Polling Systems with Station Breakdowns

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Abstract

None of the works in the literature on Polling Systems deals with the important phenomenon of failing nodes. This paper addresses this issue.

We assume that there are two modes of station breakdowns: node-dependent and general. Each node is subject to a random failure process and a failure of a node during a visit of the server causes the latter to move to the next station. We carry performance analyses of both the Gated and the Exhaustive service regimes and derive probability generating functions, Laplace transforms and means of variables such as queue sizes at polling and at arbitrary instants, durations of busy periods, number of jobs left behind at server's station-departure moments, and cycle times. The analysis of the Exhaustive case is achieved via an explicit solution of an infinite set of linear equations, where the unknowns are the state-dependent joint transforms of two key characteristics: the number of arrivals to a station during a busy period starting with a given number of jobs, and the number of jobs successfully served during that period. Stability conditions are indicated for each regime.

1 Introduction

Polling systems have been studied extensively in recent years (see, for example, Takagi [6], [7], [8], Yechiali [9] and references there) due to their wide applicability as models for numerous processes in telecommunications, flexible manufacturing systems, reliability, traffic control and alike. However, none of the works in the literature deals with the important phenomenon of breaking, or failing, nodes, as is the case in real systems. This paper addresses this issue.

We consider polling systems with an infinite-capacity queue at each node (station, channel), Poisson arrival processes, general independent service times and general independent switchover times. We assume that there are two modes of station breakdowns: node-dependent, and general. If the server enters a non-empty station and starts serving customers (jobs, messages) present there, the station may fail due to a node-dependent breakdown process. We assume that a breakdown (if occurs) is observed only at the end of a service (e.g. transmission of a message). The server then switches on to the next station, while the job (message) 'suffering' the breakdown will have to be processed (transmitted) again during the next server's visit. We further assume that due to an overall general 'noise' in the system at large, whenever the server switches to station i, the station is found to be "*up*" with probability p_i, and "*down*" with probability $(1-p_i)$. These probabilities are independent of whether the station was up or down when last visited. If the server finds that the station is down, it immediately switches on to the next node.

In Section 2 we analyze the system under the Gated service protocol where, if a station is up and non-empty when polled, the server will attempt to serve only those jobs present in the queue at the polling instant. We define the evolution equations of the system-state at polling instants and derive its Probability Generating Functions (PGFs), as well as the Laplace-Stieltjes transform (LST) and mean of the length of the busy period at each queue. We also derive the PGF of the number of jobs left at a queue at the end of a server's visit, and calculate its mean. We further derive the PGF of the system state at any *arbitrary* moment and obtain a set of equations for the calculations of mean number of jobs at the various queues at polling instants. We conclude the section with a formula for the mean cycle time.

In Section 3 we study the system under the Exhaustive service protocol in which, if the server enters a non-failed and non-empty station, it continues serving customers there until either all work at the station is exhausted or a breakdown is observed, whichever occurs first. We show that, under both the Gated and Exhaustive service procedures, the mean cycle time is the same. The analysis of the Exhaustive case is achieved via an explicit solution of an infinite set of linear equations where the unknowns are the state-dependent joint transforms of the number of arrivals to a station during a busy period starting with a given number of jobs, and the number of jobs successfully served during that period. We derive the PGF of the system-state at polling instants, and calculate the mean length of a busy period in each queue, as well as the mean number of customers served during a busy period. We conclude with a derivation of the PGF of the system-state at an arbitrary moment.

We will use the following notation and assumptions:

- N = number of stations, indexed $i = 1, 2, \ldots, N$.

- $A^i(t)$ = number of customers (jobs, messages) arriving to node i during a time interval of length t. The arrival process is Poisson with intensity λ_i.

- T_i = time to breakdown of station i. T_i is Exponentially distributed with parameter γ_i. This station-dependent process is regenerated at the beginning of every new visit of the server.

- $(B_{im})_{m=1}^{\infty}$ = sequence of i.i.d. service times at queue i, all distributed like B_i with mean b_i.

- D_i = switchover time from queue i to queue $i+1$, with $E[D_i] = d_i$.

- $S^*(w) \equiv E[e^{-wS}]$ = LST of a non-negative continuous random variable S.

- The server visits the stations in a *cyclic* order, i.e.

$$1 \to 2 \to \cdots \to i \to i+1 \to \cdots \to N-1 \to N \to 1 \to \cdots$$

- 'Polling instant' of node i is the moment when the server enters station i, after completion of a switchover period, D_{i-1}, from station $i-1$ to station i.

- X_i^j = number of jobs at queue $j (j = 1, 2, \ldots, N)$ when node i is polled.

- $\underline{X}_i \equiv (X_i^1, X_i^2, \ldots, X_i^j, \ldots, X_i^N)$ = system-state at polling instant of node i.

- $G_i(z_1, z_2, \ldots, z_N) = G_i(\underline{z}) \equiv E[\prod_{j=1}^{N} z_j^{X_i^j}]$ = PGF of the system-state at a polling instant of node i.

- $\widehat{X}(z) \equiv E[z^X]$ = PGF of a non-negative, integer-valued, random variable X.

2 The Gated Regime

For the analysis of the Gated regime we need the following variables:

- $Y_i(k)$ = number of customers that are *successfully* served in queue i during a visit of the server, given that node i is up and k customers are present at the polling instant. Clearly, $Y_i(0) = 0$, but for $k \geq 1$, $Y_i(k) = k$ only if station i does *not* break down during the visit of the server.

- $\widehat{Y}_i(k) = \min(Y_i(k) + 1, k)$. $\widehat{Y}_i(k)$ is the number of service attempts if node i is up and there are k customers present at the polling instant of station i.

2.1 Evolution Equations

The evolution equations for the system state $\underline{X}$ at polling instants depends on whether a node is *up* or is *down* when polled.

If node i is *down* when polled, the server immediately moves to the next station, and hence

$$X_{i+1}^j = X_i^j + A^j(D_i) \qquad i, j = 1, 2, \ldots, N\ . \tag{2.1}$$

If node i is *up* when polled, then

$$X_{i+1}^j = \begin{cases} X_i^j + A^j\Big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\Big) + A^j(D_i), & j \neq i \\ X_i^i + A^i\Big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\Big) + A^i(D_i) - Y_i(X_i^i), & j = i \end{cases} \tag{2.2}$$

If node i is *up* but $X_i^i = 0$, the server switches over to the next station.

Let $\theta_{X_i^i}^i$ denote the length of a busy period in station i, *starting* with X_i^i jobs. Clearly,

$$\theta_{X_i^i}^i = \sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\ .$$

As seen from Equation (2.2), $\theta^i_{X^i_i}$ plays an important role in the analysis. We turn to derive its distribution and mean.

2.2 The Busy Period $\theta^i_{X^i_i}$

For fixed r, $r = 0, 1, 2 \dots$ let θ^i_r be the duration of a busy period in station i, under the Gated regime (and with possible failing node), *initiated* with r jobs.

Let $\sigma^i_k \equiv \sum_{i=1}^{k} B_{im}$ be the duration of k consecutive services in station i, $k = 1, 2, 3 \dots$, and set $\sigma^i_0 = 0$. Also, let R^i_r be the number of *remaining* jobs at the end of a busy period θ^i_r, with $R^i_0 = 0$.

Define

$$\begin{aligned} &\Phi^i_r(w, z) = E\Big[e^{-\omega\theta^i_r} z^{R^i_r}\Big] \\ &\Phi^i_0(\omega, z) = 1 \end{aligned} \tag{2.3}$$

Suppressing i, we have

$$(\theta_r, R_r) = \sum_{k=1}^{r} (\sigma_k, r-(k-1)+A(\sigma_k)) 1_{(\sigma_{k-1},\sigma_k)}(T) + (\sigma_r, A(\sigma_r)) 1_{(\sigma_r,\infty)}(T)$$

where, $1_{(a,b)}(T) = \begin{cases} 1 & \text{if } a < T < b \\ 0, & \text{otherwise} \end{cases}$

Now,

$$\begin{aligned} &E\Big[e^{-\omega\sigma_k} z^{r-(k-1)+A(\sigma_k)} 1_{(\sigma_{k-1},\sigma_k)}(T)\Big] \\ &\quad = z^{r-(k-1)} E\Big[e^{-\omega\sigma_k} E\Big[z^{A(\sigma_k)} | B_1, \dots, B_k\Big] P(\sigma_{k-1} < T < \sigma_k | B_1, \dots, B_k)\Big] \\ &\quad = z^{r-(k-1)} E\Big[e^{-\omega\sigma_k} e^{-\lambda(1-z)\sigma_k} (e^{-\gamma\sigma_{k-1}} - e^{-\gamma\sigma_k})\Big] \\ &\quad = z^{r-(k-1)} E\Big[e^{-(\omega+\lambda(1-z)+\gamma)\sigma_{k-1}} \cdot e^{-(\omega+\lambda(1-z))B_k} - e^{-(\omega+\lambda(1-z)+\gamma)\sigma_k}\Big] \\ &\quad = z^{r-(k-1)} \Big[\sigma^*_{k-1}(\omega+\lambda(1-z)+\gamma) B^*(\omega+\lambda(1-z)) - \sigma^*_k(\omega+\lambda(1-z)+\gamma)\Big] \\ &\quad = z^r \Big[B^*(\omega+\lambda(1-z)) - B^*(\omega+\lambda(1-z)+\gamma)\Big] \Big[\frac{B^*(\omega+\lambda(1-z)+\gamma)}{z}\Big]^{k-1} \\ &\quad = z^r \Big[B^*(\delta^-) - B^*(\delta)\Big] \Big(\frac{B^*(\delta)}{z}\Big)^{k-1} \equiv z^r \cdot \beta \cdot \Big(\frac{B^*(\delta)}{z}\Big)^{k-1}, \end{aligned}$$

where $\delta \equiv \omega + \lambda(1-z) + \gamma$; $\delta^- \equiv \omega + \lambda(1-z)$; $\beta \equiv B^*(\delta^-) - B^*(\delta)$.

