A Note on Minimizing the Variability of the Waiting Times in a Globally Gated Elevator Polling System

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A polling system comprising n queues and a single server is considered. Service is performed according to an elevator scheme under the Globally Gated regime. The problem of arranging the channels to minimize a measure of the variability of the waiting times is addressed.

Keywords: Polling system, elevator polling, Globally Gated, $M/G/1$ queue

1. INTRODUCTION

A model commonly encountered in manufacturing, telecommunications, computer systems, and other applications is the following. A single server (e.g. a robot in a manufacturing process, a token in a local area communications ring, a head for reading hard disks in a computer memory) attends n queues of jobs (viz. work stations, channels, and disk tracks, respectively, in the above examples). The order of visits to the queues is of the *elevator* type: the server scans (*polls*) the queues sequentially, serving queues $1, 2, ..., n-1, n$ (the "up" cycle), then serving queues $n, n-1, ..., 2, 1$ (the "down" cycle), then restarting with queue 1. The time spent serving each queue is determined by the queue discipline. A recently introduced service discipline is that of *global gating* [1,2]: at the beginning of each cycle, the "gate" on each and every queue is "closed", i.e. no further arrivals are permitted to enter any of the queues until the end of that cycle. The server then completes a cycle by serving all "gated" jobs. At the end of the cycle, all new arrivals are "gated" simultaneously over each queue, and so forth. A surprising result that was recently discovered [1] is that if each queue is of the $M/G/1$ type (the queues may have different parameters), then the mean waiting times of all customers are equal. The elevator system with global gating is the only known nonsymmetric polling system to exhibit such a *fairness* phenomenon.

Let E_k^U (E_k^D) denote the expected waiting time for a customer in the kth queue given that the customer arrives during an up (down) cycle, $k = 1, 2, ..., n$. Since the length of a cycle under the *global gating* regime is independent of the order of service within a cycle, the customer is equally likely to arrive in either cycle. It has been shown in [2] that

$$E_1^D < E_2^D < \cdots < E_n^D \quad \text{whereas} \quad E_1^U > E_2^U > \cdots E_n^U \; .$$

Thus, a natural measure of the variability of the waiting times of the customers in queue k is $\Delta_k = E_k^D - E_k^U$, $k = 1, 2, \ldots, n$. In many applications the time taken for the server to travel from one queue to the next is negligible with respect to individual service times. It can then be shown [1] that

$$\Delta_k = \sum_{i=1}^{k-1} a_i - \sum_{i=k+1}^{n} a_i \tag{1}$$

where $a_i = 2\mu\rho_i$ and where μ is the mean backwards recurrent time of a cycle and ρ_i is the traffic intensity of the ith queue (i.e. the mean amount of work flowing into queue i per unit time), $i = 1, 2, \ldots n$. The empty sum is interpreted as zero.

One of the objectives in designing a globally gated elevator polling system is to reduce the variability of the waiting times of the customers according to some suitable criterion. Altman, Khamisy and Yechiali [1] show that, if we wish to minimize $\max_{1\le k\le n} |\Delta_k|$, an optimal permutation $(\pi(1), \pi(2), \ldots, \pi(n))$ of the n queues satisfies the condition that $\pi(1)$ is the queue with the largest traffic intensity and $\pi(n)$ is the queue with the second largest traffic intensity (or vice-versa by symmetry). This result also holds when the travel times of the server are *not* negligible.

Clearly, finding an order that follows the criterion: minimize $\{\max_{1\le k\le n} |\Delta_k|\}$ gives only a partial answer to the problem of minimizing the *variability* of waiting times in the entire system. Indeed, in modern telecommunication networks, for example, the grade of service is determined both by the mean waiting time (delay) and by its variability.

It is therefore natural to consider a general goal of reducing the *overall* variability. Such an objective can be achieved by ordering the queues so as to minimize $\sum_{k=1}^{n} |\Delta_k|$.

This problem is addressed in the present note.

