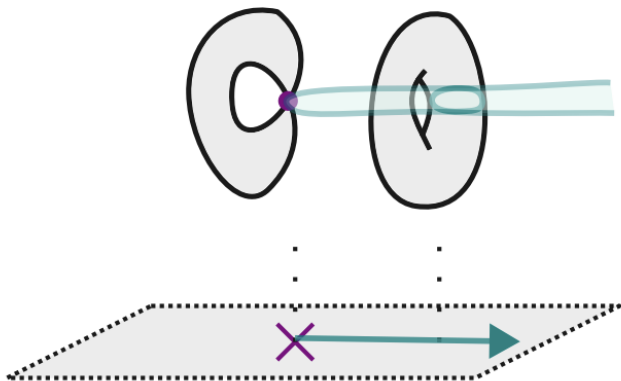


Example: $X = T^*S^2$, $\Phi : X \rightarrow B$ the standard integrable system for spherical pendulum given by energy and angular momentum + Arnold-Liouville theorem

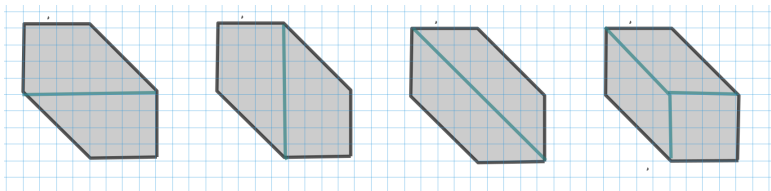
$\Phi(X) = \text{Cone}((1, 1), (-1, 1))$, $\text{Critval}(\Phi) = \{(0, 0), (0, 1)\}$ stable and unstable equilibria. Unstable equilibrium is a focus-focus value. The zero section is a tropical Lagrangian with image connecting $(0, 0)$ and $(0, 1)$.



The fiber over the unstable critical value is all the states with the same energy and angular momentum as the unstable equilibrium. These are the unstable equilibrium plus the $\mathbb{R}$ -trajectories connecting the unstable equilibrium with itself - i.e. a point plus a cylinder i.e. a nodal torus.



Example: $X = \text{Bl}^3(\mathbb{P}^2)$, $B = \mathbb{R}^2$, $\Phi(X)$ a hexagon.



There are two interesting types of tropical Lagrangians, with one or three edges respectively.

The first has one edge, and is the inverse of the diagonal in $X \rightarrow \text{Bl}^2(\mathbb{P}^1 \times \mathbb{P}^1)$. There are three such maps, so this gives three tropical Lagrangians.

The second has three edges. The local model near the vertex is obtained from the holomorphic curve $1 + z_1 + z_2 = 0$ and hyperKähler rotation.

MANIN CONFIGURATIONS

A del Pezzo surface is a compact complex surface with positive first Chern class. Any del Pezzo surface is obtained from $\mathbb{P}^2$ by blowing up at most eight points in general position, or is $\mathbb{P}^1 \times \mathbb{P}^1$. (Castelnuovo)

Example: The cubic surface $\{z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0\}$ in $\mathbb{P}^3$ is the blow-up of $\mathbb{P}^2$ at six points.

Manin: The del Pezzo surfaces of low degree correspond to root systems of ADE type. E.g. the cubic surface corresponds to the root system E_6 .

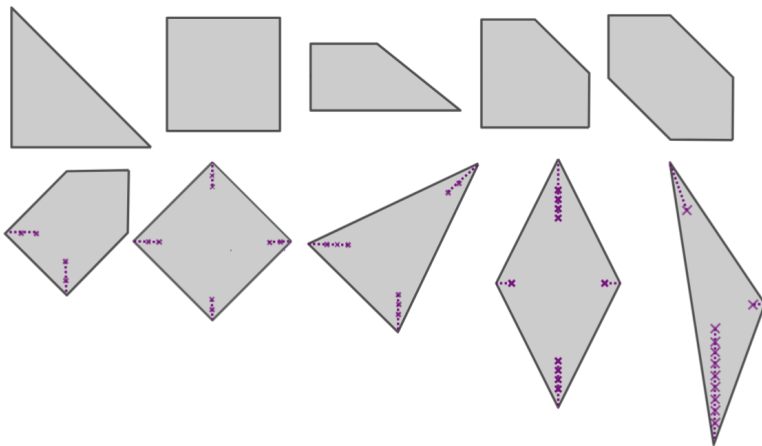
The second cohomology perpendicular to the canonical class is a semisimple root system of ADE type.