We further have,

$$\begin{aligned} & E\Big[e^{-\omega\sigma_r} z^{A(\sigma_r)} 1_{(\sigma_r,\infty)}(T)\Big] = E\Big[e^{-\omega\sigma_r} E\Big[z^{A(\sigma_r)} | \sigma_r\Big] \cdot P(T > \sigma_r | \sigma_r)\Big] \\ = \; & E\Big[e^{-\omega\sigma_r} e^{-\lambda(1-z)\sigma_r} e^{-\gamma\sigma_r}\Big] = \sigma^*_r(\omega+\lambda(1-z)+\gamma) = B^*(\delta)^r \end{aligned}$$

Thus,

$$\begin{aligned}\Phi_r(\omega, z) &= \sum_{k=1}^{r} z^r \beta \Big(\frac{B^*(\delta)}{z}\Big)^{k-1} + B^*(\delta)^r \\ &= z^r \beta \Big[\frac{1-(B^*(\delta)/z)^r}{1-B^*(\delta)/z}\Big] + B^*(\delta)^r = \frac{z\beta}{z-B^*(\delta)}(z^r - B^*(\delta)^r) + B^*(\delta)^r \ .\end{aligned}$$

To summarize,

$$\Phi_r(\omega, z) = \varphi_1(\omega, z) \cdot z^r + (1-\varphi_1(\omega, z))B^*(\delta)^r \tag{2.4}$$

where

$$\varphi_1(\omega, z) = \frac{z[B^*(\delta^-) - B^*(\delta)]}{z - B^*(\delta)} \ . \tag{2.5}$$

Finally, taking expectation over X_i^i, we obtain

$$E\Big[\Phi^i_{X_i^i}(w, z)\Big] = \varphi_1^i(\omega, z)\widehat{X}_i^i(z) + (1-\varphi_1^i(\omega, z))\widehat{X}_i^i(B_i^*(\delta_i))$$

where

$$\varphi_1^i(\omega, z) = \frac{z[B_i^*(\omega + \lambda_i(1-z)) - B_i^*(\omega + \lambda_i(1-z) + \gamma_i)]}{z - B_i^*(\omega + \lambda_i(1-z) + \gamma_i)} \ .$$

Now,

$$\begin{aligned}&E\Big[e^{-\omega\theta_r^i}\Big] = \Phi_r^i(\omega, 1) \\ &\varphi_1^i(\omega, 1) = \frac{B_i^*(\omega) - B_i^*(\omega+\gamma_i)}{1 - B_i^*(\omega + \gamma_i)} \quad , \quad \varphi_1^i(0,1) = 1 \\ &\frac{d}{d\omega}\varphi_1^i(\omega, 1)\Big|_{\omega=0} = \frac{-b_i}{1-B_i^*(\gamma_i)} \\ &-\frac{d}{d\omega}\Phi_r^i(\omega, 1)\Big|_{\omega=0} = -\frac{d}{d\omega}\varphi_1^i(\omega, 1)\Big|_{\omega=0} \cdot \Big[1 - B_i^*(\gamma_i)^r\Big] \ .\end{aligned}$$

Thus,

$$E[\theta_r^i] = \frac{b_i}{1 - B_i^*(\gamma_i)} \cdot [1 - B_i^*(\gamma_i)^r] \ . \tag{2.6}$$

Finally,

$$E[\theta^i(\text{Gated})] \equiv E[E\Big[\theta^i_{X_i^i}|X_i^i\Big]\Big] = \frac{b_i}{1 - B_i^*(\gamma_i)}[1 - \widehat{X}_i^i(B_i^*(\gamma_i))\Big] \ . \tag{2.7}$$

Interpretation

Suppressing i, we have

$$\begin{aligned}&E[\text{Min}(B, T)] = E\Big[B \cdot 1_{[B<T]} + T \cdot 1_{[B\geq T]}\Big] \\ = \ &E\Big[Be^{-\gamma B}\Big] + E\Big[\int_0^B t\gamma e^{-\gamma t} dt\Big] = E\Big[Be^{-\gamma B}\Big] + E\Big[-Be^{-\gamma B} + \frac{1}{\gamma}(1-e^{-\gamma B})\Big] = \frac{1-B^*(\gamma)}{\gamma} \ .\end{aligned}$$

Similarly,

$$E\Big[\mathrm{Min}(\sigma_X, T)\Big] = E\left[\frac{1-\sigma_X^*(\gamma)}{\gamma}\right] = \frac{E[1-B^*(\gamma)^X]}{\gamma} = \frac{1-\widehat{X}(B^*(\gamma))}{\gamma} \ .$$

Thus,

$$E\Big[\theta^i(\mathrm{Gated})\Big] = b_i \cdot \frac{E\Big[\mathrm{Min}\Big(\sigma^i_{X_i^i}, T_i\Big)\Big]}{E\Big[\mathrm{Min}(B_i, T_i)\Big]} \ . \tag{2.8}$$

2.3 The Distribution of a Successful and Unsuccessful Service Attempt

Suppressing i, let S be the duration of a *successful* service attempt, i.e. the duration of a service that has been completed before T. Then $S \sim B\Big|_{B<T}$. Now,

$$\begin{aligned} E\Big[e^{-\omega S}\Big] \equiv S^*(\omega) &= E\Big[e^{-\omega B} \mid B < T\Big] = \frac{E\Big[e^{-\omega B}\cdot 1_{[B<T]}\Big]}{P(B<T)} \\ &= \frac{E\Big[e^{-\omega B}\cdot P(B<T|B)\Big]}{E[P(B<T \mid B)]} = \frac{E\Big[e^{-\omega B}\cdot e^{-\gamma B}\Big]}{E[e^{-\gamma B}]} = \frac{B^*(\omega+\gamma)}{B^*(\gamma)} \ . \end{aligned}$$

Thus,

$$S_i^*(w) = \frac{B_i^*(\omega+\gamma_i)}{B_i^*(\gamma_i)} \tag{2.9}$$

and

$$E[S_i] = -\frac{B_i^{*\prime}(\gamma_i)}{(B_i^*(\gamma_i))} = \frac{E\Big[Be^{-\gamma_i B_i}\Big]}{B_i^*(\gamma_i)} \ . \tag{2.10}$$

Let $\widehat{S} \sim B\Big|_{B\geq T}$ be the duration of an *unsuccessful* service attempt. Then

$$\begin{aligned} E\Big[e^{-\omega\widehat{S}}\Big] \equiv \widehat{S}^*(\omega) &= E\Big[e^{-\omega B} \mid B \geq T] = \frac{E\Big[e^{-\omega B}\cdot 1_{[B\geq T]}\Big]}{P(B\geq T)} \\ &= \frac{E\Big[e^{-\omega B}P(B\geq T|B)\Big]}{E[P(B\geq T|B]} = \frac{E\Big[e^{-\omega B}(1-e^{-\gamma B})\Big]}{E[1-e^{-\gamma B}]} \ . \end{aligned}$$

Thus,

$$\widehat{S}_i^*(\omega) = \frac{B_i^*(\omega)-B_i^*(\omega+\gamma_i)}{1-B_i^*(\gamma_i)} \ . \tag{2.11}$$

Hence,

$$E[\widehat{S}_i] = \frac{b_i + B_i^{*\prime}(\gamma_i)}{1-B_i^*(\gamma_i)} \ . \tag{2.12}$$

Using (2.10) and (2.12), it is easy to check that

$$E[B_i] = E[S_i]P[B_i < T_i] + E[\widehat{S}_i]P[B_i \geq T_i] = b_i \ .$$

2.4 An Alternative Derivation of $E[\theta^i(\text{Gated})]$

With the above expressions for the distributions and means of S_i and $\widehat{S}_i$ one can derive $E[\theta^i(\text{Gated})]$ 'directly' as follows. The probability of a successful service in station i is

$$a_i \equiv P[T_i > B_i] = E[P[T_i > B_i | B_i]] = E\Big[e^{-\gamma_i B_i}\Big] = B_i^*(\gamma_i) \ .$$

Because of the memoryless property of the breakdown process, the probability distribution of $Y_i(X_i^i)$, the number of successful service completions during a visit of the server in station i, is given by

$$P[Y_i(X_i^i) = \ell] = \begin{cases} a_i^\ell(1-a_i) \ , & \ell = 0, 1, 2, \ldots, X_i^i - 1 \\ a_i^{X_i^i} \ , & \ell = X_i^i \end{cases} \tag{2.13}$$

Clearly, if $X_i^i = 0$ then $P[Y_i(0) = 0] = P[\widehat{Y}_i(0) = 0] = 1$.

Now, if $Y_i(X_i^i) = \ell < X_i^i$, there were exactly ℓ *successful* services and *one* unsuccessful service attempt, and if $Y_i(X_i^i) = X_i^i$, then *all* service attempts were successful. Hence,

$$\begin{aligned} E\Big[\theta^i_{X_i^i} | X_i^i\Big] &= \sum_{\ell=0}^{X_i^i - 1} \Big[\ell \cdot E[S_i] + E[\widehat{S}_i]\Big] a_i^\ell (1-a_i) + X_i^i E[S_i] a_i^{X_i^i} \\ &= E[S_i]\left[\frac{a_i(1 - a_i^{X_i^i - 1})}{1-a_i} - (X_i^i - 1)a_i^{X_i^i}\right] + E[\widehat{S}_i](1 - a_i^{X_i^i}) + X_i^i E[S_i] a_i^{X_i^i} \\ &= E[S_i]\left(\frac{a_i(1 - a_i^{X_i^i})}{1-a_i}\right) + E[\widehat{S}_i](1 - a_i^{X_i^i}) \ . \end{aligned}$$

Substituting the values of $E[S_i]$ and $E[\widehat{S}_i]$ From (2.10) and (2.12), we get

$$\begin{aligned} E\Big[\theta^i_{X_i^i} | X_i^i\Big] &= \frac{-B_i^{*\prime}(\gamma_i)}{a_i}\Big(\frac{a_i - a_i^{X_i^i + 1}}{1-a_i}\Big) + \frac{b_i + B^{*\prime}(\gamma_i)}{1-a_i}(1 - a_i^{X_i^i}) \\ &= \frac{b_i}{1-a_i}\Big[1 - a_i^{X_i^i}\Big] \ . \end{aligned}$$

It follows that $E\Big[\theta^i(\text{Gated})\Big] = \frac{b_i}{1-a_i}\Big[1 - \widehat{X}_i^i(B_i^*(\gamma_i))\Big]$, as given by Equation (2.7).

