Polling under the Randomly Timed Gated Regime *

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Abstract

Polling systems under the Randomly Timed Gated (RTG) regime are studied and analyzed, and various performance measures are derived. The RTG protocol operates as follows: whenever the server enters a station, a Timer is activated. If the server empties the queue before the Timer's expiration, it moves on to the next node. Otherwise (i.e., if there is still work in the station when the Timer expires), the server obeys one of the following rules, each leading to a different model: (1) The server completes *all* the work accumulated up to the Timer's expiration and then moves on to the next node. (2) The server completes only the service of the job *currently* being served, and moves on. (3) The server stops working *immediately* and moves on. The RTG protocol defines a wide class of time-based control mechanisms: under rule (1) it spans an entire spectrum of regimes lying between the Gated and the Exhaustive, while under rule (2) it spans an entire spectrum between the 1-Limited and the Exhaustive protocols.

Keywords: Randomly Timed Gated; Timers; $M/G/1$; Exhaustive; Gated; 1-Limited.

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1 INTRODUCTION

Polling systems are queueing networks consisting of n stations (channels) attended by a single server which incurs switch-over times when moving from one station to another. Such configurations are very common in real life and are used as a central model for the analysis of a wide variety of applications in areas such as computer networks, telecommunications, flexible manufacturing, vehicle traffic control and repair problems. Very often the sojourn time of the server in each station is constrained by some protocol. A useful procedure is to use Timers as a control mechanism. However, most of the analytical studies on polling systems consider either the (unconstrained) Exhaustive discipline, or the job-based Gated, Globally-Gated and Limited regimes.

Recently, Eliazar and Yechiali [3] introduced a Timer-based control mechanism, called Randomly Timed Gated (RTG) discipline, and employed it on a single-node $M/G/1$-type queue with general Intermission Intervals. (An Intermission Interval is a random time-period, often system dependent, during which the server is not available for service.) Applying the RTG protocol to a polling system leads to the following procedure: whenever the server enters a station (node, channel), an exponential Timer is activated. If the server empties the queue before the Timer's expiration, it moves on to the next node. Otherwise (i.e., if there is still work in the queue when the Timer expires), the server obeys one of the following rules, each leading to a different model:

(1) The server completes *all* the work accumulated up to the Timer's expiration and then moves on to the next node.

(2) The server completes only the service of the job *currently* being served, and moves on.

(3) The server stops working *immediately* and moves on (the service of the preempted job will have to be repeated).

The RTG model defines a wide class of time-based control mechanisms, for which the classical Gated, Exhaustive and 1-Limited regimes are special cases. Indeed, using the Timer's duration as a control, rule (1) spans an entire spectrum of regimes lying between the Gated and the Exhaustive, while rule (2) spans an entire spectrum between the 1-Limited and the Exhaustive.

It is evident that each channel in the above polling systems is a single-node, $M/G/1$-type queue, with general Intermission Intervals, operated under the RTG protocol. Hence, it may appear that the results obtained in [3] can be easily extended to these polling models. However, this is not the case, since all channels are heavily dependent on each other, implying that no channel can be analyzed in isolation.

As indicated above, works on polling systems governed by time controls are few, and we mention Coffman, Fayolle and Mitrani [2], Leung [5], and Leung and Lucantoni [6]. In Coffman et al. a single server alternates between two $M/G/1$-type queues, where each queue is equipped with an exponential timer and obeys rule (3) above. The model is transformed into a boundary value problem. Leung [5] investigated a polling system controlled by rule (2) above. By using discrete Fourier transforms, the queue-length and delay distributions are numerically solved. Two vacation models with constant time-limited service and vacation-dependent, time-limited service are proposed by Leung and Lucantoni [6] for the performance analysis of stations in timed-token networks. By the use of exponential time-stages to approximate the constant time limits, they applied the matrix analytic method to solve the vacation models for the queue length distribution and the moments of the sojourn time.

In this work we study, almost simultaneously, the three models described above (henceforth called versions), and obtain various analytical results. In Section 2 we describe the system and write its laws of motion. In Section 3 we derive a set of equations for the Probability Generating Functions (PGFs) of the system state at polling instants of the nodes. Performance measures, needed for efficient operation of the systems, are derived in Section 4, while in Section 5 we obtain formulae for the PGF and mean of the queue size at each node at an arbitrary instant. In Section 6 we derive stability conditions, as well as pseudo-conservation laws, for the non-preemptive versions 1 and 2. Finally, in Section 7, we show how the general RTG model reduces to three of its special cases – the Exhaustive, Gated and 1-Limited regimes.