2. REFORMULATION OF THE PROBLEM

The problem is to find a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ of $(1, 2, \ldots, n)$ which minimizes $\sum_{k=1}^{n} |\Delta_{\pi(k)}|$ where

$$\Delta_{\pi(k)} = \sum_{i=1}^{k-1} a_{\pi(i)} - \sum_{i=k+1}^{n} a_{\pi(i)} \ . \tag{2}$$

It is convenient to reformulate the problem as follows.

Observe, firstly, that since the a_i's are clearly nonnegative, $\Delta_{\pi(k)}$ is nondecreasing in k with $\Delta_{\pi(1)} < 0$ and $\Delta_{\pi(n)} > 0$ for any permutation π. Define $S^- = \{\pi(j) | \Delta_{\pi(j)} < 0\}$ and $S^+ = \{\pi(j) | \Delta_{\pi(j)} \geq 0\}$. Further, let $m = |S^-|$, the cardinality of S^-, so that $|S^+| = n - m$; clearly, $n > m > 0$.

PROPOSITION 2.1:

(i) Any permutation π for which $a_{\pi(1)} \geq a_{\pi(2)} \geq \cdots \geq a_{\pi(m)}$ minimizes $\sum_{j=1}^{m} |\Delta_{\pi(j)}|$.

(ii) Any permutation π for which $a_{\pi(m+1)} \leq a_{\pi(m+2)} \leq \cdots \leq a_{\pi(n)}$ minimizes $\sum_{j=m+1}^{n} |\Delta_{\pi(j)}|$.

PROOF: Observe that for any permutation π,

$$\begin{aligned}
\sum_{j=1}^{m} \Delta_{\pi(j)} &= -\sum_{i=2}^{n} a_{\pi(i)} + \sum_{k=2}^{m-1} \left(\sum_{i=1}^{k-1} a_{\pi(i)} - \sum_{i=k+1}^{n} a_{\pi(i)} \right) + \sum_{i=1}^{m-1} a_{\pi(i)} - \sum_{i=m+1}^{n} a_{\pi(i)} \\
&= (m-1)a_{\pi(1)} + (m-2)a_{\pi(2)} + \cdots + a_{\pi(m-1)} \\
&\quad -(a_{\pi(2)} + 2a_{\pi(3)} + \cdots + (m-1)a_{\pi(m)}) - m \sum_{i=m+1}^{n} a_{\pi(i)} \\
&= \sum_{j=1}^{m} (m - 2j + 1) a_{\pi(j)} - m \sum_{j=m+1}^{n} a_{\pi(j)} \ .
\end{aligned}$$

Since $\Delta_{\pi(j)} < 0$ for $j = 1, 2, \ldots, m$, we see that

$$\sum_{j=1}^{m} |\Delta_{\pi(j)}| = -\sum_{j=1}^{m} \Delta_{\pi(j)} = m \sum_{j=m+1}^{n} a_{\pi(j)} + \sum_{j=1}^{m} (2j - m - 1) a_{\pi(j)} \ . \tag{3}$$

Since $(2j - m - 1)$ is increasing in j, it follows immediately that a permutation π for which $a_{\pi(1)} \geq a_{\pi(2)} \geq \cdots \geq a_{\pi(m)}$ minimizes $\sum_{j=1}^{m} |\Delta_{\pi(j)}|$.

(ii) By a similar calculation, it is readily seen that

$$\sum_{j=m+1}^{n} |\Delta_{\pi(j)}| = (n - m) \sum_{j=1}^{m} a_{\pi(j)} + \sum_{j=m+1}^{n} (n - 2j + m + 1) a_{\pi(j)} \ , \tag{4}$$

from which the result follows by an analogous argument. □

THEOREM 2.2: A permutation π which minimizes $\sum_{j=1}^{n} |\Delta_{\pi(j)}|$ also maximizes

$$C_\pi = \sum_{j=1}^{m}(m-j)a_{\pi(j)} + \sum_{j=m+1}^{n}(j-m-1)a_{\pi(j)} \ . \tag{5}$$