2.5 Expected Number of Jobs Left at the End of a Visit

The generating function of R_r^i is derived From (2.3):

$$\widehat{R}_r^i(z) = E\Big[z^{R_r^i}\Big] = \Phi_r^i(0, z) \ .$$

From (2.4)

$$\Phi_r(0, z) = \varphi_1(0, z) z^r + (1 - \varphi_1(0, z))\Big[B^*(\lambda(1-z) + \gamma)\Big]^r$$

where, from (2.5)

$$\varphi_1(0,z) = \frac{z[B^*(\lambda(1-z)) - B^*(\lambda(1-z)+\gamma)]}{z - B^*(\lambda(1-z)+\gamma)} \ .$$

Also, $\frac{d}{dz}\varphi_1(0,z)\Big|_{z=1} = \frac{\lambda b - B^*(\gamma)}{1-B^*(\gamma)}$. Hence,

$$\begin{aligned} E[R_r] &= \frac{d}{dz}\Phi_r(0,z)\Big|_{z=1} = \frac{\lambda b - B^*(\gamma)}{1-B^*(\gamma)} + r\varphi_1(0,1) - \frac{\lambda b - B^*(\gamma)}{1-B^*(\gamma)}[B^*(\gamma)]^r \\ &= \frac{\lambda b - B^*(\gamma)}{1-B^*(\gamma)}[1 - B^*(\gamma)^r] + r \ . \end{aligned}$$

It follows that

$$E\Big[R^i_{X^i_i}\Big] = \frac{\lambda_i b_i - B^*_i(\gamma_i)}{1-B^*_i(\gamma_i)}[1 - \widehat{X}^i_i(B^*_i(\gamma_i))] + E[X^i_i] \ . \tag{2.14}$$

Going back to Equation (2.2) we can write

$$X^i_{i+1} = R^i_{X^i_i} + A^i(D_i) \ ,$$

that is

$$R^i_{X^i_i} = X^i_i + A^i\Bigg(\sum_{m=1}^{\widehat{Y}_i(X^i_i)} B_{im}\Bigg) - Y_i(X^i_i) \tag{2.15}$$

so that

$$E\Big[R^i_{X^i_i}\Big] = E[X^i_i] + E\Bigg[A^i\Big(\sum_{m=1}^{\widehat{Y}_i(X^i_i)} B_{im}\Big)\Bigg] - E\Big[Y_i(X^i_i)\Big] \ .$$

Applying (2.13) we get

$$\begin{aligned} E\Big[Y_i(X^i_i)\Big] &= E\Bigg[\sum_{\ell=0}^{X^i_i-1} \ell a_i^\ell(1-a_i) + X^i_i a_i^{X^i_i}\Bigg] \\ &= E\Bigg[\frac{a_i}{1-a_i}\Big(1 - a_i^{X^i_i}\Big)\Bigg] = \frac{a_i}{1-a_i}\Big[1 - \widehat{X}_i(a_i)\Big] \ . \end{aligned} \tag{2.16}$$

From (2.14) and (2.16) it follows that (recall, $a_i \equiv B^*_i(\gamma_i)$)

$$E\Bigg[A^i\Big(\sum_{m=1}^{\widehat{Y}_i(X^i_i)} B_{im}\Big)\Bigg] = \frac{\lambda_i b_i}{1-a_i}\Big[1 - \widehat{X}_i(a_i)\Big] \ . \tag{2.17}$$

In view of (2.17) it is interesting to calculate $E[\widehat{Y}_i(X^i_i)]$, where $\widehat{Y}_i(X^i_i)$ is the total number of service attempts during a visit of station i (before the server leaves, either because all X^i_i jobs are successfully completed, or the station breaks down).

$$\begin{aligned} E\Big[\widehat{Y}_i(X^i_i)\Big] &= E\Bigg[\sum_{\ell=0}^{X^i_i-1} (\ell+1) a_i^\ell(1-a_i) + X^i_i a_i^{X^i_i}\Bigg] \\ &= E\Bigg[\frac{1}{1-a_i}\Big(1 - a_i^{X^i_i}\Big)\Bigg] = \frac{1}{1-a_i}\Big[1 - \widehat{X}_i(a_i)\Big] \ . \end{aligned} \tag{2.18}$$

It follows that

$$E\Big[A^i\Big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\Big]\Big] = \lambda_i b_i E\Big[\widehat{Y}_i(X_i^i)\Big] \tag{2.19}$$

as if there is *no* dependence between the durations of the service times, B_i's, and the number of service attempts, $\widehat{Y}_i(X_i^i)$. At the same time, however, for any $j = 1, 2, \ldots, N$

$$E\Big[A^j\Big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\Big)\Big] = E\Big[A^j\Big(\theta^i_{X_i^i}\Big)\Big] = \lambda_j \cdot E\Big[\theta^i_{X_i^i}\Big] \tag{2.20}$$

(see Equations (2.7) and (2.17)). This is indeed so since, for the Gated regime, the length of the busy period $\theta^i_{X_i^i}$ is determined *only* by X_i^i, $B_{im}s$ and T_i, and is independent of the arrival process.

2.6 Generating Functions at Polling Instants

We now derive a system of equations for $\{G_i(\underline{z})\}$, the Probability Generating Functions (PGFs) of the system state at polling instants. We will express $G_{i+1}(\underline{z})$ in terms of $G_i(\underline{z})$.

Using Equations (2.1) and (2.2) and exploiting the Poisson property of the arrival process leads to

$$\begin{aligned}
G_{i+1}(\underline{z}) &= E\Big[\prod_{j=1}^{N} z_j^{X_{i+1}^j}\Big] \\
&= (1-p_i)E\Big[\prod_{j=1}^{N} z_j^{X_i^j + A^j(D_i)}\Big] \\
&\quad + p_i E\Big\{\Big[\prod_{j=1}^{N} z_j^{X_i^j + A^j(D_i)}\Big]\Big[\prod_{j=1}^{N} z_j^{A^j\big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\big)}\Big]\Big[z_i^{-Y_i(X_i^i)}\Big]\Big\} \qquad (2.21) \\
&= D_i^*(\Sigma)\Big\{(1-p_i)G_i(\underline{z}) + p_i E\Big[\prod_{j=1}^{N} z_j^{X_i^j} \cdot E\Big[\prod_{j=1}^{N} z_j^{A^j(\theta^i_{X_i^i})} z_i^{-Y_i(X_i^i)} | \underline{X}_i\Big]\Big]\Big\}
\end{aligned}$$

where

$$\Sigma \equiv \sum_{k=1}^{N} \lambda_k (1 - z_k) \ .$$

Now,

$$\begin{aligned}
K &\equiv E\Big[\prod_{j=1}^{N} z_j^{A^j\big(\sum_{m=1}^{\widehat{Y}_i(X_i^i)} B_{im}\big)} z_i^{-Y_i(X_i^i)} \big| \underline{X}_i\Big] \\
&= E\Big[E\Big[\prod_{j=1}^{N} z_j^{A^j\big(\sum_{m=1}^{\widehat{Y}_i} B_{im}\big)} \big|\widehat{Y}_i\Big] z_i^{-Y_i}\Big] = E\Big[S_i^*(\Sigma)^{Y_i} \cdot \widehat{S}_i^*(\Sigma) z_i^{-Y_i} 1_{\{Y_i < X_i^i\}}\Big]
\end{aligned}$$

$$+E\Big[S_i^*(\Sigma)^{Y_i} z_i^{-Y_i}\cdot 1_{\{Y_i=X_i^i\}}\Big] \equiv K_1+K_2$$

where $S_i^*(\cdot)$ and $\widehat{S}_i^*(\cdot)$ are given in (2.9) and (2.11).

$$\begin{aligned}
K_1 &= \sum_{\ell=0}^{X_i^i-1} S_i^*(\Sigma)^\ell \widehat{S}_i^*(\Sigma) z_i^{-\ell} a_i^\ell (1-a_i) = \widehat{S}_i^*(\Sigma)(1-a_i)\frac{1-(a_i S_i^*(\Sigma)/z_i)^{X_i^i}}{1-a_i S_i^*(\Sigma)/z_i} \\
&= (B_i^*(\Sigma)-B_i^*(\Sigma+\gamma_i))\frac{1-[B_i^*(\Sigma+\gamma_i)z_i^{-1}]^{X_i^i}}{1-B_i^*(\Sigma+\gamma_i)z_i^{-1}} \\
K_2 &= S_i^*(\Sigma)^{X_i^i} z_i^{-X_i^i} a_i^{X_i^i} = \Big[B_i^*(\Sigma+\gamma_i)z_i^{-1}\Big]^{X_i^i} .
\end{aligned}$$

Hence,

$$\begin{aligned}
G_{i+1}(\underline{z}) &= D_i^*(\Sigma)\Bigg\{(1-p_i)G_i(\underline{z}) \\
&\quad +p_i\Bigg[\frac{z_i(B_i^*(\Sigma)-B_i^*(\Sigma+\gamma_i))}{z_i-B_i^*(\Sigma+\gamma_i)}\Big[G_i(\underline{z})-G_i(z_1,\ldots,z_{i-1},B_i^*(\Sigma+\gamma_i),\ldots,z_N)\Big] \\
&\quad +G_i(z_1,z_2,\ldots,z_{i-1},B_i^*(\Sigma+\gamma_i),z_{i+1},\ldots,z_N)\Bigg]\Bigg\} . \qquad (2.22)
\end{aligned}$$

Finally, we write

$$\begin{aligned}
G_{i+1}(\underline{z}) &= D_i^*(\Sigma)\Bigg\{(1-p_i)G_i(\underline{z}) \\
&\quad +p_i\Bigg[\eta_i(\Sigma,\gamma_i)G_i(\underline{z}) \\
&\quad +(1-\eta_i(\Sigma,\gamma_i))\cdot G_i(z_1,z_2,\ldots,z_{i-1},B_i^*(\Sigma+\gamma_i),\ldots,z_N)\Bigg]\Bigg\} \qquad (2.23)
\end{aligned}$$

where

$$\eta_i(\omega,\gamma)=\frac{z_i(B_i^*(\omega)-B_i^*(\omega+\gamma))}{z_i-B_i^*(\omega+\gamma)} . \qquad (2.24)$$

2.7 PGF of the System State at an Arbitrary Moment

Let $F(\underline{z})=E\Big[\prod_{j=1}^{N} z_j^{X_j}\Big]$ be the PGF of the system state $\underline{X}=(X_1,X_2,\ldots,X_N)$ at an arbitrary moment. Then, as in Eisenberg [2], Altman & Yechiali [1],

$$F(\underline{z})=\frac{1}{E[C]}\Bigg\{\sum_{i=1}^{N}\Bigg[g_i^i b_i F\Big(\underline{z}\Big|\begin{matrix}\text{service}\\ \text{period } i\end{matrix}\Big)+d_i F\Big(\underline{z}\Big|\begin{matrix}\text{switch}-\text{over}\\ \text{period } i\end{matrix}\Big)\Bigg]\Bigg\} \qquad (2.25)$$

where $g_i^i=p_i[E\widehat{Y}_i(X_i^i)]$ is the mean number of service beginnings in node i per cycle. Let $\overline{G}_i(\underline{z})$ be the PGF of the system-state at the *end* of a visit to node i. Then,

$$G_{i+1}(\underline{z})=\overline{G}_i(\underline{z})D_i^*(\Sigma) . \qquad (2.26)$$