2 DESCRIPTION OF THE SYSTEM AND BASIC OBSERVATIONS

We consider a polling system consisting of n channels, indexed $i = 1, \ldots, n$, and a single *cycling* server, incurring switchover time $D^{(i)}$ when moving from channel i to channel $i+1$. Each channel is an $M/G/1$-type queue with jobs arriving at channel i according to a Poisson process $(A_i(t))_{t \geq 0}$ with rate λ_i. Jobs at node i require service time of duration V_i, inducing a flow of work at rate $\rho_i = \lambda_i E[V_i]$. We denote by θ_i the length of a (standard) busy period of such an $M/G/1$ queue. An Exponential Timer with intensity μ_i controls the server's visit time according to one of the three versions described in the introduction. We assume that the arrival processes, timers, service durations and switchover times are all mutually independent, and we analyze the system in steady state, where we use the following

Definitions and notation.

Polling instant:	The moment when the server enters (polls) a channel.
i-cycle:	The time interval between two successive polling instants of channel i.
$C^{(i)}$:	Length of an i-cycle.
$\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_n^{(i)})$:	System state at a polling instant of channel i, where $X_j^{(i)}$ is the number of jobs in queue j when station i is polled $(i, j = 1, 2, \ldots, n)$.
$B^{(i)}$:	Channel i's busy period, i.e., the time interval, during an i-cycle, in which the server is busy at channel i.
$N^{(i)}$:	Number of type-i jobs served during an i-cycle.
$Y^{(i)}$:	Queue size at channel i at the end of $B^{(i)}$.

Basic observations

The system's *law of motion* is given by

$$X_j^{(i+1)} = \begin{cases} X_j^{(i)} + A_j(B^{(i)}) + A_j(D^{(i)}) & j \neq i \\ \qquad Y^{(i)} \quad + A_i(D^{(i)}) & j = i \end{cases} \tag{2.1}$$

In vector form:

$$\begin{bmatrix} X_1^{(i+1)} \\ \vdots \\ X_{i-1}^{(i+1)} \\ X_i^{(i+1)} \\ X_{i+1}^{(i+1)} \\ \vdots \\ X_n^{(i+1)} \end{bmatrix} = \begin{bmatrix} X_1^{(i)} \\ \vdots \\ X_{i-1}^{(i)} \\ 0 \\ X_{i+1}^{(i)} \\ \vdots \\ X_n^{(i)} \end{bmatrix} + \begin{bmatrix} A_1(B^{(i)}) \\ \vdots \\ A_{i-1}(B^{(i)}) \\ Y^{(i)} \\ A_{i+1}(B^{(i)}) \\ \vdots \\ A_n(B^{(i)}) \end{bmatrix} + \begin{bmatrix} A_1(D^{(i)}) \\ \vdots \\ A_{i-1}(D^{(i)}) \\ A_i(D^{(i)}) \\ A_{i+1}(D^{(i)}) \\ \vdots \\ A_n(D^{(i)}) \end{bmatrix} \tag{2.2}$$

Furthermore, consider a fixed channel i. Its queue size at polling instants of channel j $(j = i+1, i+2, \ldots, n, 1, \ldots, i-1, i)$, calculated from the moment when the server left channel i, is given by:

$$X_i^{(j)} = Y^{(i)} + A_i(D^{(i)}) + \sum_{k=i+1}^{j-1} \left(A_i(B^{(k)}) + A_i(D^{(k)}) \right) \tag{2.3}$$

where the summation (here and throughout the paper) is cyclic. On the other hand, tracking the server's motion from channel j to channel i (through channels $j+1, j+2, \ldots, i-1$), we have

$$X_i^{(j)} + \sum_{k=j}^{i-1} \left(A_i(B^{(k)}) + A_i(D^{(k)}) \right) = X_i^{(i)} \tag{2.4}$$

Moreover, the length of an i-cycle is

$$C^{(i)} = \sum_{k=i}^{i-1} (B^{(k)} + D^{(k)}) \tag{2.5}$$

In what follows, $\widehat{R}(z)$ will denote the Probability Generating Function (PGF), $E[z^R]$, of a discrete r.v. $R \in \{0, 1, 2, \cdots\}$, where $|z| \leq 1$, while $\widetilde{R}(w)$ will denote the Laplace-Stieltjes Transform (LST), $E[e^{-wR}]$, of a nonnegative r.v. R, where $\mathrm{Re}(w) \geq 0$.

3 PROBABILITY GENERATING FUNCTIONS

Our goal is to compute the PGF of $\mathbf{X}^{(i)}$ $(i = 1, \ldots, n)$, defined as

$$G_i(\mathbf{z}) = G_i(z_1, \ldots, z_n) = E\left[\prod_{j=1}^{n} z_j^{X_j^{(i)}}\right], \quad |z_j| \leq 1 \quad \forall j \tag{3.1}$$

We express $G_{i+1}(\cdot)$ in terms of $G_i(\cdot)$ as follows:

Using the law of motion (2.1) we write

$$\begin{aligned}
G_{i+1}(\mathbf{z}) &= E\left[\prod_{j=1}^{n} z_j^{X_j^{(i+1)}}\right] = E\left[\prod_{j\neq i} z_j^{X_j^{(i)}+A_j(B^{(i)})+A_j(D^{(i)})} \cdot z_i^{Y^{(i)}+A_i(D^{(i)})}\right] \\
&= E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot \prod_{j\neq i} z_j^{A_j(B^{(i)})} z_i^{Y^{(i)}} \cdot \prod_{j=1}^{n} z_j^{A_j(D^{(i)})}\right] \\
&= E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot \prod_{j\neq i} z_j^{A_j(B^{(i)})} z_i^{Y^{(i)}}\right] \cdot E\left[\prod_{j=1}^{n} z_j^{A_j(D^{(i)})}\right] \\
&= E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot E\left(\prod_{j\neq i} z_j^{A_j(B^{(i)})} z_i^{Y^{(i)}} \Big| \mathbf{X}^{(i)}\right)\right] \cdot E\left[\prod_{j=1}^{n} z_j^{A_j(D^{(i)})}\right] .
\end{aligned}$$

Now,

$$\begin{aligned}
E\left[\prod_{j=1}^{n} z_j^{A_j(D^{(i)})}\right] &= E\left[E\left[\prod_{j=1}^{n} z_j^{A_j(D^{(i)})} \Big| D^{(i)}\right]\right] = \\
= E\left[e^{-\left(\sum_{j=1}^{n} \lambda_j(1-z_j)\right) D^{(i)}}\right] &= \widetilde{D}^{(i)}(s), \quad \text{where} \quad s = \sum_{j=1}^{n} \lambda_j(1-z_j) .
\end{aligned}$$

Define the joint transform of $(B^{(i)}, Y^{(i)})$, given that $X_i^{(i)} = r$, as

$$\Phi_r^{(i)}(s, z) = E[e^{-sB^{(i)}} \cdot z^{Y^{(i)}} | X_i^{(i)} = r] \tag{3.2}$$

We have

$$
\begin{aligned}
& E\left[\prod_{j\neq i} z_j^{A_j(B^{(i)})} z_i^{Y^{(i)}} \Big| \mathbf{X}^{(i)}\right] = E\left[E\left(\prod_{j\neq i} z_j^{A_j(B^{(i)})} z_i^{Y^{(i)}} \Big| B^{(i)}, Y^{(i)}, \mathbf{X}^{(i)}\right) \Big| \mathbf{X}^{(i)}\right] \\
= \; & E\left[\prod_{j\neq i} E\big[z_j^{A_j(B^{(i)})} \mid B^{(i)}, Y^{(i)}, \mathbf{X}^{(i)}\big] \cdot E\big[z_i^{Y^{(i)}} \mid B^{(i)}, Y^{(i)}, \mathbf{X}^{(i)}\big] \Big| \mathbf{X}^{(i)}\right] \\
= \; & E\left[\prod_{j\neq i} E\big[z_j^{A_j(B^{(i)})} \mid B^{(i)}\big] \cdot z_i^{Y^{(i)}} \Big| \mathbf{X}^{(i)}\right] = E\left[\prod_{j\neq i} e^{-\lambda_j(1-z_j)B^{(i)}} \cdot z_i^{Y^{(i)}} \Big| \mathbf{X}^{(i)}\right] \\
= \; & E\left[e^{-\left(\sum\limits_{j\neq i} \lambda_j(1-z_j)\right)B^{(i)}} \cdot z_i^{Y^{(i)}} \Big| \mathbf{X}^{(i)}\right] \\
= \; & E[e^{-s_i B^{(i)}} z_i^{Y^{(i)}} | \mathbf{X}^{(i)}] = E[e^{-s_i B^{(i)}} z_i^{Y^{(i)}} | X_i^{(i)}] = \Phi^{(i)}_{X_i^{(i)}}(s_i, z_i)
\end{aligned}
$$

where $s_i = \sum\limits_{j\neq i} \lambda_j(1-z_j)$.

Adapting Theorem 1 from Eliazar and Yechiali [3] to our polling structure, we can immediately write that the joint transform $\Phi_r^{(i)}$ satisfies

$$
\Phi_r^{(i)}(s_i, z_i) = \big(1 - \varphi_i(s_i, z_i)\big)\left(\widetilde{\theta}_i(s_i + \mu_i)\right)^r + \varphi_i(s_i, z_i)\big(\xi_i(s_i, z_i)\big)^r
$$

where the values of the functions $\varphi_i(w, z)$ and $\xi_i(w, z)$ are given in Table 1 below.

