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Rough Sets and 3-Valued Logics

Abstract. In the paper we explore the idea of describing Pawlak's rough sets using three-valued logic, whereby the value **t** corresponds to the positive region of a set, the value **f** — to the negative region, and the undefined value **u** — to the border of the set. Due to the properties of the above regions in rough set theory, the semantics of the logic is described using a non-deterministic matrix (Nmatrix). With the strong semantics, where only the value **t** is treated as designated, the above logic is a "common denominator" for Kleene and Łukasiewicz 3-valued logics, which represent its two different "determinizations". In turn, the weak semantics — where both **t** and **u** are treated as designated — represents such a "common denominator" for two major 3-valued paraconsistent logics.

We give sound and complete, cut-free sequent calculi for both versions of the logic generated by the rough set Nmatrix. Then we derive from these calculi sequent calculi with the same properties for the various "determinizations" of those two versions of the logic (including Łukasiewicz 3-valued logic). Finally, we show how to embed the four above-mentioned determinizations in extensions of the basic rough set logics obtained by adding to those logics a special two-valued "definedness" or "crispness" operator.

Keywords: rough sets, three-valued logics, non-deterministic matrices, sequent calculi.

1. Introduction

Next to Zadeh's fuzzy sets, Pawlak's rough sets [20, 21] are one of the two most famous notions used to describe vague information. Since their introduction in the early 1980s, they have been the subject of an impressive body of research. They have also found numerous practical applications, like the control of manufacturing processes [16], development of decisions tables [22], data mining [16], data analysis [23], knowledge discovery [18, 19], and so on.

The present paper has been motivated by the idea of proposing a new logical approach to this paradigm: namely, to try to describe rough sets using multiple-valued logics. Accordingly, we develop here a three-valued logic for rough sets. In this logic, the value **t** corresponds to the positive region of a set, the value **f** — to the negative region, and the undefined value **u** — to the border of the set.

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However, it turns out that, due to the properties of the above regions in rough set theory, the semantics of such a logic cannot be described with help of an ordinary logical matrix because of the non-deterministic properties of this semantics. As a result, we base the semantics of the rough set logic developed here on a 3-valued, non-deterministic matrix (Nmatrix) [4, 3].

The above logic turns out to have interesting relationships with some well-known 3-valued logics. Namely, with the strong version of the semantics (where only the value **t** is treated as designated) the rough set logic is a "common denominator" for Kleene and Łukasiewicz 3-valued logics, which represent its two different "determinizations". In turn, with a weak semantics (where both **t** and **u** are treated as designated) the logic is such a "common denominator" for two major 3-valued paraconsistent logics. In the paper we explore in detail the above-mentioned relationships between the two variants of the rough-set logic and the four known logics, and provide sound and complete, cut-free sequent calculi for all the logics discussed in this paper.

The paper is organized as follows. Following this introduction, in Section 2 we give an outline of Pawlak's rough set theory. Section 3 gives the motivation for using a three-valued logic to describe rough sets, and Section 4 — the motivation for the use of non-deterministic matrices. Section 5 introduces simple predicate languages, suitable for describing membership of elements in rough sets, while Section 6 describes the 3-valued rough set logic itself. In Section 7 we give a cut-free sequent calculus for the strong version of the propositional logic generated by the Nmatrix underlying the rough set logic, and prove its soundness and completeness. The relationships between this logic and Kleene and Łukasiewicz 3-valued logics are examined in Section 8, where sequent calculi for the latter logics are also developed. (In the case of Łukasiewicz this is a new cut-free calculus.) Section 9 deals with the weak version of the logic and its relationships with two major paraconsistent three-valued logics. Finally, Section 10 is devoted to conclusions and plans for further research.