Let $V_i(\underline{z})$ ($\overline{V}_i(\underline{z})$) be the PGF of the system-state at a customer's service-beginning (service-endings) instant in queue i. Then

$$\begin{aligned}
\overline{V}_i(\underline{z}) &= a_i V_i(\underline{z}) \frac{B_i^*(\Sigma)}{z_i} + (1-a_i) V_i(\underline{z}) B_i^*(\Sigma) \\
&= V_i(\underline{z}) B_i^*(\Sigma) \Big[\frac{a_i}{z_i} + (1-a_i)\Big] \qquad (2.27) \\
F\Big(\underline{z}\Big| \begin{matrix}\text{service} \\ \text{period } i\end{matrix}\Big) &= V_i(\underline{z}) \frac{1-B_i^*(\Sigma)}{b_i \cdot \Sigma} \qquad (2.28)
\end{aligned}$$

where $\frac{1-B_i^*(\Sigma)}{b_i \Sigma}$ is the PGF of the number of arrivals to the various stations during the time elapsed from the start of the current service. Similarly,

$$F\Big(\underline{z}\Big| \begin{matrix}\text{switch} - \text{over} \\ \text{period } i\end{matrix}\Big) = \overline{G}_i(\underline{z}) \frac{1-D_i^*(\Sigma)}{d_i \cdot \Sigma} \ . \qquad (2.29)$$

All the above PGFs are related through Eisenberg's [2] observation that the union of visit beginnings and all service endings in node i coincides with the union of all service beginnings and visit endings there. This observation leads to

$$G_i(\underline{z}) + g_i^i \overline{V}_i(\underline{z}) = g_i^i V_i(\underline{z}) + \overline{G}_i(\underline{z}) \ . \qquad (2.30)$$

Combining with (2.26) and (2.27), equation (2.30) is written as

$$V_i(\underline{z}) = \frac{\Big[G_i(\underline{z}) D_i^*(\Sigma) - G_{i+1}(\underline{z})\Big]}{D_i^*(\Sigma) g_i^i \Big[1 - B_i^*(\Sigma)\Big(\frac{a_i}{z_i} + (1-a_i)\Big)\Big]} \qquad (2.31)$$

Using (2.28), (2.29) and the above, we finally have for (2.25)

$$\begin{aligned}
F(\underline{z}) &= \frac{1}{E[C]} \Bigg[\sum_{i=1}^{N} \Bigg[g_i^i b_i V_i(\underline{z}) \cdot \frac{1-B_i^*(\Sigma)}{b_i \cdot \Sigma} + d_i \frac{G_{i+1}(\underline{z})}{D_i^*(\Sigma)} \cdot \frac{1-D_i^*(\Sigma)}{d_i \cdot \Sigma}\Bigg]\Bigg] \\
&= \frac{1}{E[C]} \Bigg[\sum_{i=1}^{N} \Bigg[\frac{[G_i(\underline{z}) D_i^*(\Sigma) - G_{i+1}(\underline{z})]}{D_i^*(\Sigma)\Big[1 - B_i^*(\Sigma)\Big(\frac{a_i}{z_i} + (1-a_i)\Big)\Big]} \cdot \frac{1-B_i^*(\Sigma)}{\Sigma} \\
&\qquad + \frac{G_{i+1}(z)}{D_i^*(\Sigma)} \cdot \frac{1-D_i^*(\Sigma)}{\Sigma}\Bigg]\Bigg] \ . \qquad (2.32)
\end{aligned}$$

The relation between $G_{i+1}(\underline{z})$ and $G_i(\underline{z})$ is expressed in Eq. (2.23).

2.8 Waiting Times

The distribution of waiting times in each queue is obtained in the classical way. Let W_i be the waiting time of an arbitrary job at station i, with $W_i^*(\cdot)$ its LST. Then, since the distribution of the number of jobs in the system is the same at arrival and departure instants, and with the use of PASTA,

$$F(1,1,\ldots,1,z_i,1,\ldots,1) = W_i^*(\lambda_i(1-z_i)) \ . \tag{2.33}$$

2.9 First-Order Moments

We now derive a set of equations for the calculation of the mean number of jobs at the various queues at polling instants.

Define $f_i^j = E[X_i^j]$. Then, for any fixed i $(i = 1,2,\ldots,N)$, taking expectations of both sides of (2.1) and (2.2) leads to

$$f_{i+1}^j = f_i^j + \lambda_j d_i + p_i \begin{cases} \lambda_j E[\theta^i_{X_i^i}] \ , & i \neq j \\ \left\{\lambda_i E[\theta^i_{X_i^i}] - E[Y_i(X_i^i)]\right\} \ , & i = j \end{cases} \tag{2.34}$$

Substituting results (2.7) and (2.16) and (2.34) yields.

$$f_{i+1}^j = f_i^j + \lambda_j d_i + p_i \begin{cases} \frac{\lambda_j b_i}{1-a_i}[1 - \widehat{X}_i^i(a_i)] \ , & i \neq j \\ \frac{\lambda_i b_i - a_i}{1-a_i}[1 - \widehat{X}_i^i(a_i)] \ , & i = j \end{cases} \tag{2.35}$$

Remark. If $\gamma_i = 0$, i.e. $E[T_i] = 1/\gamma_i = \infty$, then $a_i = B^*(\gamma_i) = 1$ and $\widehat{X}_i^i(a_i) = 1$. Thus, by using L'Hospital rule,

$$\lim_{a_i \to 1} \frac{1 - \widehat{X}_i^i(a_i)}{1 - a_i} = \frac{d}{da_i}\widehat{X}_i^i(a_i)\Big|_{a_i=1} = E[X_i^i] = f_i^i \ .$$

Hence, if $p_i = 1$, Equation (2.35) reduces to the classical Gated regime (See Takagi [6], Yechiali [9]):

$$f_{i+1}^j = \begin{cases} f_i^j + \lambda_j d_i + \lambda_j b_i f_i^i & j \neq i \\ \lambda_i d_i + \lambda_i b_i f_i^i & j = i \end{cases}$$

Next we turn to find the unknown quantity $\widehat{X}_i^i(a_i)$ appearing in (2.35).

Let $x_i \equiv \frac{p_i b_i}{1-a_i}[1 - \widehat{X}_i^i(a_i)] = p_i b_i E[\widehat{Y}_i(X_i^i)]$ (see (2.18)), and $\alpha_i \equiv \frac{a_i}{\lambda_i b_i}$. That is, x_i is the expected duration of a visit of the server in station i. Dividing by λ_j, Equation (2.35) can be written as

$$f_{i+1}^j/\lambda_j = f_i^j/\lambda_j + d_i + \begin{cases} x_i & j \neq i \\ (1-\alpha_j)x_j & j = i \end{cases} \tag{2.36}$$

Now, for every j, by summing (2.36) over all i, we obtain

$$0 = d + \sum_{\substack{i=1 \\ i\neq j}}^{N} x_i + (1-\alpha_j)x_j \quad j = 1,2,\ldots,N \tag{2.37}$$

or

$$A\underline{x} = d \cdot \underline{1} \tag{2.38}$$

where A is an $N \times N$ matrix

$$A = \begin{bmatrix} (\alpha_1 - 1) & -1 & -1 & \cdots & -1 \\ -1 & (\alpha_2 - 1) & -1 & \cdots & -1 \\ -1 & -1 & (\alpha_3 - 1) & \cdots & -1 \\ & & & & \\ -1 & -1 & -1 & & \alpha_N - 1 \end{bmatrix}$$

while $\underline{1}$ and $\underline{x}$ are each an N-dimensional vector:

$$\underline{x}^T = [x_1, x_2, \ldots, x_N] \; ,$$
$$\underline{1}^T = [1, 1, \ldots, 1] \; .$$

Since $a_i < 1$, $\widehat{X}_i^i(a_i) = G_i(1, 1, \ldots, 1_{i-1}, a_i, 1_{i+1}, \ldots, 1) < 1$, and therefore $x_i > 0$. It follows that a necessary condition for stability of station i is $\alpha_i - 1 > 0$, i.e. $a_i > \lambda_i b_i$. Indeed, since $a_i = B_i^*(\gamma_i)$ is the probability of a successful service attempt at node i, the expected number of attempts until service completion is $1/a_i$. Thus, the mean amount of work ‘flowing’ to node i per unit time is $\lambda_i b_i / a_i$. This quantity should not exceed 1.

It is clear that with $\alpha_i > 1$ for every $i = 1, 2, \ldots, N$, Equation (2.38) has a *unique* solution $\underline{x}^0$. Hence, the unknown quantity $\widehat{X}_i^i(a_i)$ in Equation (2.35) is given by

$$\widehat{X}_i^i(a_i) = 1 - \frac{1 - B_i^*(\gamma_i)}{p_i b_i} x_i^0 \; . \tag{2.39}$$

2.10 Cycle Time

The Cycle duration, C, is given by

$$C = \sum_{i=1}^{N} D_i + \sum_{i=0}^{N} p_i E\left[\widehat{Y}_i(X_i^i)\right] b_i \; . \tag{2.40}$$

Using (2.18), the mean cycle time is

$$E[C] = d + \sum_{i=1}^{N} \frac{p_i b_i}{1 - a_i} \left[1 - \widehat{X}_i^i(a_i)\right] = d + \sum_{i=1}^{N} x_i^0 \; . \tag{2.41}$$

It follows from (2.37) that, for every j,

$$\alpha_j x_j^0 = d + \sum_{i=1}^{N} x_i^0 = E[C] \; . \tag{2.42}$$

Indeed, the mean number of arrivals to node j during a cycle is $\lambda_j E[C]$, each requiring b_j/a_j amount of work. Thus, the mean visit time in node j is $x_j^0 = \lambda_j E[C] \cdot \frac{b_j}{a_j} = E[C]/\alpha_j$. Alternatively, from (2.16), the mean number of jobs served per cycle in station i is

$$p_i E\Big[Y_i(X_i^i)\Big] = \frac{a_i}{b_i} x_i^0 = \lambda_i \alpha_i x_i^0 = \lambda_i E[C] \ . \tag{2.43}$$

Equation (2.43) simply says that the average outflow of jobs from a node, per cycle, equals the average inflow of jobs to that node. Now, from (2.42),

$$x_j^0 = E[C]/\alpha_j = E[C]\frac{\lambda_j b_j}{a_j} \tag{i}$$

$$\sum_{j=1}^{N} x_j^0 = E[C] - d \ . \tag{ii}$$

Hence,

$$\sum_{j=1}^{N} x_j^0 = E[C] \sum_{j=1}^{N} \frac{\lambda_j b_j}{a_j} = E[C] - d \ ,$$

implying that

$$E[C] = \frac{d}{1 - \sum\limits_{j=1}^{N} \lambda_j (\frac{b_j}{a_j})} \equiv \frac{d}{1-\rho} \tag{2.44}$$

where $\rho \equiv \sum_{j=1}^{N} \rho_j = \sum_{j=1}^{N} \lambda_j \left(\frac{b_j}{a_j}\right)$ is the total inflow rate of work to the system. It is clear that a necessary condition for system's stability is $\rho = \sum_{j=1}^{N} \lambda_j (b_j/a_j) < 1$.