Table 1: Values of $\varphi_i(w, z)$ and $\xi_i(w, z)$

Version	$\varphi_i(w, z)$	$\xi_i(w, z)$
1	$\dfrac{\mu_i}{\mu_i + \lambda_i(z - \widetilde{V}_i(w + \lambda_i(1-z)))}$	$\widetilde{V}_i(w + \lambda_i(1-z))$
2	$\dfrac{\widetilde{V}_i(w + \lambda_i(1-z)) - \widetilde{V}_i(\mu_i + w + \lambda_i(1-z))}{z - \widetilde{V}_i(\mu_i + w + \lambda_i(1-z))}$	z
3	$\dfrac{\mu_i z}{\mu_i + w + \lambda_i(1-z)} \cdot \dfrac{1 - \widetilde{V}_i(\mu_i + w + \lambda_i(1-z))}{z - \widetilde{V}_i(\mu_i + w + \lambda_i(1-z))}$	z

By combining the above we obtain

$$\begin{aligned}
G_{i+1}(\mathbf{z}) &= E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot \left[\big(1-\varphi_i(s_i,z_i)\big)\big(\tilde{\theta}_i(s_i+\mu_i)\big)^{X_i^{(i)}}\right.\right. \\
&\qquad \left.\left. +\varphi_i(s_i,z_i)\big(\xi_i(s_i,z_i)\big)^{X_i^{(i)}}\right]\right] \widetilde{D}^{(i)}(s) \\
&= \big(1-\varphi_i(s_i,z_i)\big)E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot \big(\widetilde{\theta}_i(s_i+\mu_i)\big)^{X_i^{(i)}}\right] \widetilde{D}^{(i)}(s) \\
&\qquad +\varphi_i(s_i,z_i)E\left[\prod_{j\neq i} z_j^{X_j^{(i)}} \cdot \big(\xi_i(s_i,z_i)\big)^{X_i^{(i)}}\right] \widetilde{D}^{(i)}(s) \\
&= \big(1-\varphi_i(s_i,z_i)\big)G_i\big(z_1,\ldots,z_{i-1},\widetilde{\theta}_i(s_i+\mu_i),z_{i+1},\ldots,z_n\big)\widetilde{D}^{(i)}(s) \\
&\qquad +\varphi_i(s_i,z_i)G_i\big(z_1,\ldots,z_{i-1},\xi_i(s_i,z_i),z_{i+1},\ldots,z_n\big)\widetilde{D}^{(i)}(s)\ .
\end{aligned}$$

We can therefore state

Theorem 1 *For each version, the PGFs of* $\mathbf{X}^{(i)}, G_i(\cdot)$, $i = 1,\ldots,n$ *are given implicitly by the following set of* n *equations:*

Version 1:

$$\begin{aligned}
G_{i+1}(\mathbf{z}) = \Big\{&\big(1-g_i(\mathbf{z})\big)G_i(z_1,\ldots,z_{i-1},\widetilde{\theta}_i(s_i+\mu_i),z_{i+1},\ldots,z_n) \\
&+g_i(\mathbf{z})G_i\big(z_1,\ldots,z_{i-1},\widetilde{V}_i(s),z_{i+1},\ldots,z_n\big)\Big\}\,\widetilde{D}^{(i)}(s)
\end{aligned} \tag{3.3}$$

Versions 2,3:

$$\begin{aligned}
G_{i+1}(\mathbf{z}) &= \Big\{\big(1-g_i(\mathbf{z})\big)G_i(z_1,\ldots,z_{i-1},\widetilde{\theta}_i(s_i+\mu_i),z_{i+1},\ldots,z_n) \\
&\quad +g_i(\mathbf{z})G_i(\mathbf{z})\}\,\widetilde{D}^{(i)}(s)
\end{aligned} \tag{3.4}$$

where, for all versions,

$$s_i = \sum_{j\neq i}\lambda_j(1-z_j), \quad s = \sum_{j=1}^{n}\lambda_j(1-z_j) \tag{3.5}$$

and

$$g_i(\mathbf{z}) = \begin{cases} \dfrac{\mu_i}{\mu_i + \lambda_i(z_i - \widetilde{V}_i(s))} & \textit{Version 1} \\[2ex] \dfrac{\widetilde{V}_i(s) - \widetilde{V}_i(s+\mu_i)}{z_i - \widetilde{V}_i(s+\mu_i)} & \textit{Version 2} \\[2ex] \dfrac{\mu_i z_i}{s+\mu_i} \cdot \dfrac{1 - \widetilde{V}_i(s+\mu_i)}{z_i - \widetilde{V}_i(s+\mu_i)} & \textit{Version 3} \end{cases} \tag{3.6}$$

Symmetric Systems

In the case of a symmetric system when

$$\lambda_i = \lambda, \quad V_i = V, \quad \mu_i = \mu, \quad D^{(i)} = D \qquad (i = 1, 2, , \ldots, n)$$

each system of equations (3.3) and (3.4) reduces to a single equation. Indeed, due to the symmetry, we need only to find $G_1(\mathbf{z})$. To do so note that (due to the cyclic symmetry)

$$G_n(t_1, \ldots, t_{n-1}, t_n) = G_1(t_n, t_1, \ldots, t_{n-1}) \tag{3.7}$$

$$\widetilde{\theta}_n(s_n + \mu_n) = \widetilde{\theta}(s_n + \mu), \quad s_n = \lambda \sum_{j=1}^{n-1} (1 - z_j) \tag{3.8}$$

$$\widetilde{V}_n(s) = \widetilde{V}(s), \quad \widetilde{D}^{(n)}(s) = \widetilde{D}(s), \quad s = \lambda \sum_{j=1}^{n} (1 - z_j)\ . \tag{3.9}$$