Manin root systems: Take the lattice perpendicular to $c_1(X)$ in $H^2(X)$. The classes with self-intersection -2 form a root system of ADE type.

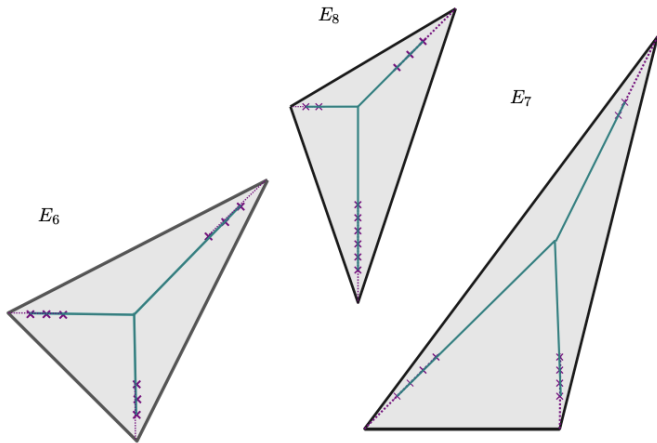
X	Manin Root System	$\approx \text{Spec}(c_1 \star \in \text{End}(QH(X)))$
$\mathbb{P}^2$		$3\alpha, \alpha^3 = 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	A_1	$4, 0^{\oplus 2}, -4$
$B1^1 \mathbb{P}^2$		$-0.33, 3.8, -2.23 \pm 1.94I$
$B1^2 \mathbb{P}^2$	A_1	$(-1)^{\oplus 2}, 4.73, -2.86 \pm 0.94I$
$B1^3 \mathbb{P}^2$	$A_1 \oplus A_2$	$(-2)^{\oplus 3}, (-3)^{\oplus 2}, 6$
$B1^4 \mathbb{P}^2$	A_4	$(-3)^{\oplus 5}, 8.09, -3.09$
$B1^5 \mathbb{P}^2$	D_5	$(-4)^{\oplus 7}, 12$
$B1^6 \mathbb{P}^2$	E_6	$(-6)^{\oplus 8}, 21$
$B1^7 \mathbb{P}^2$	E_7	$(-12)^{\oplus 9}, 52$
$B1^8 \mathbb{P}^2$	E_8	$(-60)^{\oplus 10}, 372.$

VIANNA'S DIAGRAMS FOR DEL PEZZOS

Vianna has shown that every del Pezzo admits an almost toric structure a la Symington.



From the diagrams, you can easily construct Manin's root systems.



DISK POTENTIALS

Let L be a compact spin monotone tropical Lagrangian in a compact symplectic X .

$W_L : \text{Hom}(\pi_1(L), \mathbb{C}^\times) \rightarrow \mathbb{C}^\times$ the *disk potential* of L counting Maslov index two disks passing through a generic point, weighted by holonomies.

If L is a Lagrangian sphere, W_L is just an integer.

Thm going back to Kontsevich, see e.g. Sheridan: If L is a torus then any non-degenerate critical value of W_L is eigenvalue of c_1^* on $QH(X)$. Same holds for even-dimensional spheres.

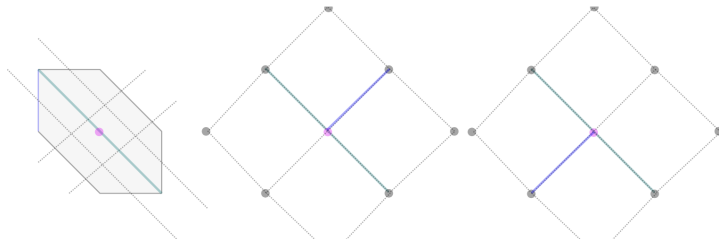
If a particular spectral value in the table above is an integer, one can wonder whether it's generated by a Lagrangian sphere.