2. Pawlak's Rough Set Theory

In its most general formulation, Pawlak's rough set theory [21] is based on the observation that we can view knowledge as represented by an equivalence relation. Namely, the relation representing our knowledge partitions a universe of objects into disjoint equivalence classes containing objects of which we have the same knowledge. In other words, using our knowledge, we cannot distinguish between any two objects belonging to the same equivalence class. The semantic framework for general rough set theory is based on the notion of a knowledge base:

DEFINITION 1. *A* knowledge base *is a tuple* $K = (U, \mathbf{R})$, *where:*

- U *is a* universe *of objects;*
- $\mathbf{R}$ *is a set of* equivalence relations *on objects in* U.

For any equivalence relation $R \in \mathbf{R}$ and any object x in the universe U, we denote by $[x]_R$ the unique equivalence class of R which contains x. The set of all equivalence classes of R is denoted by U/R.

Basically, rough sets are subsets of the universe U defined up to their lower and upper approximations with respect to a given equivalence relation in the knowledge base. The said approximations are defined as follows:

DEFINITION 2. *Let* $R \in \mathbf{R}$ *be any equivalence relation in the knowledge base* $K = (U, \mathbf{R})$, *and let* X *be any subset of* U. *Then:*

- *The* lower approximation of the set X with respect to the relation R *is the set*

$$\underline{R}X = \bigcup\{Y \in U/R \mid Y \subseteq X\} = \{x \in U \mid [x]_R \subseteq X\}$$

- *The* upper approximation of the set X with respect to the relation R *is the set*

$$\overline{R}X = \bigcup\{Y \in U/R \mid Y \cap X \neq \emptyset\} = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$$

Obviously, in view of the above definition, one can say that under the knowledge R:

- $\underline{R}X$ is the set of all the objects in U which *certainly* belong to X;
- $\overline{R}X$ is the set of all the objects in U which *might* belong to X;

3. Motivation for Three-Valued Logic

In view of the definitions presented in the previous section, with any set X we can associate the three following regions in U, representing three basic statuses, or degrees, of membership of an object of the universe U in the set $X \subseteq U$:

DEFINITION 3. *Let* $K = (U, \mathbf{R})$, $R \in \mathbf{R}$ *and* $X \subseteq U$. *Then:*

- *The R*-positive region *of X with respect to the relation R is*

$$POS_R(X) = \underline{R}X$$

- *The R*-negative region *of X with respect to the relation R is the set*

$$NEG_R(X) = U - \overline{R}X$$

- *The R*-boundary region *of X with respect to the relation R is the set*

$$BN_R(X) = \overline{R}X - \underline{R}X$$

Based on the knowledge R, we can say that

- The elements of $POS_R(X)$ *certainly belong* to X;
- The elements of $NEG_R(X)$ *certainly do not belong* to X;
- *We cannot tell if* the elements of $BN_R(X)$ *belong* to X or not

This suggests a natural way of describing rough sets with help of a simple three-valued logic $\mathcal{L}_{rs}$, defined informally as follows:

- The formulas of $\mathcal{L}_{rs}$ are all expressions of the form Ax, where:
  - A is an expression representing a subset of U
  - x is a variable representing an object in U
- The semantics of $\mathcal{L}_{rs}$ uses logical values in

  $$\mathcal{T} = \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$$

  where:
  - **t** represents the classical value **true**,
  - **f** represents the classical value **false**,
  - **u** represents a non-classical value **unknown**.

  As designated values we can take either $\{\mathbf{u}, \mathbf{t}\}$ — obtaining a weak logic — or only $\{\mathbf{t}\}$, obtaining a strong logic. However, as we will show later, both options are dual from the viewpoint of satisfaction and proof theory. Accordingly, we will first examine in detail the strong logic, and then use the duality to obtain the corresponding results for the weak logic.
- The truth-values of formulas in $\mathcal{L}_{rs}$ with respect to a knowledge base $K = (U, \mathbf{R})$, a relation $R \in \mathbf{R}$, an interpretation $|\cdot|$ of set expressions, and a valuation v of object variables, are as follows:

$$||Ax||_v = \begin{cases} \mathbf{t} & \text{if } v(x) \in POS_R(|A|) \\ \mathbf{f} & \text{if } v(x) \in NEG_R(|A|) \\ \mathbf{u} & \text{if } v(x) \in BN_R(|A|) \end{cases} \qquad (1)$$