Substituting (3.7)–(3.9) into (3.3) and (3.4) (for $i = n$) we obtain, respectively
Version 1:

$$\begin{aligned} G_1(\mathbf{z}) \ &= \Big\{ \big(1 - g_n(\mathbf{z})\big) G_1(\widetilde{\theta}(s_n + \mu), z_1, \ldots, z_{n-1}) + \\ &\quad + g_n(\mathbf{z}) G_1\big(\widetilde{V}(s), z_1, \ldots, z_{n-1}\big) \Big\} \widetilde{D}(s) \end{aligned} \tag{3.10}$$

Versions 2,3:

$$\begin{aligned} G_1(\mathbf{z}) \ &= \Big\{ \big(1 - g_n(\mathbf{z})\big) G_1(\widetilde{\theta}(s_n + \mu), z_1, \ldots, z_{n-1}) + \\ &\quad + g_n(\mathbf{z}) G_1\big(z_n, z_1, \ldots, z_{n-1}\big) \Big\} \widetilde{D}(s) \end{aligned} \tag{3.11}$$

where

$$g_n(\mathbf{z}) = \begin{cases} \dfrac{\mu}{\mu + \lambda(z_n - \widetilde{V}(s))} & \text{Version 1} \\[2ex] \dfrac{\widetilde{V}(s) - \widetilde{V}(s+\mu)}{z_n - \widetilde{V}(s+\mu)} & \text{Version 2} \\[2ex] \dfrac{\mu z_n}{s+\mu} \cdot \dfrac{1 - \widetilde{V}(s+\mu)}{z_n - \widetilde{V}(s+\mu)} & \text{Version 3} \end{cases} \tag{3.12}$$

4 PERFORMANCE MEASURES

Considering channel i $(i = 1, 2, \ldots, n)$ as a single $M/G/1$-type queue with general Intermission Intervals, we summarize in Table 2 below the relevant performance measures from Eliazar and Yechiali [3]. We will use these results to derive performance measures for our polling systems. Equations (4.1)–(4.7) below are derived simultaneously for all three versions.

Table 2: Performance Measures for Channel i, Viewed as a Single-Node RTG Queue

	Version 1	Version 2	Version 3
$\rho_i^* \triangleq \frac{E[B^{(i)}]}{E[C^{(i)}]}$	ρ_i	ρ_i	$\lambda_i \frac{1-\widetilde{V}_i(\mu_i)}{\mu_i \widetilde{V}_i(\mu_i)}$
$E[B^{(i)}]$	$E[V_i]E[X_i^{(i)}] + \rho_i\gamma_i$	$E[V_i]\delta_i$	γ_i
$E[C^{(i)}]$	$\frac{1}{\lambda_i}E[X_i^{(i)}] + \gamma_i$	$\frac{1}{\lambda_i}\delta_i$	$\frac{1}{\lambda_i}\widetilde{V}_i(\mu_i)\delta_i$
$E[N^{(i)}]$	$E[X_i^{(i)}] + \lambda_i\gamma_i$	δ_i	$\widetilde{V}_i(\mu_i)\delta_i$
$E[X_i^{(i)}] - E[Y_i^{(i)}]$	$(1-\rho_i)E[N^{(i)}]$	$(1-\rho_i)E[N^{(i)}]$	$(1-\rho_i^*)E[N^{(i)}]$

In the above table we use the notation

$$\gamma_i = \frac{1-\hat{X}_i^{(i)}(\widetilde{\theta}_i(\mu_i))}{\mu_i}, \qquad \delta_i = \frac{1-\hat{X}_i^{(i)}(\widetilde{\theta}_i(\mu_i))}{1-\widetilde{V}_i(\mu_i)} ,$$

where $\hat{X}_i^{(i)}(\cdot)$ is the PGF of $X_i^{(i)}$, given by the marginal transform $G_i(1,\ldots,1,\cdot,1,\ldots,1)$.