GOAL: COMPUTE W_L BY “DEGENERATION” AKA MULTI-DIRECTIONAL SFT

Input: A polyhedral decomposition $\mathcal{P} = \{P\}$ of B .

A *gluing datum* consisting of dual polytopes $P^\vee$ satisfying certain conditions, forming the *dual complex* $B^\vee = \cup_{P \in \mathcal{P}} P^\vee$.

Example: Suppose we want to compute disks bounding one of the Lagrangian spheres in $X = \text{Bl}^3(\mathbb{P}^2)$ passing through a generic point. A natural decomposition to take, and the corresponding dual complex, is shown below.



Main Thm: Suppose X is almost toric. Then W_L is a sum over tropical graphs Γ , with the contribution of each Γ given as a product over vertices v of some multiplicities $m(v)$ by a Mikhalkin-type formula

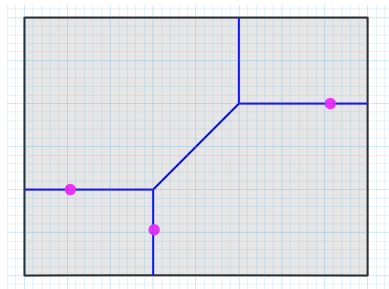
$$W_L = \sum_{\Gamma} \# \operatorname{Aut}(\Gamma)^{-1} \prod_{v \in \operatorname{Vert}(\Gamma)} m(v)$$

and in dimension 4 the multiplicities are explicitly computable in many cases.

The same formula holds for any other similar genus zero invariant, e.g. counts of spheres passing through fixed number of points.

DUAL COMPLEX

The tropical graphs Γ live in the *dual complex* $B^\vee$. It's convenient to draw the graphs on top of the moment polytope (i.e. identify B with $B^\vee$) in which case we call them *cartoon diagrams*.

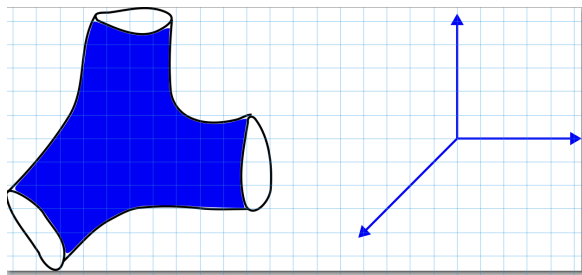


This picture shows the number of holomorphic spheres of degree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through three generic points. I.e. the number of isomorphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ with three prescribed values.

MIKHALKIN MULTIPLICITIES, TORIC CASE

The special case of sphere counts is due to Mikhalkin.

In this case there are only two multiplicities needed: At a trivalent vertex v , $m(v)$ is the area parallelogram spanned by the directions if the outgoing directions of the edges at v sum to zero, and vanishes otherwise.



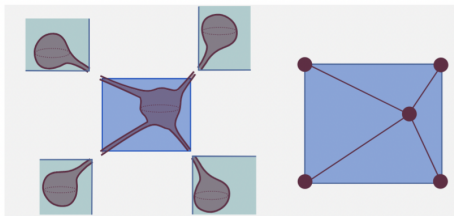
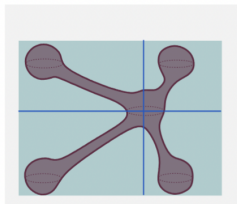
At a univalent vertex v mapping to the boundary of the moment polytope, $m(v) = 1$ and the direction must be normal to the boundary.

SFT-STYLE PROOF (FOLLOWING B. PARKER)

Consider a neck-stretching family of almost complex structure, in which the inverse images $\Phi^{-1}(P)$ are replaced by increasingly long cylinders $\Phi^{-1}(P) \times P^\vee$.