PROOF: On adding (3) and (4), we see that, for a permutation π,

$$\begin{aligned}
\sum_{j=1}^{n} |\Delta_{\pi(j)}| &= \sum_{j=1}^{m}(2j-m-1)a_{\pi(j)} + m\sum_{j=m+1}^{n} a_{\pi(j)} + \sum_{j=m+1}^{n}(n-2j+m+1)a_{\pi(j)} \\
&\quad +(n-m)\sum_{j=1}^{m} a_{\pi(j)} \\
&= \sum_{j=1}^{m}(n+2j-2m-1)a_{\pi(j)} + \sum_{j=m+1}^{n}(n-2j+2m+1)a_{\pi(j)} \\
&= (n-1)\sum_{j=1}^{n} a_{\pi(j)} - 2\left[\sum_{j=1}^{m}(m-j)a_{\pi(j)} + \sum_{j=m+1}^{n}(j-(m+1))a_{\pi(j)}\right]
\end{aligned}$$

on rearrangement. Since $(n-1)\sum_{j=1}^{n} a_{\pi(j)}$ is constant over all permutations π, it follows that minimizing $\sum_{j=1}^{n} |\Delta_{\pi(j)}|$ is equivalent to maximizing

$$\sum_{j=1}^{m}(m-j)a_{\pi(j)} + \sum_{j=m+1}^{n}(j-m-1)a_{\pi(j)} \ ,$$

as claimed. □

In the remainder of the paper, it will generally prove to be more convenient to maximize C_π than to minimize $\sum_{j=1}^{n} |\Delta_{\pi(j)}|$.

3. PROPERTIES OF THE OPTIMAL SOLUTION

There does not appear to exist a method for identifying a permutation π which minimizes C_π other than by complete enumeration. Thus, our approach is to study some properties of the optimal solution and to investigate special cases when n is small (see Section 4).

Our first result characterizes the shape of an optimal solution.

THEOREM 3.1: A V-shaped permutation π, i.e. one that satisfies the inequalities $a_{\pi(1)} \geq a_{\pi(2)} \geq \cdots \geq a_{\pi(m)}$ and $a_{\pi(m)} \leq a_{\pi(m+1)} \leq \cdots \leq a_{\pi(n)}$ is optimal.

PROOF: The result follows immediately on observing that to maximize

$$C_\pi = \sum_{j=1}^{m}(m-j)a_{\pi(j)} + \sum_{j=m+1}^{n}(j-m-1)a_{\pi(j)} \ ,$$

it is sufficient that $a_{\pi(j)}$ decrease in j for $1 \leq j \leq m$ and increase in j for $m+1 \leq j \leq n$. □

We define the following: Given any permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, the *reverse permutation* is $\tilde{\pi} = (\pi(n), \pi(n-1), \ldots, \pi(1))$.

PROPOSITION 3.2: $C_\pi = C_{\tilde{\pi}}$ for any permutation π.

PROOF: This is straightforward. □

PROPOSITION 3.3: Suppose that $1 \leq j \leq m$ and that $m+1 \leq k \leq n$. If $0 \neq m-j \leq k-m-1$ and $a_{\pi(j)} > a_{\pi(k)}$ or if $m-j < k-m-1$ and $a_{\pi(j)} \geq a_{\pi(k)}$, then interchanging queues j and k will improve the solution.

PROOF: Let π be any permutation and suppose that $1 \leq j \leq m$ and $m+1 \leq k \leq n$. Let π' be the permutation obtained on interchanging $\pi(j)$ and $\pi(k)$. This interchange improves the solution if and only if $C_{\pi'} > C_\pi$, i.e. if and only if

$$(m-j)a_{\pi(k)} + (k-m-1)a_{\pi(j)} > (m-j)a_{\pi(j)} + (k-m-1)a_{\pi(k)}$$

or, equivalently, on rearrangement,

$$a_{\pi(k)}(2m+1-j-k) > a_{\pi(j)}(2m+1-j-k) \ .$$

This last inequality holds if and only if $a_{\pi(j)} > a_{\pi(k)}$ and $0 \neq m-j \leq k-m-1$ or $a_{\pi(j)} \geq a_{\pi(k)}$ and $m-j < k-m-1$ as claimed. □

Proposition 3.3.gives us a simple criterion for deciding whether or not an improved solution will be obtained by interchanging the positions of two queues.