2.11 Higher-Order Moments

Higher-order moments may be derived by repeatedly differentiating both sides of Equation (2.23). Numerical solutions would have to be employed to solve the set of equations resulting from the differentiation. For example, to obtain the second-order and mixed moments, it would be required to solve a set of N^3 equations in N^3 unknowns $E[X_i^j X_i^k]$, $k \neq j$, and $E[(X_i^i)^2]$. Various numerical methods to solve such equations have been developed, see, for example, Sarkar & Zangwill [5] and Konheim, Levy & Srinivasan [4].

3 The Exhaustive Regime

Under the Exhaustive service discipline, if node i is *up* when polled, the server starts serving the jobs present there (if any), and continues to do so until either *all* work accumulated is done (including service of new arrivals) *or* the node fails, whichever occurs first. The server then moves to node $i+1$. As before, we assume that a failure is discovered *only* at the *end* of a service period of a job.

Let θ^i_r denote the length of a busy period at node i *starting with* r *jobs*. θ^i_r is *not* an r-fold regular busy period in an $M/G/1$ queue with arrival rate λ_i and service times B_i because it may terminate, due to a failure, *before* all work is exhausted.

Let $Y^E_i(k)$ denote the number of jobs *successfully* served at station i during a visit of the server, under the Exhaustive regime, if $X^i_i = k$. Then, the evolution law of $\underline{X}_i$ is given by:

(i) if node i is *down* upon polling, then

$$X^j_{i+1} = X^j_i + A^j(D_i) \tag{3.1}$$

(ii) if node i is *up* upon polling, then

$$X^j_{i+1} = \begin{cases} X^j_i + A^j(\theta^i_{X^i_i}) + A^j(D_i) & j \neq i \\ X^i_i + A^i(\theta^i_{X^i_i}) + A^i(D_i) - Y^E_i(X^i_i) & j = i \end{cases} \tag{3.2}$$

We'll use equations (3.1) and (3.2) to derive the PGFs of the system states $\underline{X}_i$'s. The analysis will be achieved via an explicit solution of an infinite set of linear equations where the unknowns are the state-dependent joint transforms of the number of arrivals to a station during a busy period starting with a given number of jobs, and the number of jobs successfully served during that period.

3.1 Generating Functions at Polling Instants

As before, let $G_i(\underline{z}) = E\Big[\prod_{j=1}^{N} z_j^{X^j_i}\Big]$. Then, by using (3.1) and (3.2),

$$\begin{aligned} G_{i+1}(\underline{z}) &= E\Big[\prod_{j=1}^{N} z_j^{X^j_{i+1}}\Big] \\ &= (1-p_i)E\Big[\prod_{j=1}^{N} z_j^{X^j_i + A^j(D_i)}\Big] \\ &\quad + p_i E\Big[\Big(\prod_{j=1}^{N} z_j^{X^j_i}\Big)\Big(\prod_{j=1}^{N} z_j^{A^j(D_i)}\Big)\Big(\prod_{j=1}^{N} z_j^{A_j(\theta^i_{X^i_i})}\Big) z_i^{-Y^E_i(X^i_i)}\Big] \\ &= D^*_i(\Sigma)\Big\{(1-p_i)G_i(\underline{z}) + p_i E\Big[\prod_{j=1}^{N} z_j^{X^j_i} \cdot E\Big[\prod_{j=1}^{N} z_j^{A^j(\theta^i_{X^i_i})} z_i^{-Y^E_i(X^i_i)} \Big| \underline{X}_i\Big]\Big]\Big\}\ . \end{aligned} \tag{3.3}$$

Thus, we need to calculate, for all k,

$$\Omega_k^i(\underline{z}) = E\left[\prod_{j=1}^{N} z_j^{A^j(\theta_k^i)} z_i^{-Y_i^E(k)}\right] . \tag{3.4}$$

Clearly, $\Omega_0^i(\underline{z}) = 1$ since $\theta_0^i = Y_i^E(0) = 0$. Furthermore, for $k = 1, 2, \ldots$

$$\begin{aligned} \theta_k^i &= B_i + 1_{\{B_i<T_i\}}\theta^i_{k-1+A^i(B_i)} \\ Y_i^E(k) &= 1_{\{B_i<T_i\}}\Big(1 + Y_i^E(k-1+A^i(B_i))\Big) \end{aligned} \tag{3.5}$$

where $1_{\{\cdot\}}$ is the indicator function of an event $\{\cdot\}$.

Now, using (3.5), (and suppressing for a while the index i from B_i and T_i),

$$\begin{aligned} &\prod_{j=1}^{N} z_j^{A^j(\theta_k^i)} z_i^{-Y_i^E(k)} = \\ =\; &\prod_{j=1}^{N} z_j^{A^j\left(B+1_{\{B<T\}}\theta^i_{k-1+A^i(B)}\right)} z_i^{-1_{\{B<T\}}\left(1+Y_i^E(k-1+A^i(B))\right)} \\ =\; &\prod_{j=1}^{N} z_j^{A^j(B)} \cdot \left[1_{\{B\ge T\}} + 1_{\{B<T\}}\prod_{j=1}^{N} z_j^{A^j\left(\theta^i_{k-1+A^i(B)}\right)}\right] \\ &\cdot\left[1_{\{B\ge T\}} + 1_{\{B<T\}} z_i^{-1} z_i^{-Y_i^E(k-1+A^i(B))}\right] \\ =\; &\prod_{j=1}^{N} z_j^{A^j(B)}\left[1_{\{B\ge T\}} + 1_{\{B<T\}}\prod_{j=1}^{N} z_j^{A^j(\theta^i_{k-1+A^i(B)})} z_i^{-1} z_i^{-Y_i^E(k-1+A^i(B))}\right] \end{aligned} \tag{3.6}$$

where we used the facts that

(i) $$\prod_{j=1}^{N} z_j^{A^j(1_{\{B<T\}}\theta)} = 1_{\{B\ge T\}} + 1_{\{B<T\}}\prod_{j=1}^{N} z_j^{A^j(\theta)}$$

and

(ii) $$z_i^{-1_{\{B<T\}}Y} = 1_{\{B\ge T\}} + 1_{\{B<T\}} z_i^{-Y} .$$

Observe that, due to the independent increment property of the Poisson process, the random variables $A^j(B)$ and $A^j(\theta^i_{k-1+A^i(B)})$, appearing in equation (3.6), are *independent*.

It therefore follows that

$$\Omega_k^i(\underline{z}) = E\left[\prod_{j=1}^{N} z_j^{A^j(B)}\Big[1_{\{B\ge T\}} + 1_{\{B<T\}}\prod_{j=1}^{N} z_j^{A^j(\theta_{k-1+A^i(B)})} z_i^{-1} z_i^{-Y_i^E(k-1+A^i(B))}\Big]\right] . \tag{3.7}$$

The first term on the RHS of (3.7) is calculated to be

$$
\begin{aligned}
& E\Big[\prod_{j=1}^{N} z_j^{A^j(B)} 1_{\{B\ge T\}}\Big] = E\Big[E\big[1_{\{B\ge T\}}\big|B\big]\cdot E\big[\prod_{j=1}^{N} z_j^{A^j(B)}\big|B\big]\Big] \\
= \; & E\Big[\Big(1-e^{-\gamma_i B}\Big)e^{-\sum_{j=1}^{N}\lambda_j(1-z_j)B}\Big] = B^*(\Sigma) - B^*(\Sigma+\gamma_i)
\end{aligned}
\tag{3.8}
$$

where, as before, $\Sigma \equiv \sum_{j=1}^{N} \lambda_j(1-z_j)$.

The calculation of the second term on the RHS of (3.7) requires more effort. We have

$$
\begin{aligned}
& E\Big[E\big[\prod_{j=1}^{N} z_j^{A^j(B)} \prod_{j=1}^{N} z_j^{A^j(\theta^i_{k-1+A^i(B)})} z_i^{-1} z_i^{-Y_i^E(k-1+A^i(B))}\big|B\big]\Big] \\
= \; & E\Big[\prod_{j=1}^{N}\sum_{\ell_j=0}^{\infty} z_j^{\ell_j} e^{-\lambda_j B}\frac{(\lambda_j B)^{\ell_j}}{\ell_j!}\prod_{j=1}^{N} z_j^{A^j(\theta^i_{k-1+\ell_i})} z_i^{-1} z_i^{-Y_i^E(k-1+\ell_i)}\Big] \\
= \; & E\Big[e^{-\Sigma_{j\neq i}\lambda_j(1-z_j)B}\sum_{\ell_i=0}^{\infty} e^{-\lambda_i B}\frac{(\lambda_i z_i B)^{\ell_i}}{\ell_i!}\prod_{j=1}^{N} z_j^{A^j(\theta^i_{k-1+\ell_i})} z_i^{-1} z_i^{-Y_i^E(k-1+\ell_i)}\Big] \equiv M \; .
\end{aligned}
\tag{3.9}
$$

Hence,

$$
\begin{aligned}
& E\Big[1_{\{B<T\}}\cdot M\Big] = E\Big[e^{-\gamma_i B}\cdot M\Big] \\
= \; & E\Big[e^{-\gamma_i B} e^{-\sum_{j\neq i}\lambda_j(1-z_j)B}\sum_{\ell_i=0}^{\infty} e^{-\lambda_i B}\frac{(\lambda_i z_i B)^{\ell_i}}{\ell_i!} \\
& \cdot E\big[\prod_{j=1}^{N} z_j^{A^j(\theta^i_{k-1+\ell_i})} z_i^{-1} z_i^{-Y_i^E(k-1+\ell_i)}\big]\Big] \\
= \; & \sum_{\ell=0}^{\infty} E\Big[\frac{(\lambda_i z_i B)^{\ell}}{\ell!}\exp\Big\{-\big[\lambda_i+\gamma_i+\sum_{j\neq i}\lambda_j(1-z_j)\big]B\Big\}\Big]\Omega^i_{k-1+\ell}(\underline{z})\cdot z_i^{-1} \; .
\end{aligned}
\tag{3.10}
$$

Set

$$
a^i_\ell(\underline{z}) \equiv E\Big[\frac{(\lambda_i z_i B_i)^{\ell}}{\ell!} e^{-\big(\lambda_i+\gamma_i+\sum_{j\neq i}\lambda_j(1-z_j)\big)B_i}\Big] \; . \tag{3.11}
$$

Then, combining (3.7), (3.8) and (3.10) we have an infinite set of (linear) equations in $\Omega^i_k(\underline{z})$, namely

$$
\begin{aligned}
& \Omega^i_0(\underline{z}) = 1 \\
& \Omega^i_k(\underline{z}) = \Big[B_i^*(\Sigma) - B_i^*(\Sigma+\gamma_i)\Big] + z_i^{-1}\sum_{\ell=0}^{\infty} a^i_\ell(\underline{z})\Omega_{k-1+\ell}(\underline{z}), \;\; k=1,2,3\ldots
\end{aligned}
\tag{3.12}
$$

The existence and uniqueness of a solution for the system of equation (3.12) can be shown using the same arguments as in Eliazar & Yechiali [3].