Define $\mathfrak{X} = \sqcup_{P \in \mathcal{P}} \Phi^{-1}(P) \times \text{span}(P^\vee)$. This is an almost complex manifold with an action of a vector space $\mathfrak{t}_p^\vee$ of dimension $P^\vee$.

Each sequence u_ν of maps contributing to the count has a broken limit u_∞ in $\mathfrak{X}$.

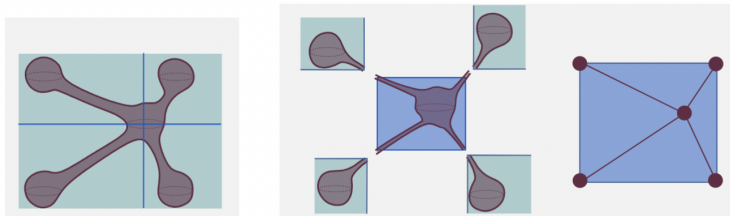


THE TROPICAL LIMIT

Define a tropical graph Γ by assigning to each component u_v of u_∞ a vertex v , and each node in u_∞ an edge.

The maps u_ν look like the components u_v glued together by long cylinders, which look like $z \rightarrow z^{T(e)}x$ for some *direction* $T(e) \in \mathfrak{t}_p^\vee$.

The map u_∞ is called by us a *broken map*; Parker calls these *exploded maps* and he has a slicker formulation; we need something quite explicit for Floer theory.



DISTRIBUTION OF CONSTRAINTS

The moduli space of broken maps $M(\mathfrak{X})$ is the subset of $\prod_{v \in \text{Vert}(\Gamma)} M(\mathfrak{X}_{P(v)})$ satisfying matching conditions at the nodes.

A special feature in the genus zero $\dim(X) = 4$ case:

Each fiber product $M = M_1 \times_{M_{12}} M_2$ is actually a Cartesian product after adding constraints.

Since $\dim(M) = 0$, without loss of generality $\dim(M_1) = 0$, and the image of M_{12} in M_2 provides a constraint Y_2 so that the constrained moduli space $\dim(M_2(Y_2)) = 0$. So

$$M(X) = \bigcup_{\Gamma} \prod_{v \in \text{Vert}(\Gamma)} M(X_{P(v)}, Y_v)$$

There is something like this in higher dimension, but it's more complicated (Fulton-Sturmfels splitting of diagonal.)

MULTIPLICITIES AT FOCUS-FOCUS VALUES

There is just one additional vertex for a sphere with direction going to a focus-focus singularity perpendicular to a branch cut direction.

The multiplicity is the Bryan-Pandharipande formula $m(v) = (-1)^{d-1}/d^2$, where d is the lattice length.

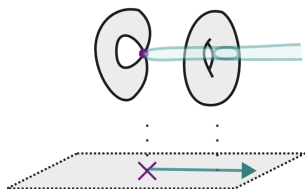
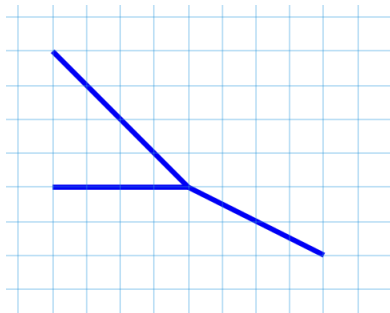
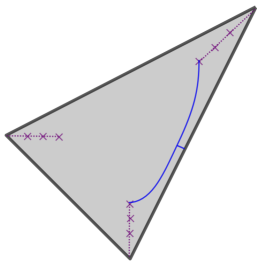