4. Motivation for the Use of Non-Deterministic Matrices

Unfortunately, the logic $\mathcal{L}_{rs}$ proposed informally above has a major drawback: its semantics is not decompositional. This follows from the fact that the lower and upper approximations of a set obey the rules

$$\begin{array}{ll} \overline{R}(A \cup B) = \overline{R}A \cup \overline{R}B & \underline{R}(A \cup B) \supseteq \underline{R}A \cup \underline{R}B \\ \underline{R}(A \cap B) = \underline{R}A \cap \underline{R}B & \overline{R}(A \cap B) \subseteq \overline{R}A \cap \overline{R}B \\ \underline{R}(-A) = -\overline{R}A & \overline{R}(-A) = -\underline{R}A \end{array} \qquad (2)$$

where the inclusions cannot be in general replaced by equalities.

Clearly, these inclusions imply that the values of $(A \cup B)x$ and $(A \cap B)x$ are not always uniquely determined by the values of Ax and Bx, which is exactly the factor that makes the semantics of $\mathcal{L}_{rs}$ non-decompositional. Namely, we have:

If $||Ax|| = \mathbf{u}$ and $||Bx|| = \mathbf{u}$, then
$||(A \cup B)x|| \in \{\mathbf{u}, \mathbf{t}\}$ and $||(A \cap B)x|| \in \{\mathbf{f}, \mathbf{u}\}$

and we cannot in general say in advance which of the respective two values will be assigned by the interpretation to the considered two formulas.

In view of the above, it is evident that the semantics of $\mathcal{L}_{rs}$ cannot be defined using an ordinary logical matrix. A solution to this problem is to use instead a *non-deterministic logical matrix (Nmatrix)*, which is a generalization of an ordinary matrix modelling non-determinism, with interpretations of logical connectives returning sets of logical values instead of single values.

Definition 4. [4] *A* non-deterministic matrix *(*Nmatrix*) for a propositional language L is a tuple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where:*

- *$\mathcal{T}$ is a non-empty set of truth values.*
- *$\emptyset \subset \mathcal{D} \subseteq \mathcal{T}$ is the set of designated values.*
- *For every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \{\emptyset\}$.*

Let W be the set of well-formed formulas of L. A (legal) valuation in an Nmatrix $\mathcal{M}$ is a function $v : W \rightarrow \mathcal{T}$ that satisfies the following condition:

$$v(\diamond(\psi_1, \cdots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1) \cdots, v(\psi_n))$$

for every n-ary connective $\diamond$ of L and any $\psi_1 \cdots, \psi_n \in W$.

Let $\mathcal{V}_M$ denote the set of all valuations in the Nmatrix $\mathcal{M}$. The notions of satisfaction under a valuation, validity, and consequence relation are defined as follows:

- *A formula $\varphi \in W$ is* satisfied *by a valuation $v \in \mathcal{V}_M$, in symbols $v \models \varphi$, if $v(\varphi) \in \mathcal{D}$*
- *A sequent $\Sigma = \Gamma \Rightarrow \Delta$ is* satisfied *by a valuation $v \in \mathcal{V}_M$, in symbols $v \models \Sigma$, iff either v does not satisfy some formula in Γ or v satisfies some formula in Δ*
- *A sequent Σ is* valid*, in symbols $\models \Sigma$, if it is satisfied by all valuations $v \in \mathcal{V}_M$*
- *The consequence relation on W defined by $\mathcal{M}$ is the relation $\vdash_{\mathcal{M}}$ on sets of formulas in W such that, for any $T, S \subseteq W$, $T \vdash_{\mathcal{M}} S$ iff there exist finite sets $\Gamma \subseteq T, \Delta \subseteq S$ such that the sequent $\Gamma \Rightarrow \Delta$ is valid.*

Though an Nmatrix is basically defined for a standard propositional language, we will see below that it can be easily used to define the semantics for the simple predicate-type of language needed to describe rough sets.