For the remainder of the paper, it is assumed, without loss of generality, that the queues are ordered such that $a_1 \geq a_2 \geq \cdots \geq a_n$.

PROPOSITION 3.4: Suppose that $a_1 \geq \sum_{j=2}^{n} a_j$. Then an optimal permutation is $(1, n, n-1, \ldots, 3, 2)$.

PROOF: Suppose that the given condition holds. Then we must have $m = 1$, i.e. the only element of S^- is 1. Since, by Theorem 3.1, an optimal permutation is V-shaped, the queues in S^+ must be ordered as claimed. □

The following corollary is an immediate consequence of Propositions 3.2 and 3.4.

COROLLARY: If $a_1 \geq \sum_{j=2}^{n} a_j$, an optimal permutation is $(2, 3, \ldots, n-1, n, 1)$.

Perhaps the principal difficulty in constructing an optimal solution is that $m = |S^-|$ is unknown. An advantage of knowledge of the value of m is given by the following proposition.

PROPOSITION 3.5: Suppose that n is even and that it is known that $m = n/2$. Then two optimal permutations are $(1, 3, 5, \ldots, n-1, n, n-2, \ldots, 6, 4, 2)$ and its reverse.

PROOF: The proof follows from repeated application of Proposition 3.3 and from Proposition 3.2. $\square$

4. SPECIAL CASES

In this section we derive optimal permutations for the special cases $n = 1, 2, 3, 4$ and 5. The cases $n = 1$ and $n = 2$ are trivial, and so we commence by investigating the case $n = 3$. Recall that we have adopted the convention that $a_1 \geq a_2 \geq \cdots \geq a_n$.

PROPOSITION 4.1: For $n = 3$, an optimal permutation is (1,3,2) (and its reverse).

PROOF: Since an optimal permutation is V-shaped (Theorem 3.1), the only two possible optimal permutations are $\pi_1 = (1, 3, 2)$ and $\pi_2 = (1, 2, 3)$ (and their reverses which, to simplify the presentation, we ignore). Choose $\mathbf{a} = (a_1, a_2, a_3)$ arbitrarily. Since $a_1 \geq a_2 \geq a_3$, we have

$$\sum_{i=1}^{3} |\Delta_{\pi_1(i)}| = |-(a_2 + a_3)| + |a_1 - a_2| + |a_1 + a_3| = 2(a_1 + a_3)$$

whereas

$$\sum_{i=1}^{3} |\Delta_{\pi_2(i)}| = |-(a_2 + a_3)| + |a_1 - a_3| + |a_1 + a_2| = 2(a_1 + a_2) \ ,$$

i.e. $\sum_{i=1}^{3} |\Delta_{\pi_1(i)}| \leq \sum_{i=1}^{3} |\Delta_{\pi_2(i)}|$ for all vectors $\mathbf{a}$ as claimed. $\square$

PROPOSITION 4.2: For $n = 4$, an optimal permutation is (1,4,3,2) (and its reverse).

PROOF: Since an optimal permutation is V-shaped (Theorem 3.1), the only possible optimal permutations are $\pi_1 = (1, 4, 3, 2)$, $\pi_2 = (1, 3, 4, 2)$, $\pi_3 = (1, 2, 4, 3)$, and $\pi_4 = (1, 2, 3, 4)$ (and their reverses). In order to prove the proposition, it is thus sufficient to verify the following three inequalities: (i) $C_{\pi_1} \leq C_{\pi_2}$, (ii) $C_{\pi_1} \leq C_{\pi_3}$, and (iii) $C_{\pi_1} \leq C_{\pi_4}$ for arbitrary $\mathbf{a} = (a_1, a_2, a_3, a_4)$. We prove (i).