FIGURE 3. A focus-focus singularity

HOLOMORPHIC SPHERES IN DEL PEZZOS

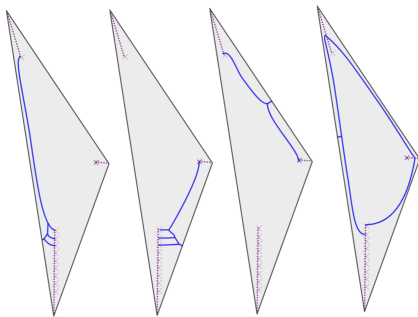
Example: A famous formula of Cayley-Salmon is the count of twenty seven lines (lowest degree curves) in a cubic surface. The 27 lines in the cubic are all variations on the following graph:



3 outgoing directions $\times$ 3 focus-focus singularities to choose from $\times$ 3 focus-focus singularities to choose from = 27 tropical lines

HOLOMORPHIC SPHERES FOR DEL PEZZOS

There is a less well-known count of 252 degree one curves in the del Pezzo of degree one:

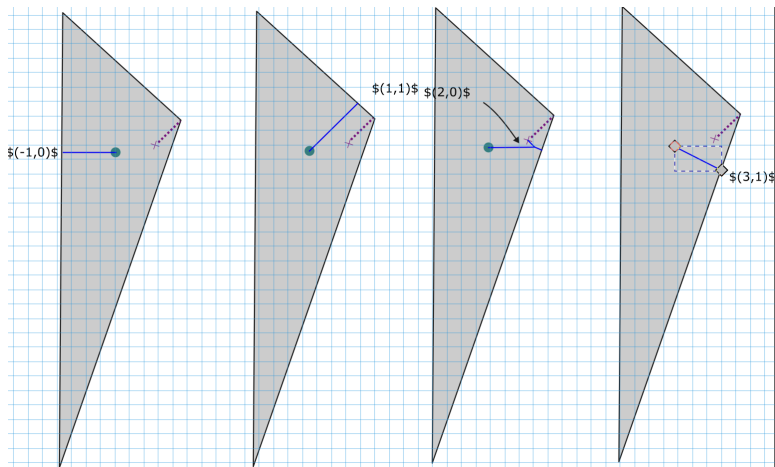


$$84 + 84 + 3 + 81 = 252$$

Of these 240 are embedded (and correspond to the roots of E_8).
The extra 12 are half of the 24 nodal fibers in a $K3$.

EXAMPLE: THE CHEKANOV TORUS

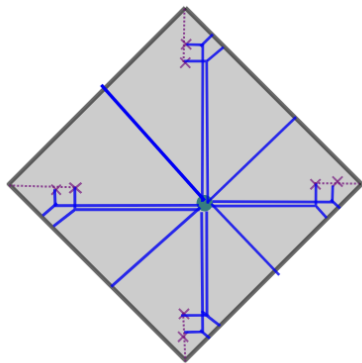
The Chekanov torus is a moment fiber for a non-standard almost toric structure on $\mathbb{P}^2$ shown below. There are four tropical disks, shown below; one has Mikhalkin multiplicity two.



EXAMPLE: DEGREE FOUR DEL PEZZO

For the degree four del Pezzo has potential

$W_L(y_1, y_2) = (1 + y_1)^2(1 + y_2)^2/(y_1 y_2) - 4$, arising from the tropical graphs (shown as cartoon diagrams)



The critical value $W_L = 12$ is easy to see from this picture, since $y = 1$ is a critical value by symmetry. What about the other critical value -4 ?

POTENTIALS OF DEL PEZZOS

Using the tropical formula one can reproduce the table of potentials due to Pascaleff-Tonkonog (see also Coates et al)