In what follows we will define a three-valued logic for modelling rough sets with semantics based on an Nmatrix.

5. Simple Predicate Languages

Since the language of our rough set logic will be a predicate-level one, with atomic formulas expressing membership of objects in sets, we will first provide a general definition of such a language, together with its semantics based on an Nmatrix.

We start with defining the syntax of that language:

Definition 5. *The alphabet of a* simple predicate language *L_P contains:*

- *a set $\mathbf{P}_n$ of n-ary predicate symbols for $n = 0, 1, 2, \ldots$;*
- *a set $\mathbf{O}_n^k$ of symbols for k-ary operations on n-ary predicates for $n, k = 0, 1, 2, \ldots$*
- *a set $\mathbf{V}$ of individual variables.*

The set $\mathbf{E}_n$ of predicate expressions of arity n *is the least set such that:*

- *$\mathbf{P}_n \subseteq \mathbf{E}_n$;*
- *if $\diamond \in \mathbf{O}_n^k$ and $e_1, \ldots, e_k \in \mathbf{E}_n$, then $\diamond(e_1, \ldots, e_k) \in \mathbf{E}_n$.*

The set W of well-formed formulas of L_P consists of all expressions of the form

$$e(x_1, \dots, x_n)$$

where $n \geq 0, e \in \mathbf{E}_n$ and $x_1, \dots, x_n \in \mathbf{V}$.

The semantics of L_P is defined based on an Nmatrix for L_P and a structure for L_P:

Definition 6. *An Nmatrix for L_P is a non-deterministic matrix $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ with $\mathcal{O}$ containing an interpretation $\widetilde{\diamond} : \mathcal{T}^k \to 2^{\mathcal{T}} \setminus \{\emptyset\}$ for every k-ary operation $\diamond$ on n-ary predicates in $\mathbf{O}^k_n, n \geq 0$.*

Definition 7. *A $\mathcal{T}$-*structure *for L_P is a pair $\mathbf{M} = (X, |\cdot|)$, where:*

- *X is a non-empty set;*
- *$|\cdot|$ is an interpretation of predicate symbols, with $|p| : X^n \to \mathcal{T}$ for any $p \in P_n, n \geq 0$.*

To define the interpretation of formulas of L_P in a given structure, we use a non-deterministic matrix for interpreting the operators of that language, adapting the definition of a legal valuation under an Nmatrix for a propositional language given in Definition 4 to the syntax of L_P as follows:

Definition 8. *Let $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ be an Nmatrix for L_P, and let $\mathbf{M} = (X, |\cdot|)$ be a $\mathcal{T}$-structure for $\mathcal{L}_P$.*

An interpretation *of L_P under the Nmatrix $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ in the structure $\mathbf{M} = (X, |\cdot|)$ for a valuation $v : V \to X$ is a function $||\cdot||^M_v : W \to \mathcal{T}$ such that:*

- $||p(x_1, \dots, x_n)||^M_v = |p|(v(x_1), \dots, v(x_n))$ *for any $p \in \mathbf{P_n}, n \geq 0$,*
- $|| \diamond (e_1, \dots, e_k)(x_1, \dots, x_n)||^M_v$
  $\in \widetilde{\diamond}(||(e_1(x_1, \dots, x_n))||^M_v, \dots, ||(e_k(x_1, \dots, x_n)||^M_v)$
  for any k-ary operation on n-ary predicates $\diamond \in \mathbf{O}^k_n$, any n-ary predicate expressions $e_1, \dots, e_k \in \mathbf{E}_n$, and any individual variables $x_i \in \mathbf{V}, i = 1, \dots, n$.