We claim:

Theorem 3.1. *The solution of system (3.12) is given by*

$$\Omega_k^i(\underline{z}) = \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})\cdot z_i^{-1}} + \left(1 - \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})\cdot z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^k, k=0,1,2\ldots \tag{3.13}$$

where

$$\begin{aligned} \Sigma_i^- &= \sum_{j\neq i}\lambda_j(1-z_j) \\ \beta_i(\underline{z}) &= B_i^*(\Sigma) - B_i^*(\Sigma+\gamma_i) \\ \alpha_i(\underline{z}) &= \sum_{\ell=0}^{\infty} a_\ell^i(\underline{z}) = B_i^*(\Sigma+\gamma_i) \end{aligned} \tag{3.14}$$

and $\theta_i^*(\cdot)$ is the Laplace Stieltjes Transform (LST) of a regular busy period in a $M/G/1$ queue with arrival rate λ_i and service times B_i.

Proof: Using the fact that $\theta_i^*(s) = B_i^*(s+\lambda_i(1-\theta_i^*(s)))$, and substituting $\Omega_k^i(\underline{z})$ from (3.13), we have (suppressing for a while the index i from $\alpha_i(\underline{z})$ and $B_i(\underline{z})$),

$$\begin{aligned} &\sum_{\ell=0}^{\infty} a_\ell^i(\underline{z})\Omega_{k-1+\ell}^i(\underline{z}) = \\ = \; & E\Big[\sum_{\ell=0}^{\infty}\frac{(\lambda_i z_i B_i)^\ell}{\ell!}e^{-(\lambda_i+\gamma_i+\Sigma_i^-)B_i}\Big[\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}} + \Big(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\Big)\Big(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\Big)^{k-1+\ell}\Big]\Big] \\ = \; & \frac{\alpha(\underline{z})\beta(\underline{z}}{1-\alpha(\underline{z})z_i^{-1}} + \left(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\right)E\Big[\sum_{\ell=0}^{\infty}\frac{(\lambda_i z_i B_i)^\ell}{\ell!}e^{-(\lambda_i+\gamma_i+\Sigma_i^-)B_i}\Big(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\Big)^{k-1+\ell}\Big] \\ = \; & \frac{\alpha(\underline{z})\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}} + \left(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^{k-1} \\ & \cdot E\Big[\sum_{\ell=0}^{\infty}\frac{(\lambda_i B_i\theta_i^*(\gamma_i+\Sigma_i^-))^\ell}{\ell!}e^{-(\lambda_i+\gamma_i+\Sigma_i^-)B_i}\Big] \\ = \; & \frac{\alpha(\underline{z})\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}} + \left(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^{k-1} B_i^*\Big(\gamma_i+\Sigma_i^- + \lambda_i(1-\theta_i^*(\gamma_i+\Sigma_i^-))\Big) \\ = \; & \frac{\alpha(\underline{z})\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}} + \left(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^{k-1} \theta_i^*(\gamma_i+\Sigma_i^-) \\ = \; & \frac{\alpha(\underline{z})\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}} + \left(1-\frac{\beta(\underline{z})}{1-\alpha(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^{k} z_i \;. \end{aligned}$$

Thus, the RHS of (3.12) becomes,

$$\beta_i(\underline{z}) \quad + \quad z_i^{-1}\sum_{\ell=0}^{\infty} a_\ell^i(\underline{z})\Omega_{k-1+\ell}^i(\underline{z})$$

$$= \beta_i(\underline{z}) + \frac{z_i^{-1}\alpha_i(\underline{z})\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}} + z_i^{-1}\left(1 - \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^k \cdot z_i$$

$$= \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}} + \left(1 - \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}}\right)\left(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\right)^k = \Omega_k^i(\underline{z}) \ .$$

Hence, $\Omega_k^i(\underline{z})$ as expressed in (3.13) is indeed the solution of the system of equations (3.12).

Q.E.D.

Finally, the PGF of the system-state at polling instant of station $i+1$ is expressed in terms of the PGF of the system-state at polling instant of station i. Returning to (3.3) one obtains

$$\begin{aligned}
G_{i+1}(\underline{z}) &= D_i^*(\Sigma)\Big\{(1-p_i)G_i(\underline{z}) + p_i E\Big[\prod_{j=1}^N z_j^{X_i^j}\Omega_{X_i^i}^i\Big]\Big\} \\
&= D_i^*(\Sigma)\Big\{(1-p_i)G_i(\underline{z}) + p_i E\Big[\prod_{j=1}^N z_j^{X_i^j}\Big[\frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}} + \\
&\quad + \Big(1 - \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}}\Big)\Big(\frac{\theta_i^*(\gamma_i+\Sigma_i^-)}{z_i}\Big)^{X_i^i}\Big]\Big]\Big\} \\
&= D_i^*(\Sigma)\Big\{(1-p_i)G_i(\underline{z}) + p_i\Big[\frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}}G_i(\underline{z}) \\
&\quad + \Big(1 - \frac{\beta_i(\underline{z})}{1-\alpha_i(\underline{z})z_i^{-1}}\Big)G_i\Big(z_1, z_2 \ldots z_{i-1}, \theta_i^*(\gamma_i+\Sigma_i^-), \ldots, z_n\Big)\Big]\Big\}
\end{aligned} \tag{3.15}$$

where (see (3.14), $\beta_i(\underline{z}) = B_i^*(\Sigma) - B_i^*(\Sigma+\gamma_i)$ and

$$\alpha_i(\underline{z}) = B_i^*(\Sigma+\gamma_i) \ .$$

3.2 First-Order Moments

To get the first-order moments, $f_i^j \equiv E[X_i^j]$, one can differentiate Equation (3.15) and obtain a set of N^2 linear equations in these variables. However, as we did for the Gated case, we go directly to Equations (3.1) and (3.2) and get

$$f_{i+1}^j = f_i^j + \lambda_j d_i + p_i \begin{cases} \lambda_j E[\theta_{X_i^i}^i] \ , & j \neq i \\ E[A^i(\theta_{X_i^i}^i)] - E[Y_i^E(X_i^j)] \ , & j = i \end{cases} \tag{3.16}$$

In order to be able to solve Equation (3.16) we first have to calculate $E[\theta_{X_i^i}^i]$ and $E[Y_i^E(X_i^i)]$.

We start with the derivation of $E[\theta_{X_i^i}^i]$. We'll calculate $E[\theta_k^i]$ for all $k \geq 0$ and then take expectation over k.

Clearly, $E[\theta_0^i] = 0$. For $k > 0$ we claim

Theorem 3.2.

$$E[\theta_k^i] = \frac{b_i}{1 - B_i^*(\gamma_i)}[1 - \theta_i^*(\gamma_i)^k] \ . \tag{3.17}$$

Proof: Using Equation (3.5) we have

$$\begin{aligned} E[\theta_k^i] &= b_i + E\left[e^{-\gamma_i B_i}\theta^i_{k-1+A^i(B_i)}\right] \\ &= b_i + E\left[e^{-\gamma_i B_i}\sum_{\ell=0}^{\infty}\theta^i_{k-1+\ell}e^{-\lambda_i B_i}\frac{(\lambda_i B_i)^\ell}{\ell!}\right] \\ &= b_i + \sum_{\ell=0}^{\infty} E\left[\theta^i_{k-1+\ell}\right]E\left[e^{-(\lambda_i+\gamma_i)B_i}\frac{(\lambda_i B_i)^\ell}{\ell!}\right] \ . \end{aligned}$$

Thus,

$$\begin{aligned} E[\theta_0^i] &= 0 \\ E[\theta_k^i] &= b_i + \sum_{\ell=0}^{\infty} E[\theta^i_{k-1+\ell}]a^i_\ell(\underline{1}), \quad k \geq 1 \end{aligned} \tag{3.18}$$

where $a^i_\ell(\underline{1}) = E\left[e^{-(\lambda_i+\gamma_i)B_i}\frac{(\lambda_i B_i)^\ell}{\ell!}\right]$, (see (3.11)). Substituting the postulated solution (3.17) in the RHS of Equation (3.18) and using $a_i \equiv B_i^*(\gamma_i)$ leads to

$$\begin{aligned} & b_i + \sum_{\ell=0}^{\infty} E[\theta^i_{k-1+\ell}]a^i_\ell(\underline{1}) = \\ = \ & b_i + \sum_{\ell=0}^{\infty}\frac{b_i}{1-a_i}[1-\theta_i^*(\gamma_i)^{k-1+\ell}]E\left[e^{-(\lambda_i+\gamma_i)B_i)}\frac{(\lambda_i B_i)^\ell}{\ell!}\right] \\ = \ & b^i + \frac{b_i}{1-a_i}E[e^{-\gamma_i B_i}] - \frac{b_i}{1-a_i}\theta_i^*(\gamma_i)^{k-1}E\left[e^{-[\gamma_i+\lambda_i(1-\theta_i^*(\gamma_i))]B_i}\right] \\ = \ & \frac{b_i}{1-a_i} - \frac{b_i}{1-a_i}\theta_i^*(\gamma_i)^{k-1}\cdot B_i^*(\gamma_i+\lambda_i(1-\theta_i^*(\gamma_i))) \ . \end{aligned}$$

Again, using the fact that $\theta_i^*(\gamma_i) = B_i^*(\gamma_i + \lambda_i(1-\theta_i^*(\gamma_i)))$, we finally obtain that the RHS of Equation (3.18) equals

$$b_i + \sum_{\ell=0}^{\infty} E[\theta^i_{k-1+\ell}]a^i_\ell(\underline{1}) = \frac{b_i}{1-B_i^*(\gamma_i)}\left[1-\theta_i^*(\gamma_i)^k\right] = E[\theta_k^i] \ .$$

Q.E.D.