del Pezzo surface	Manin root system	disk potential
$\mathbb{P}^2$		$y_1 + y_2 + 1/y_1 y_2$
$\mathbb{P}^1 \times \mathbb{P}^1$	A_1	$y_1 + 1/y_1 + y_2 + 1/y_2$
$B1^1 \mathbb{P}^2$		$y_1 + y_2 + 1/(y_1 y_2) + y_1 y_2$
$B1^2 \mathbb{P}^2$	A_1	$(1 + y_1 + y_2)(1 + 1/(y_1 y_2)) - 1$
$B1^3 \mathbb{P}^2$	$A_1 \oplus A_2$	$(1 + y_1)(1 + y_2)(1 + 1/(y_1 y_2)) - 2$
$B1^4 \mathbb{P}^2$	A_4	$(1 + y_1 + y_2)(1 + 1/y_1)(1 + 1/y_2) - 3$
$B1^5 \mathbb{P}^2$	D_5	$(1 + y_1)^2(1 + y_2)^2/(y_1 y_2) - 4$
$B1^6 \mathbb{P}^2$	E_6	$(1 + y_1 + y_2)^3/(y_1 y_2) - 6$
$B1^7 \mathbb{P}^2$	E_7	$(1 + y_1 + y_2)^4/(y_1 y_2) - 12$
$B1^8 \mathbb{P}^2$	E_8	$(1 + y_1 + y_2)^6/(y_1 y_2) - 60$

Table: Potentials of del Pezzo surfaces

REMARKS ON THE TORUS DISK POTENTIALS

- (a) The potentials for degrees five and higher have the right Jacobian rings, i.e. $Jac(W) = QH(X)$; the other do not.
- (b) It's fairly hard to compute the critical points explicitly, and so get the putative split-generators of the Fukaya category.
- (c) Gross-Siebert give mirror formulas for del Pezzos which are different, but not explicit for degree one.
- (d) The best mirrors (i.e. clean and with correct Jacobi rings) are Givental-Hori-Vafa mirrors but the relationship between these and the disk potentials is rather unclear.
- (e) The original proof by Pascaleff-Tonkonog used mutation invariance; we give a similar mutation-based proof as well.

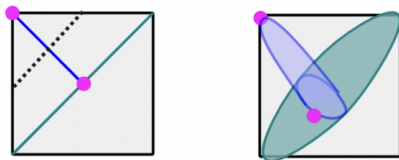
POTENTIALS FOR TROPICAL LAGRANGIAN SPHERES

Now let L be a tropical Lagrangian sphere for an almost toric manifold X . New types of vertices in the tropical formula for W_L :

(1) univalent v starting on an edge e of lattice length $\ell(e) \in \mathbb{Z}_{\geq 0}$ of Lagrangian, with

$$m(v) = (-1)^{\ell(e)}$$

corresponding to a half-cylinder, perpendicular to the tropical Lagrangian.

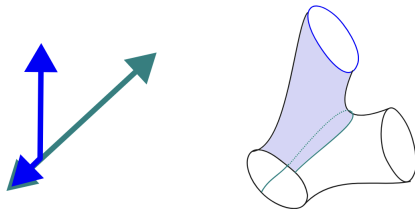


THE HOLOMORPHIC TROUSER

(2) Bivalent vertex v corresponding to the holomorphic pant (half a pair of pants) with

$$m(v) = \pm |\det(\mu_{\circ} \mu_{\bullet})| \in \mathbb{Q}$$

where $\mu_{\circ}$ resp. $\mu_{\bullet}$ is the boundary puncture resp. interior puncture direction, if the triple $\mu_{\circ}$, $\mu_{\bullet}$ and the reflection of $\mu_{\bullet}$ satisfy the balancing condition. Proof is a reduction to the case of real lines by a multiple cover argument.



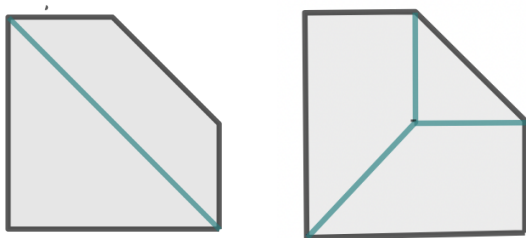
HIND-EVANS ISOTOPY

(3) Disks bounding Lagrangian pants.

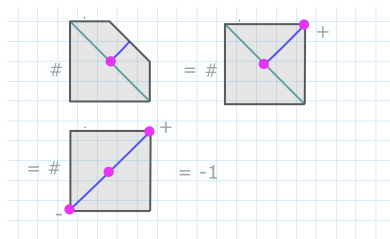
Hind, Evans: Lagrangian spheres with the same homology class in del Pezzos are Hamiltonian isotopic.