For permutation π_1, $\Delta_{\pi_1(1)} = -(a_4 + a_3 + a_2)$, $\Delta_{\pi_1(2)} = a_1 - (a_3 + a_2)$, $\Delta_{\pi_1(3)} = a_1 + a_4 - a_2$, and $\Delta_{\pi_1(4)} = a_1 + a_4 + a_3$, whereas for permutation π_2, $\Delta_{\pi_2(1)} = -(a_3 + a_4 + a_2)$, $\Delta_{\pi_2(2)} = a_1 - (a_4 + a_2)$, $\Delta_{\pi_2(3)} = a_1 + a_3 - a_2$, and $\Delta_{\pi_2(4)} = a_1 + a_3 + a_4$. There are two cases to consider: $a_1 \geq a_2 + a_3$ and $a_1 < a_2 + a_3$.

Case 1: $a_1 \geq a_2 + a_3$. In this case, $|S^-| = 1$ for permutation π_1, and so $C_{\pi_1} = 2a_2 + a_3$. Further, the assumption $a_1 \geq a_2 + a_3$ implies $a_1 \geq a_2 + a_4$, and hence $|S^-| = 1$ for permutation π_2. Thus, $C_{\pi_2} = a_4 + 2a_2 \leq 2a_2 + a_3 = C_{\pi_1}$ for any vector $\mathbf{a}$.

Case 2: $a_1 < a_2 + a_3$. In this case, $|S^-| = 2$ for permutation π_1, and so $C_{\pi_1} = a_1 + a_2$. There are two subcases to consider: (a) $a_1 \geq a_4 + a_2$ and (b) $a_1 < a_4 + a_2$. For subcase (a), $|S^-| = 1$ for permutation π_2, and so $C_{\pi_2} = a_4 + 2a_2 \leq a_1 + a_2 = C_{\pi_1}$ by assumption. For subcase (b), $|S^-| = 2$ for permutation π_2 and $C_{\pi_2} = a_1 + a_2 = C_{\pi_1}$.

Thus $C_{\pi_1} \leq C_{\pi_2}$ for all $\mathbf{a}$ as claimed. The proof of (ii) and (iii) are similar and are omitted. □

PROPOSITION 4.3: For $n = 5$, there are two possible optimal permutations: $\pi_1 = (1,5,4,3,2)$ and $\pi_2 = (1,4,5,3,2)$ (and their reverses).

PROOF: Since an optimal permutation is V-shaped, there are eight possible optimal permutations: $\pi_1 = (1,2,3,4,5)$, $\pi_2 = (1,5,4,3,2)$, $\pi_3 = (1,4,5,3,2)$, $\pi_4 = (1,2,5,4,3)$, $\pi_5 = (1,2,3,5,4)$, $\pi_6 = (1,3,5,4,2)$, $\pi_7 = (1,3,4,5,2)$ and $\pi_8 = (1,2,4,5,3)$ (and their reverses). To prove the proposition, it is thus sufficient to verify the following claims for arbitrary $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$.

(i) $C_{\pi_2} \geq C_{\pi_4} \geq C_{\pi_8} \geq C_{\pi_5} \geq C_{\pi_3}$.
(ii) $C_{\pi_2} \geq C_{\pi_6} \geq C_{\pi_7}$.
(iii) there is no order relation between C_{π_1} and C_{π_2}.

These claims may be proved by arguments similar to those of the proof of Proposition 4.2. The details are extremely tedious and are omitted. □

REMARK: The specific optimal permutations calculated for the (relatively small) cases $n = 3, 4, 5$ may suggest that one optimal permutation is $(1, n, n-1, \ldots, 2)$. However, taking $n = 6$ and the vector $\mathbf{a} = (10, 9, 8, 7, 6, 5)$, it is easy to show that the permutation $\pi_1 = (1,6,5,4,3,2)$ gives a *higher* value of $\sum_{i=1}^{6} |\Delta_{\pi(i)}|$ than the corresponding value for the permutation $\pi_2 = (1,4,5,6,3,2)$. Nevertheless, if the values of a_i are such that $a_1 \geq \sum_{i=2}^{6} a_i$ then a direct calculation reveals that permutation π_1 is better than permutation π_2. Thus, it seems that the structural permutations $(1, n, n-1, \ldots, 2)$ or $(1, n, n-1, \ldots, n-2, 3, 2)$ do *not* give a general optimal solution.

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