Finally,

$$E[\theta^i_{X_i^i}] = \frac{b_i}{1-B_i^*(\gamma_i)}\left[1-\widehat{X_i^i}(\theta_i^*(\gamma_i))\right] \tag{3.19}$$

where $\widehat{X_i^i}(z_i)$ is the generating function of X_i^i, namely

$$\widehat{X_i^i}(z_i) = E[z_i^{X_i^i}] = G_i(1_1, 1_2, \ldots, 1_{i-1}, z_i 1_{i+1}, \ldots, 1_N) \ .$$

Next we calculate $E[Y_i^E(k)]$. Clearly, $E[Y_i^E(0)] = 0$. For $k > 0$ we claim

Theorem 3.3.
$$E[Y_i^E(k)] = \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}[1-\theta_i^*(\gamma_i)^k] \ . \tag{3.20}$$

Proof: Using equation (3.5) we have

$$\begin{aligned} E[Y_i^E(k)] &= B_i^*(\gamma_i) + E\Big[e^{-\gamma_i B_i} E[Y_i^E(k-1+A^i(B_i))] \mid B_i\Big] \\ &= B_i^*(\gamma_i) + E\Bigg[e^{-\gamma_i B_i}\sum_{\ell=0}^{\infty}\Bigg[E\Big[Y_i^E(k-1+\ell)\Big]e^{-\lambda_i B_i}\frac{(\lambda_i B_i)^\ell}{\ell!}\Bigg]\Bigg] \\ &= B_i^*(\gamma_i) + \sum_{\ell=0}^{\infty} E[Y_i^E(k-1+\ell)]\cdot a_\ell^i(\underline{1}) \ . \end{aligned} \tag{3.21}$$

Substituting the postulated expression for $E[Y_i^E(k)]$ from (3.20) into the RHS of (3.21) leads to

$$\begin{aligned} E[Y_i^E(k)] &= B_i^*(\gamma_i) + \sum_{\ell=0}^{\infty}\left(\frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}\Big[1-\theta_i^*(\gamma_i)^{k-1+\ell}\Big]\right)a_\ell^i(\underline{1}) \\ &= B_i^*(\gamma_i) + \frac{B_i^*(\gamma_i)^2}{1-B_i^*(\gamma_i)} - \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}\theta_i^*(\gamma_i)^{k-1}\cdot E\Big[e^{-[\gamma_i+\lambda_i(1-\theta_i^*(\gamma_i))]B_i}\Big] \\ &= \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)} - \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}\theta_i^*(\gamma_i)^k = \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}\Big[1-\theta_i^*(\gamma_i)^k\Big] \end{aligned}$$

where we used again the fact that $\theta_i^*(\gamma_i) = B_i^*(\gamma_i+\lambda_i(1-\theta_i^*(\gamma_i)))$. Q.E.D.

Thus, indeed, (3.20) is the solution for $E[Y_i^E(k)]$, $k>0$. It follows that

$$E[Y_i^E(X_i^i)] = \frac{B_i^*(\gamma_i)}{1-B_i^*(\gamma_i)}\Big[1-\widehat{X}_i^i(\theta_i^*(\gamma_i))\Big] \ . \tag{3.22}$$

From (3.19) and (3.22) it follows that $E[\theta^i_{X_i^i}] = \frac{b_i}{B_i^*(\gamma_i)}E[Y_i^E(X_i^i)]$. Indeed, each job loads the system with $b_i/B_i^*(\gamma_i)$ units of time of work.

Next we calculate $E[A^i(\theta^i_{X_i^i}) - EY_i^E(X_i^i)]$. Define

$$\Lambda_k(z_i) = \Omega_k^i(1_1, 1_2, \ldots, 1_{i-1}, z_i, 1_{i+1}, \ldots, 1_N) = E\Big[z_i^{A^i(\theta_k^i)} z_i^{-Y_i^E(k)}\Big] \ .$$

Then,

$$\beta_i(z_i) \equiv B_i^*(\lambda_i(1-z_i)) - B_i^*(\lambda_i(1-z_i)+\gamma_i);$$
$$\beta_i(1) = 1 - B_i^*(\gamma_i) = 1-a_i \quad \alpha_i(z_i) = B_i^*(\lambda_i(1-z_i)+\gamma_i); \alpha_i(1) = B_i^*(\gamma_i) = a_i \ .$$

Suppressing i, we write (see 3.13)

$$\Lambda_k(z) = \frac{\beta(z)}{1-\alpha(z)z^{-1}} + \Big(1-\frac{\beta(z)}{1-\alpha(z)z^{-1}}\Big)\Big(\frac{\theta_i^*(\gamma_i)}{z}\Big)^k \ .$$

Taking derivatives and using $\beta_i'(1) = \lambda_i b_i + \lambda_i B^{*'}(\gamma_i)$, $\alpha_i'(1) = -\lambda_i B_i^{*'}(\gamma_i)$, we obtain

$$\frac{d}{dz}\Lambda_k(z)|_{z=1} = E[A^i(\theta_k^i)] - E[Y_i^E(k)] = \frac{\lambda_i b_i - a_i}{1 - a_i}[1 - \theta_i^*(\gamma_i)^k] \ . \tag{3.23}$$

Hence

$$E[A^i(\theta^i_{X_i^i})] - E[Y_i^E(X_i^i)] \equiv \frac{\lambda_i b_i - a_i}{1 - a_i}\Big[1 - \widehat{X}_i^i(\theta_i^*(\gamma_i))\Big] \ . \tag{3.24}$$

It follows that $E[A^i(\theta^i_{X_i^i})] = \lambda_i E[\theta^i_{X_i^i}]$, as *if* the arrival process to station i and the length of the busy period in that station are independent!

Now, for a complete knowledge of $E[\theta^i_{X_i^i}]$ and $E[Y_i^E(X_i^i)]$ we need to evaluate, for all i, the quantity

$$x_i^E \equiv \widehat{X}_i^i(\theta_i^*(\gamma_i)) = E[\theta_i^*(\gamma_i)^{X_i^i}] \ . \tag{3.25}$$

Substituting results (3.19) and (3.24) back into (3.16) yields

$$f_{i+1}^j = f_i^j + \lambda_j d_i + \begin{cases} \lambda_j p_i \frac{b_i}{1-a_i}[1 - x_i^E], & j \neq i \\ \lambda_j p_j \frac{b_j}{1-a_j}[1 - x_j^E] - \frac{p_j a_j}{1-a_j}[1 - x_j^E], & j = i \end{cases} \tag{3.26}$$

For *fixed* j, summing (3.26) over all i results in

$$0 = \lambda_j d + \lambda_j \sum_{i=1}^{N} \frac{p_i b_i}{1 - a_i}[1 - x_i^E] - \frac{p_j a_j}{1 - a_j}[1 - x_j^E] \ . \tag{3.27}$$

Equations (3.26) is *identical* to Equations (2.35), developed for the Gated regime, with the *modification* that in (3.26) we have $\widehat{X}_i^i(\theta_i^*(\gamma_i))$ instead of $\widehat{X}_i^i(B_i^*(\gamma_i))$. Hence, with this modification, we get *exactly* the *same* solution $\underline{x}^0$ for Equations (2.37) and (3.27) where, for the Exhaustive case,

$$\widehat{X}_i^i(\theta_i^*(\gamma_i)) = 1 - \frac{1 - B_i^*(\gamma_i)}{p_i b_i} x_i^0 \ . \tag{3.28}$$

Note that the necessary condition for stability of station i, $\alpha_i > 1$, remains the *same*, i.e. $B_i^*(\gamma_i) = a_i > \lambda_i b_i$, or $\lambda_i \frac{b_i}{a_i} < 1$.

Remark. If $\gamma_i \to 0$, i.e. $E[T_i] \to \infty$, then $\theta_i^*(\gamma_i) \to 1$. Thus, by using L'Hospital rule,

$$\lim_{\gamma_i \to 0}\left[\frac{1 - \widehat{X}_i^i(\theta_i^*(\gamma_i))}{1 - B_i^*(\gamma_i)}\right] = \frac{E[X_i^i] \cdot E[\theta_i]}{b_i} \ .$$

Hence,

$$E[\theta^i_{X_i^i}] = E[X_i^i] \cdot E[\theta_i] \qquad \text{(see 3.19)} \ ,$$

and

$$E[Y_i^E(X_i^i)] = \tfrac{E[X_i^i]E[\theta_i]}{b_i} \qquad \text{(see 3.22)} \ .$$

Indeed, if $\gamma_i \to 0$, i.e, there are no breakdowns, the length of a busy period starting with X_i^i awaiting jobs is the sum of X_i^i independent regular $M/G/1$ busy periods, each lasting for θ_i units of time. The expected number of jobs served is then the expected length of the busy period divided by the mean service time. If, in addition, $p_i = 1$, then Equation (3.26) reduced to

$$f_{i+1}^j = f_i^j + \lambda_j d_i + \begin{cases} \lambda_j f_i^i E[\theta_i] & i \neq j \\ \lambda_j f_j^j E[\theta_j] - \frac{f_j^j E(\theta_j)}{b_j} & i = j \end{cases} \tag{3.29}$$

However, the mean duration of a busy period in a $M/G/1$ queue is $E[\theta_j] = b_j/(1-\rho_j)$, where $\rho_j = \lambda_j b_j$. Thus, for $i = j$,

$$f_j^j + \lambda_j f_j^j E[\theta_j] - \frac{f_j^j E[\theta_j]}{b_j} = f_j^j \Big[1 + \frac{\rho_j}{1-\rho_j} - \frac{1}{1-\rho_j}\Big] = 0 \ .$$

Hence, Equation (3.29) reduces to

$$f_{i+1}^j = \begin{cases} f_i^j + \lambda_j d_i + \lambda_j f_i^i E[\theta_i] & i \neq j \\ \lambda_j d_j & i = j \end{cases} \tag{3.30}$$

Equation (3.30) is the well known relationship for the classical Exhaustive regime (see Takagi [6], Yechiali [9]).