To simplify notation, in what follows we will drop the decorations on the $||\cdot||$ symbol whenever this does not lead to confusion.

6. Propositional Rough Sets Logic

6.1. Basic Rough Sets Language

The predicate language L_{RS} for describing rough sets uses only unary predicate symbols representing sets, object variables, and the symbols $-, \cup, \cap$ representing operations on predicates corresponding to set-theoretical operations on rough sets, i.e.

- $\mathbf{P}_1 = \{P, Q, R, \ldots\}$, $\mathbf{P}_n = \emptyset$ for $n \neq 1$
- $\mathbf{O}_1^1 = \{-\}, \mathbf{O}_1^2 = \{\cup, \cap\}, \mathbf{O}_n^k = \emptyset$ otherwise.

Thus the set W_{RS} of well-formed formulas of L_{RS} contains all expressions of the form Ax, where A is a unary predicate expression representing a set, built of the predicate symbols in $\mathbf{P}_1$ and using the operation symbols $-, \cup, \cap$, while $x \in \mathbf{V}$ is an individual variable.

The semantics of L_{RS} is given by the Nmatrix $\mathcal{M}_{RS} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where $\mathcal{T} = \{\mathbf{f}, \mathbf{u}, \mathbf{t}\}$, $\mathcal{D} = \{\mathbf{t}\}$, and $-, \cup, \cap$ are interpreted as set-theoretic operations on rough sets, with their semantics given by:

$\tilde{-}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{t}$
	$\mathbf{t}$	$\mathbf{u}$	$\mathbf{f}$

$\tilde{\cup}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{u}$	$\{\mathbf{u}, \mathbf{t}\}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

$\tilde{\cap}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{u}$	$\mathbf{f}$	$\{\mathbf{f}, \mathbf{u}\}$	$\mathbf{u}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{t}$

where $\mathbf{f}, \mathbf{u}$ and $\mathbf{t}$ stand for the appropriate singleton sets.

It can be easily checked that in view of the rules (2) governing the interplay between the operations of lower and upper approximations and set-theoretic operations in the rough sets framework, the interpretation of the latter operations in the Nmatrix $\mathcal{M}_{RS}$ corresponds to the intended interpretation (1) of the Ax type of formulas in that framework. Note that complement is in fact a deterministic operation, while the semantic tables for union and intersection have just one "non-determinacy point" each: namely, the result of the operation on two undefined arguments.

6.2. Propositional Rough Set Logic

Our next goals are to produce efficient proof systems for the three-valued rough set logic $\mathcal{L}_{RS}$ defined by the language L_{RS} with the semantics given by

$\mathcal{M}_{RS}$, as well as to compare it to the two most well-known three-valued logics — namely, Kleene [12] and Łukasiewicz [17] logics. However, we cannot make such a comparison directly — for the latter logics are standard propositional ones, featuring no predicate symbols or object variables. Hence our comparison must be based not on the logic $\mathcal{L}_{RS}$ with the language L_{RS} itself, but on a propositional logic with the semantics defined by the Nmatrix $\mathcal{M}_{RS}$ derived from the rough sets framework. This does not make a real difference with regard to the proof system, since $\mathcal{L}_{RS}$ has no quantifiers or logical connectives, and a proof system for the propositional logic mentioned above can easily be translated to a proof system for $\mathcal{L}_{RS}$ (see Note 2 below).

What is more, basing on the well-known fact that conjunction and disjunction can be expressed in terms of negation and implication in both Kleene and Łukasiewicz logics, in order to simplify development of the proof system for that logic, we will switch to a propositional language using just the latter connectives. The implication in that language will be defined as

$$A \to B = \neg A \vee B$$

where the connectives $\neg$ and $\vee$ correspond to the operations $-$ and $\cup$ in $\mathcal{M}_{RS}$. (The use of such a negation-implication language will facilitate comparison with Kleene and Łukasiewicz logics, for which we will also use their negation-implication versions.). More exactly, we consider a propositional logic $\mathcal{L}^I_{RS}$ with propositional variables in $\mathbf{P} = \{p, q, r, \ldots\}$ and connectives $\neg, \to$. The formulas of $\mathcal{L}^I_{RS}$ are denoted by $A, B, C, \ldots$, and the set of all well-formed formulas — by W_I.