In particular, the Lagrangian pants L in $\text{Bl}^2(\mathbb{P}^2)$ is Hamiltonian isotopic to the inverse image L' of the diagonal (which has tropical graph an interval).

Since disk counts are invariant under Hamiltonian isotopy, $W_L = W_{L'}$.

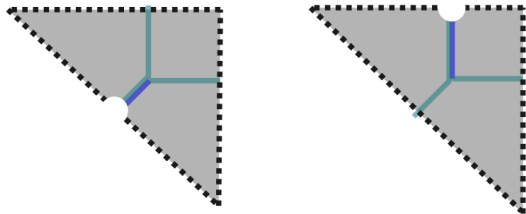


Maps in the left-hand picture can be blown down to maps to $\mathbb{P}^1 \times \mathbb{P}^1$ bounding the diagonal, and these correspond to maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ under the correspondence between spheres and disks bounding the diagonal. However, there is a sign change coming from the reversal of symplectic structure on one factor. So $W_L = -1$.



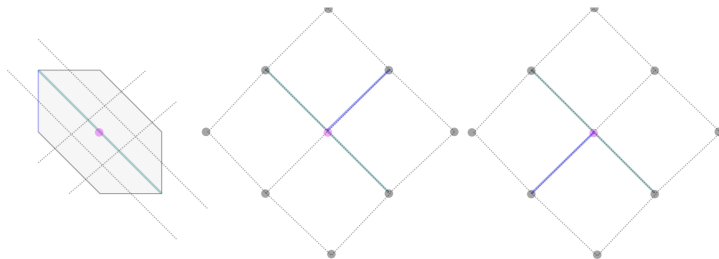
STRATEGY FOR COMPUTING MULTIPLICITIES

Find some computation for which the answer is already known, and for which the all the multiplicities except for one are known. From this we get that the multiplicities of vertices shown below is 1.

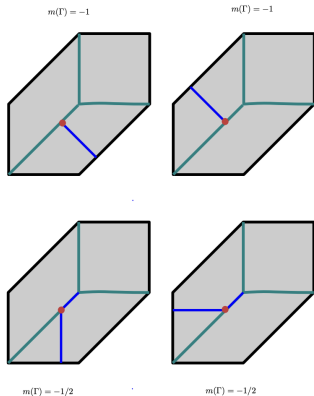


EXAMPLE FOR $X = \text{Bl}^3(\mathbb{P}^2)$.

For the A_2 -system Lagrangians $W_L = -2$ by the degeneration argument in the following picture: The tropical graphs of the holomorphic curves are drawn on the right, with the dual complex which has nine vertices.



EXAMPLE FOR $X = \text{Bl}^3(\mathbb{P}^2)$.



For the A_1 -system Lagrangians $W_L = -3$.
 Warning: The multiplicities $m(v)$ are not invariants, i.e. independent of the choice of perturbation in general, but there is a “symmetric choice” in this case.

EXAMPLE: GRASSMANNIANS (IN PROGRESS)

The Grassmannian $Gr(k, n)$ of k -dimensional subspaces of $\mathbb{C}^n$ and other Fano manifolds in higher dimension seem to have interesting almost toric structures and tropical Lagrangians and their Fukaya categories can be approached from this perspective.

Castronovo showed that Gelfand-Cetlin tori in Grassmannians sometimes splitgenerate the Fukaya category, but not if 0 is an eigenvalue of c_1 .

Work in progress w/ Y. Xiao continuing Nohara-Ueda:
Compute Floer cohomologies of non-torus Gelfand-Cetlin fibers, tropical Lagrangians. E.g. In $Gr(2k, 2n)$, the quaternionic Grassmannian $Gr_H(k, n)$ is a tropical Lagrangian and for small k, n split-generates the 0-eigencategory in $Fuk(Gr(2k, 2n))$.

Other interesting higher-dimensional examples include Chiang Lagrangian which are conjecturally tropical.