From (2.5),

$$E[C] \triangleq E[C^{(1)}] = \cdots = E[C^{(n)}] = \sum_{k=1}^{n} \left(E[B^{(k)}] + E[D^{(k)}]\right) \tag{4.1}$$

Setting $d_k = E[D^{(k)}]$, and substituting $E[B^{(k)}] = \rho_k^* E[C]$ (from Table 2) in (4.1), we get $E[C] = E[C] \sum\limits_{k=1}^{n} \rho_k^* + \sum\limits_{k=1}^{n} d_k$. Setting $\rho^* \triangleq \sum\limits_{k=1}^{n} \rho_k^*$ and $d = \sum\limits_{k=1}^{n} d_k$, gives

$$E[C] = \frac{d}{1-\rho^*} . \tag{4.2}$$

This yields immediately

$$E[B^{(i)}] = \rho_i^* \frac{d}{1-\rho^*} . \tag{4.3}$$

Since $E[N^{(i)}] = \lambda_i E[C^{(i)}]$ and $E[X_i^{(i)}] - E[Y^{(i)}] = (1-\rho_i^*)E[N^{(i)}]$ (see Table 2), we also obtain

$$E[N^{(i)}] = \lambda_i \frac{d}{1-\rho^*} \tag{4.4}$$

$$E[X_i^{(i)}] - E[Y^{(i)}] = \lambda_i (1-\rho_i) \frac{d}{1-\rho^*} . \tag{4.5}$$

Finally, from (2.4) we get

$$E[X_i^{(j)}] + \sum_{k=j}^{i-1} \left(\lambda_i E[B^{(k)}] + \lambda_i d_k\right) = E[X_i^{(i)}] \tag{4.6}$$

and hence

$$E[X_i^{(i)}] - E[X_i^{(j)}] = \lambda_i \left(\frac{d}{1-\rho^*} \sum_{k=j}^{i-1} \rho_k^* + \sum_{k=j}^{i-1} d_k \right) . \tag{4.7}$$

Observe that, for general Intermission Intervals (i.e. general, often system dependent, random time-periods in which the server is unavailable for service), almost all performance measures depend on the terms γ_i and δ_i, which are functions of the unknown quantities $\hat{X}_i^{(i)}$ $(\widetilde{\theta}_i(\mu_i))$. However, in our polling systems, knowing the structure of the Intermissions enables us to compute explicitly the values of $E[C], E[B^{(i)}], E[N^{(i)}]$, etc. This, in turn, yields the values of γ_i and δ_i. Specifically,

$$\gamma_i = \frac{d}{1-\rho} - \frac{1}{\lambda_i} E[X_i^{(i)}] \qquad \text{Version 1} \qquad (4.8)$$

$$\delta_i = \begin{cases} \lambda_i \dfrac{d}{1-\rho} & \text{Version 2} \qquad (4.9) \\ \\ \dfrac{\lambda_i}{\widetilde{V}_i(\mu_i)} \cdot \dfrac{d}{1-\rho^*} & \text{Version 3} \qquad (4.10) \end{cases}$$

and $\mu_i \gamma_i = (1 - \widetilde{V}_i(\mu_i))\delta_i$ for every version.

Observe also that in all versions the quantity $E[X_i^{(i)}]$ is unknown. Thus, in contrast to the classical Exhaustive or Gated regimes (see e.g. Yechiali [7]), where explicit expressions are obtained for $E[X_i^{(j)}]$, here one gets explicit expressions only for the *differences* $E[X_i^{(i)}] - E[X_i^{(j)}]$. Recall that the same situation arises in the 1-Limited case.

5 QUEUE SIZES AND WAITING TIMES

In this section we derive the PGF and expectation of $L^{(i)}$, the queue size in channel i at an arbitrary instant in steady state. Again, we rely heavily on the general results for the *single-node* RTG model [3]. For each version we state the general results and then give their counterparts for the Polling case.

Version 1

From [3], the general single-node results are equations (5.1) and (5.2) below:

$$\widehat{L}(z) = \frac{\widetilde{V}(\lambda(1-z))}{\mu + \lambda(z - \widetilde{V}(\lambda(1-z)))} \cdot \left\{ \mu + \frac{\lambda}{E[N]}\big(\hat{X}(z) - 1\big) + \frac{\mu}{E[N]}\left(\frac{\hat{X}(z) - \hat{X}(\widetilde{V}(\lambda(1-z)))}{z - \widetilde{V}(\lambda(1-z))} - E[X] \right) \right\} \tag{5.1}$$

$$E[L] = \rho - \frac{\lambda}{\mu}(1-\rho) + \frac{\frac{1+\rho}{2}E[X(X-1)] + \frac{\lambda}{\mu}E[X]}{E[N]} \tag{5.2}$$

For our Polling model, using (5.1) and (5.2) above, together with (4.4), yields

$$\widehat{L}^{(i)}(z) = \frac{\widetilde{V}_i(\lambda_i(1-z))}{\mu_i + \lambda_i(z - \widetilde{V}_i(\lambda_i(1-z)))} \tag{5.3}$$
$$\cdot \left\{ \mu_i + \frac{1-\rho}{d}\big(\hat{X}_i^i(z) - 1\big) + \frac{1-\rho}{d}\frac{\mu_i}{\lambda_i}\left(\frac{\hat{X}_i^{(i)}(z) - \hat{X}_i^{(i)}(\widetilde{V}_i(\lambda_i(1-z)))}{z - \widetilde{V}_i(\lambda_i(1-z))} - E[X_i^{(i)}] \right) \right\}$$

$$E[L^{(i)}] = \rho_i - \frac{\lambda_i}{\mu_i}(1-\rho_i) + \frac{1-\rho}{d}\left(\frac{1+\rho_i}{2\lambda_i} E\big[X_i^{(i)}(X_i^{(i)} - 1)\big] + \frac{1}{\mu_i}E[X_i^{(i)}] \right) \tag{5.4}$$