It can be easily seen that the Nmatrix corresponding to $\mathcal{L}^I_{RS}$ is $\mathcal{M}^I_{RS} = (\mathcal{T}, \mathcal{D}, \mathcal{O}_I)$, where $\mathcal{T}, \mathcal{D}$ are the same as before, and $\mathcal{O}_I$ contains the interpretations of $\neg$ and $\to$ defined by the following semantic tables:

$\tilde{\neg}$	**f**	**u**	**t**
	t	**u**	**f**

$\tilde{\to}$	**f**	**u**	**t**
f	**t**	**t**	**t**
u	**u**	$\{$**u**, **t**$\}$	**t**
t	**f**	**u**	**t**

(3)

It should be noted here that the two possible "determinizations" of the above Nmatrix give ordinary matrices underlying the two famous three-valued logics of Kleene and Łukasiewicz. As a result, the logic based on $\mathcal{M}^I_{RS}$ can be seen as a "common denominator" for Kleene and Łukasiewicz logics, which we can safely work with if we are not sure which of the above logics to choose for our application. Indeed: if out of the two possibilities for the value of $\mathbf{u} \to \mathbf{u}$ given in the implication table (3) we choose $\mathbf{u}$, we obtain Kleene logic, whereas the choice of $\mathbf{t}$ yields Łukasiewicz logic.

A (legal) valuation of $\mathcal{L}^I_{RS}$ under the matrix $\mathcal{M}^I_{RS}$ is defined according to Definition 4.

From now on, by the rough set logic we shall mean the implication and negation logic $\mathcal{L}^I_{RS}$ defined in this section.

7. Sequent Calculus for the Logic Generated by $\mathcal{M}^I_{RS}$

Now we shall present a proof system for the logic $\mathcal{L}^I_{RS}$ based on $\mathcal{M}^I_{RS}$. Note that this logic has no tautologies, but only valid entailments, represented by valid sequents. Accordingly, the deduction formalism best suited to $\mathcal{L}^I_{RS}$ should necessarily be a sequent calculus.

Let **IRS** be the following sequent calculus over $\mathcal{L}^I_{RS}$:

Axioms:

(A1) $A \Rightarrow A$; **(A2)** $\neg A, A \Rightarrow$

Inference rules:

Weakening on both sides, together with the following logical rules:

$(\neg\neg \Rightarrow)$ $\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$ $(\Rightarrow \neg\neg)$ $\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$

$(\Rightarrow\to I)$ $\dfrac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, A \to B}$ $(\Rightarrow\to II)$ $\dfrac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$

$(\to\Rightarrow I)$ $\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$ $(\to\Rightarrow II)$ $\dfrac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma, A \to B \Rightarrow \Delta}$

$$(\Rightarrow \neg \to) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \to B)}$$

$(\neg \to\Rightarrow I)$ $\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg(A \to B) \Rightarrow \Delta}$ $(\neg \to\Rightarrow II)$ $\dfrac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \to B) \Rightarrow \Delta}$

NOTE 1. When presenting **IRS**, we chose a set of rules where each premise has exactly one active formula. However, we can also obtain a set of rules which is more economic in size by combining certain pairs of rules. Namely, rules ($\Rightarrow\to I$) and ($\Rightarrow\to II$) above can be combined to

$$\frac{\Gamma \Rightarrow \Delta, \neg A, B}{\Gamma \Rightarrow \Delta, A \to B}$$

while rules ($\to\Rightarrow I$) and ($\to\Rightarrow II$) can be combined to

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A, \neg B}{\Gamma, A \to B \Rightarrow \Delta}$$

and, finally, rules ($\neg\to\Rightarrow I$) and ($\neg\to\Rightarrow II$) can be combined to

$$\frac{\Gamma, A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \to B) \Rightarrow \Delta}$$

As we can see (especially from the three-premise rule obtained in this way), there is a clear trade-off between the size of the set of rules and the complexity of the individual rules — so the choice of a particular option should depend on the intended application.