3.3 Cycle Time

The length of the cycle duration under the Exhaustive regime is given by

$$C^E = \sum_{i=1}^N D_i + \sum_{i=1}^N 1_{\left\{\substack{\text{node } i \\ \text{is up}}\right\}} \theta_{X_i^i}^i \ . \tag{3.31}$$

Hence, the mean cycle time is

$$E[C^E] = d + \sum_{i=1}^N p_i E[\theta_{X_i^i}^i] \ . \tag{3.32}$$

Summing equation (3.16) over all i yields, for every j,

$$p_j E[Y_j^E(X_j^j)] = \lambda_j \Big[d + \sum_{i=1}^N p_i E[\theta_{X_i^i}^i]\Big] = \lambda_j E[C^E] \ . \tag{3.33}$$

Indeed, on the average, the mean number of jobs served per node-j visit is the mean number of jobs arriving to node j during a cycle.

Now, returning to (3.16) and using the solution $\underline{x}^0$ from Equation (2.38), this time, however, $x_i^0 = \frac{p_i b_i}{1-a_i}[1 - \widehat{X}_i^i(\theta_i^*(\gamma_i))]$, we have

$$f_{i+1}^j = f_i^j + \lambda_j d_i + \begin{cases} \lambda_j x_i^0 & i \neq j \\ \lambda_j x_j^0 - \frac{a_j}{b_j} x_j^0 & i = j \end{cases} \tag{3.34}$$

This, as well as (3.33), leads to

$$\frac{a_j}{b_j} x_j^0 = \lambda_j d + \lambda_j \sum_{i=1}^{N} x_i^0 \ . \tag{3.35}$$

It follows from (3.32) that

$$E[C^E] = d + \sum_{i=1}^{N} x_i^0 \ . \tag{3.36}$$

That is, the mean cycle times for the Gated and for the Exhaustive regimes are the *same*. Explicitly, using (3.35) and (3.36), we get, similarly to the Gated case,

$$\begin{aligned} x_j^0 &= \frac{\lambda_j b_j}{a_j} E[C^E] \\ \sum_{j=1}^{N} x_j^0 &= E[C^E] \cdot \sum_{j=1}^{N} \frac{\lambda_j b_j}{a_j} \\ E[C^E] - d &= E[C^E] \cdot \sum_{j=1}^{N} \frac{\lambda_j b_j}{a_j} \\ E[C^E] &= \frac{d}{1 - \sum_{j=1}^{N} \lambda_j (\frac{b_j}{a_j})} = \frac{d}{1-\rho} \end{aligned} \tag{3.37}$$

where $\rho \equiv \sum_{j=1}^{N} \rho_j = \sum_{j=1}^{N} \lambda_j (\frac{b_j}{a_j})$ is the *mean* amount of work loaded on the system per unit time. Again, this follows since each job in node j is attempted, on the average, $\frac{1}{a_j}$ times, independent of the service regime, and each attempt requires b_j units of time. Clearly, the necessary condition for system's stability is $\rho < 1$.

3.4 PGF of the System-State at an Arbitrary Moment

Let $F^E(\underline{z}) = E[\prod_{j=1}^{N} z_j^{X_j^E}]$ be the PGF of the system-state $\underline{X}^E = (X_1^E, X_2^E, \ldots, X_N^E)$ at an arbitrary moment. Then, similarly to (2.25),

$$F^E(\underline{z}) = \frac{1}{E[C]} \left\{ \sum_{i=1}^{N} \left[g_i^E b_i F^E\left(\underline{z} \Big|_{\text{period } i}^{\text{service}}\right) + d_i F^E(\underline{z}) \Big|_{\text{period } i}^{\text{switch-over}} \right] \right\} \tag{3.38}$$

where g_i^E = expected number of service attempts (under the Exhaustive regime) during a visit of the server at station i. Clearly,

$$g_i^E = p_i E[\widehat{Y}_i^E(X_i^i)] \ .$$

Calculation of $E[\widehat{Y}_i^E(X_i^i)]$

$$\begin{aligned}
\widehat{Y}_i^E(k) &= 1 + 1_{\{B_i<T_i\}}\widehat{Y}_i^E(k-1+A^i(B_i)) \qquad k=1,2\ldots \\
\widehat{Y}_i^E(0) &= 0 \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
E[\widehat{Y}_i^E(k)] &= 1 + E\left[e^{-\gamma_i B_i}\cdot E\left[\widehat{Y}_i^E(k-1+A^i(B_i)) \mid B_i\right]\right] \\
&= 1 + E\left[e^{-\gamma_i B_i}\sum_{\ell=0}^{\infty} E\left[\widehat{Y}_i^E(k-1+\ell)\right]e^{-\lambda_i B_i}\frac{(\lambda_i B_i)^{\ell}}{\ell!}\right] \qquad (3.39) \\
&= 1 + \sum_{\ell=0}^{\infty} E[\widehat{Y}_i^E(k-1+\ell)]a_\ell^i(\underline{1})
\end{aligned}$$

where $a_\ell^i(\underline{1}) = E\left[e^{-(\lambda_i+\gamma_i)B_i}\frac{(\lambda_i B_i)^{\ell}}{\ell!}\right]$ (see (3.11)).

We claim:

Theorem 3.4. $E[\widehat{Y}_i^E(k)] = \frac{1}{1-a_i}[1-\theta_i^*(\gamma_i)^k]$.

Proof: Substituting the above in the RHS of (3.39) and using again the fact that for a $M/G/1$ queue, $\theta_i^*(\gamma_i) = E[e^{-[\gamma_i+\lambda_i(1-\theta_i^*(\gamma_i))]B_i}]$, we have

$$\begin{aligned}
1 \ &+ \ \sum_{\ell=0}^{\infty}\frac{1}{1-a_i}[1-\theta_i^*(\gamma_i)^{k-1+\ell}]E[e^{-(\lambda_i+\gamma_i)B_i}\frac{(\lambda_i B_i)^{\ell}}{\ell!}] \\
&= 1 + \frac{1}{1-a_i}E[e^{-\gamma_i B_i}] - \frac{1}{1-a_i}\theta_i^*(\gamma_i)^{k-1}E\left[e^{-[\gamma_i B_i+\lambda_i(1-\theta_i^*(\gamma_i))]B_i}\right] \\
&= \frac{1}{1-a_i} - \frac{1}{1-a_i}\theta_i^*(\gamma_i)^{k-1}\cdot\theta_i^*(\gamma_i) \\
&= \frac{1}{1-a_i}[1-\theta_i^*(\gamma_i)^k] \ .
\end{aligned}$$

Q.E.D.

It follows that

$$E[\widehat{Y}_i^E(X_i^i)] = \frac{1}{1-a_i}[1-\widehat{X}_i^i(\theta_i^*(\gamma_i))] \ . \qquad (3.40)$$

Let $\overline{G}_i(\underline{z}), \overline{V}_i(\underline{z}), V_i(\underline{z})$ be defined as for the Gated case. Then

$$G_{i+1}(\underline{z}) = \overline{G}_i(\underline{z})D_i^*(\Sigma) \qquad \text{(see (2.26))}$$

$$\overline{V}_i(\underline{z}) = a_i V_i(\underline{z})\frac{B_i^*(\Sigma)}{z_i} + (1-a_i)V_i(\underline{z})B_i^*(\Sigma) \quad \text{(see (2.27))}$$

$$F\left(\underline{z}\Big|^{\text{service}}_{\text{period } i}\right) = V_i(\underline{z})\frac{1-B_i^*(\Sigma)}{b_i\cdot\Sigma} \quad \text{(see (2.28))}$$

$$F\left(\underline{z}\Big|^{\text{switch-over}}_{\text{period } i}\right) = \overline{G}_i(\underline{z})\frac{1-D_i^*(\Sigma)}{d_i\cdot\Sigma} \quad \text{(see (2.29))}$$

Eisenberg's observation implies that

$$G_i(\underline{z}) + g_i^E\overline{V}_i(\underline{z}) = g_i^E V_i(\underline{z}) + \overline{G}_i(\underline{z}) \qquad \text{(see 2.30))}$$

where $V_i(\underline{z})$ is given by Equation (2.31) (with $G_i(\underline{z})$ of the Exhaustive case).

Finally, $F^E(\underline{z})$ is given by (2.32) (with $G_i(\underline{z})$ of the Exhaustive case).

3.5 Waiting Times

Arguing as in the Gated case,

$$F^E(1,1,\ldots,1,z_i,1,\ldots,1) = W_i^*(\lambda_i(1-z_i) \ .$$

Conclusion

We have studied Polling Systems with failing nodes, and analyzed both the Gated and Exhaustive Service regimes. Various performance measures concerning lengths of busy periods, number of remaining jobs at the termination of a busy period, distributions of successful and of unsuccessful service attempts, multi-dimensional generating functions of the system-state at polling and at arbitrary instants, as well as mean durations of cycle times, have been derived and calculated.

A further step in the study and analysis of such systems is to investigate their evolution under some time-dependent repair process, and to search for optimal visit procedures of the server that will increase both efficiency and fairness.

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References

[1] Altman, E. and Yechiali, U. (1994). "Polling in a Closed Network". *Probability in the Engineering and Informational Sciences* **8**, 327-343.

[2] Eisenberg, M. (1972). "Queues with Periodic Service and Changeover Time". *Operations Research* **20**, 440-451.

[3] Eliazar, I. and Yechiali, U. (1995). "Randomly Timed Gated Queueing Systems". Technical Report, School of Mathematical Sciences, Tel-Aviv University, Israel.

[4] Konheim, A.G., Levy, H. and Srinivasan (1994). "Descendant Set: an Efficient Approach for the Analysis of Polling Systems". *IEEE Transactions on Communications* **42**, 1245-1253.

[5] Sarkar, O. and Zangwill, W.I. (1989). "Expected Waiting Time for Nonsymmetric Cyclic Queueing Systems – Exact Results and Applications. *Management Science* **35**, 1463-1474.

[6] Takagi, H. (1986). *Analysis of Polling Systems*. MIT Press.

[7] Takagi, H. (1990). "Queueing Analysis of Polling Models: an Update". In: *Stochastic Analysis of Computer and Communications Systems* (H. Takagi, Ed.), pp. 267-318, North Holland.

[8] Takagi, H. (1994). "Queueing Analysis of Polling Models: Progress in 1990-1993". Technical Report No. 584, Institute of Socio-Economic Planning, University of Tsukuba, Japan.

[9] Yechiali, U. (1993). "Analysis and Control of Polling Systems". In: *Performance Evaluation of Computer and Communications Systems* (L. Donatiello and R. Nelson, Eds.), pp. 630-650, Springer Verlag.