Version 2

The general single-node results [3] are:

$$\widehat{L}(z) = \frac{\widetilde{V}(\lambda(1-z))}{z - \widetilde{V}(\mu + \lambda(1-z))}\left(\big(1 - \widetilde{V}(\mu)\big) - \frac{1 - \hat{X}(z)}{\delta} \right) \tag{5.5}$$

$$E[L] = \frac{1}{1 - \widetilde{V}(\mu)}\left(\frac{E[X]}{\delta} - \widetilde{V}(\mu) - \lambda \widetilde{V}'(\mu) \right) - (1-\rho) \tag{5.6}$$

For the Polling model, using (5.5) and (5.6) above together with (4.9), yields

$$\widehat{L}^{(i)}(z) = \frac{\widetilde{V}_i(\lambda_i(1-z))}{z - \widetilde{V}(\mu_i + \lambda_i(1-z))} \left(\big(1 - \widetilde{V}_i(\mu_i)\big) - \frac{1-\rho}{d} \frac{1 - \hat{X}_i^{(i)}(z)}{\lambda_i} \right) \tag{5.7}$$

$$E[L^i] = \frac{1}{1 - \widetilde{V}_i(\mu_i)} \left(\frac{1-\rho}{d} \frac{E[X_i^i]}{\lambda_i} - \widetilde{V}_i(\mu_i) - \lambda_i \widetilde{V}_i'(\mu_i) \right) - (1 - \rho_i) \tag{5.8}$$

Version 3

The general single-node results [3] are:

$$\widehat{L}(z) = \frac{1}{\widetilde{V}(\mu)} \frac{\widetilde{V}(\mu + \lambda(1-z))}{z - \widetilde{V}(\mu + \lambda(1-z))} \left(\big(1 - \widetilde{V}(\mu)\big) - \frac{1 - \hat{X}(z)}{\delta} \right) \tag{5.9}$$

$$E[L] = \frac{1}{1 - \widetilde{V}(\mu)} \left(\frac{E[X]}{\delta} - 1 - \lambda \frac{\widetilde{V}'(\mu)}{\widetilde{V}(\mu)} \right) \tag{5.10}$$

The corresponding Polling model formulas are obtained by using (5.9) and (5.10) above, together with (4.10):

$$\widehat{L}^{(i)}(z) = \frac{\widetilde{V}_i(\mu_i + \lambda_i(1-z))}{z - \widetilde{V}_i(\mu_i + \lambda_i(1-z))} \left(\frac{1 - \widetilde{V}_i(\mu_i)}{\widetilde{V}_i(\mu_i)} - \frac{1-\rho^*}{d} \frac{1 - \hat{X}_i^{(i)}(z)}{\lambda_i} \right) \tag{5.11}$$

$$E[L^{(i)}] = \frac{1}{1 - \widetilde{V}_i(\mu_i)} \left(\frac{1-\rho^*}{d} \frac{\widetilde{V}_i(\mu_i)}{\lambda_i} E[X_i^{(i)}] - 1 - \lambda_i \frac{\widetilde{V}_i'(\mu_i)}{\widetilde{V}_i(\mu_i)} \right) \tag{5.12}$$

Waiting Times

For all three versions, being $M/G/1$-type systems, and assuming First Come First Served order of service, the waiting (sojourn) time of a job in channel i, $W^{(i)}$, obeys

$$\widetilde{W}^{(i)}\big(\lambda_i(1-z)\big) = \widehat{L}^{(i)}(z), \quad E[W^{(i)}] = E[L^{(i)}]/\lambda_i \tag{5.13}$$

6 STABILITY CONDITIONS AND PSEUDO CONSERVATION LAWS

In this section we derive necessary and sufficient conditions for stability, as well as pseudo conservation laws, for the non-preemptive Versions 1 and 2.

We rely on Fricker and Jaïbi's [4] result stating that a necessary and sufficient condition for the stability of a periodic polling system is

$$\rho + d \cdot \max_{1 \le i \le n} \left\{ \frac{\lambda_i}{m_i} \right\} < 1 \tag{6.1}$$

where m_i is the mean number of type-i jobs that would be served during a cycle if there were an *infinite* number of type-i jobs present at channel i at the start of the cycle.

In our case this means that

$$m_i = E[N^{(i)} | X_i^{(i)} = \infty] \ . \tag{6.2}$$

From Table 1 it readily follows that for Version 1, $m_i = \infty$ so that the necessary and sufficient condition is $\rho < 1$.

For Version 2, $m_i = \frac{1}{1-\widetilde{V}_i(\mu_i)}$, leading to the condition

$$\rho + d \cdot \max_{1 \le i \le n} \left\{ \lambda_i \big(1 - \widetilde{V}_i(\mu_i) \big) \right\} < 1 \ .$$

Boxma and Groenedijk [1] derived the following pseudo conservation law for the quantity $\overline{W} \equiv \sum_{i=1}^{n} \rho_i E[W^{(i)}] = \sum_{i=1}^{n} E[V_i] E[L^{(i)}]$ in non-preemptive polling systems:

$$\overline{W} = \rho \frac{\sum_{i=1}^{n} \lambda_i E[V_i^2]}{2(1-\rho)} + \rho \frac{E[D^2]}{2E[D]} + \frac{E[D]}{2(1-\rho)} \left(\rho^2 - \sum_{i=1}^{n} \rho_i^2 \right) + \sum_{i=1}^{n} u_i \tag{6.3}$$

where $D = \sum_{i=1}^{n} D^{(i)}$, and u_i is the expected amount of *unfinished* work in channel i at the instant of the server's departure from that channel, i.e. $u_i = E\left[\sum_{k=1}^{Y^{(i)}} V_{ik} \right]$, where V_{ik} are i.i.d. $\sim V_i$.

For Versions 1 and 2, using (4.5),

$$u_i = E[V_i]E[Y^{(i)}] = E[V_i]E[X_i^i] - \rho_i(1-\rho_i)\frac{d}{1-\rho} .$$

Hence, for both versions (each with its corresponding $E[X_i^{(i)}]$),

$$\begin{aligned}\overline{W} = \rho\frac{\sum\limits_{i=1}^{n}\lambda_i E[V_i^2]}{2(1-\rho)} + \rho\frac{E[D^2]}{2d} \\ +\frac{d}{2(1-\rho)}\left(\sum_{i=1}^{n}\rho_i^2+\rho^2-2\rho\right)+\sum_{i=1}^{n}E[V_i]E[X_i^{(i)}]\end{aligned} \tag{6.4}$$

It follows, from the definition of $\overline{W}$ and (6.4), that

$$\sum_{i=1}^{n} E[V_i]\big(E[L^{(i)}] - E[X_i^{(i)}]\big) \equiv \text{constant} \tag{6.5}$$

where the constant's value is the sum of the first three terms in the right-hand side of (6.4).

7 SPECIAL CASES

In this section we show how the well known Exhaustive, Gated and 1-Limited regimes are obtained as limiting cases of the general RTG protocol.

The *Exhaustive* regime is obtained by extending the Timers' durations indefinitely (i.e. $\mu_i \to 0$ for $i = 1, 2, \ldots, n$). Indeed, in Theorem 1, $g_i(\mathbf{z}) \to 0$ ($i = 1, 2, \ldots, n$) for all three Versions. Hence, results (3.3) for Version 1, as well as (3.4) for Versions 2 and 3, reduce to

$$G_{i+1}(\mathbf{z}) = G_i\big(z_1, z_2, \ldots, z_{i-1}, \widetilde{\theta}_i(s_i), z_{i+1}, \ldots, z_n\big)\widetilde{D}^{(i)}(s) \tag{7.1}$$

which is the well known system of equations for the pure Exhaustive regime (see Yechiali [7]).

The *Gated* regime is obtained from Version 1 by shrinking the Timers' durations to zero (i.e. $\mu_i \to \infty$). Then, $g_i(\mathbf{z}) \to 1$, $\widetilde{\theta}_i(s_i + \mu_i) \to 0$, so that (3.3) becomes

$$G_{i+1}(\mathbf{z}) = G_i\big(z_1, z_2, \ldots, z_{i-1}, \widetilde{V}_i(s), z_{i+1}, \ldots, z_n\big)\widetilde{D}^{(i)}(s) . \tag{7.2}$$

Again, (8.2) is the well known set of equations for the pure Gated regime. Finally, the *1-Limited* protocol is obtained from Version 2 by shrinking the Timers' duration to zero. This results in $g_i(\mathbf{z}) \to \frac{\widetilde{V}_i(s)}{z_i}$, $\widetilde{\theta}_i(s_i + \mu_i) \to 0$, so that (3.4) becomes the well known set of equations for the 1-Limited case, i.e.

$$\begin{aligned} G_{i+1}(\mathbf{z}) &= \left\{ \left(1 - \frac{\widetilde{V}_i(s)}{z_i}\right) G_i(z_1, z_2, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_n) \right. \\ &\qquad \left. + \frac{\widetilde{V}_i(s)}{z_i} G_i(\mathbf{z}) \right\} \cdot \widetilde{D}^{(i)}(s) \ . \qquad (7.3) \end{aligned}$$

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