LEMMA 1.

1. *The axioms of the system* **IRS** *are valid.*
2. *For any inference rule r of* **IRS** *and any valuation v, if v satisfies all the premises of r then v satisfies the conclusion of r.*

PROOF. Both parts can be easily verified based on the truth tables of $\mathcal{M}^I_{RS}$. By way of example, we shall prove the second part for the rule:

$$(\Rightarrow\neg\to) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \to B)}$$

Suppose $v \models \Gamma \Rightarrow \Delta, A$ and $v \models \Gamma \Rightarrow \Delta, \neg B$. Then either (1) $v(\gamma) \neq \mathbf{t}$ for some $\gamma \in \Gamma$ or $v(\delta) = \mathbf{t}$ for some $\delta \in \Delta$, or (2) $v(A) = \mathbf{t}$ and $v(\neg B) = \mathbf{t}$ (and so $v(B) = \mathbf{f}$). If (1) holds, then clearly $v \models \Gamma \Rightarrow \Delta, \neg(A \to B)$. If (2) holds, then from the truth table of $\to$ in $\mathcal{M}^I_{RS}$ it follows that $v(A \to B) = \mathbf{f}$, whence $v(\neg(A \to B)) = \mathbf{t}$, and again $v \models \Gamma \Rightarrow \Delta, \neg(A \to B)$. ■

Clearly, from the above Lemma we immediately conclude:

COROLLARY 1. *The inference rules of* **IRS** *are sound, i.e. they preserve the validity of sequents.*

THEOREM 1. *The sequent calculus* **IRS** *is sound and complete for* $\vdash_{\mathcal{M}^I_{RS}}$.

PROOF. Let us denote provability in **IRS** by $\vdash_{\mathrm{IRS}}$. More exactly, for any sequent Σ over $\mathcal{L}^I_{RS}$

$$\vdash_{\mathrm{IRS}} \Sigma \text{ if } \Sigma \text{ has a proof in } \mathbf{IRS}$$

We have to prove that, for any sequent Σ over $\mathcal{L}^I_{RS}$:

$$\models \Sigma \text{ iff } \vdash_{\mathrm{IRS}} \Sigma \tag{4}$$

The backward implication, representing soundness of the system, follows immediately from Lemma 1 and Corollary 1.

To prove the forward implication (completeness), we argue by contradiction. Suppose Σ is a sequent over $\mathcal{L}^I_{RS}$ such that $\not\vdash_{\mathrm{IRS}} \Sigma$. We shall prove that $\not\models \Sigma$.

Let us assume that inclusion and union of sequents are defined componentwise, i.e.:

$$(\Gamma' \Rightarrow \Delta') \subseteq (\Gamma'' \Rightarrow \Delta'') \text{ iff } \Gamma' \subseteq \Gamma'' \text{ and } \Delta' \subseteq \Delta''$$

$$(\Gamma' \Rightarrow \Delta') \cup (\Gamma'' \Rightarrow \Delta'') = (\Gamma', \Gamma'' \Rightarrow \Delta', \Delta'')$$

Call a sequent Σ_0 *saturated* if is closed under all the rules in **IRS** applied backwards. More exactly, for any rule r in **IRS** whose conclusion is contained in Σ_0, one of its premises must also be contained in Σ_0 (for a single premise rule, this means its only premise must be contained in Σ_0). For example, if $\Sigma_0 = (\Gamma_0 \to \Delta_0)$ is saturated and $(A \to B) \in \Delta_0$, then in view of the rules $(\Rightarrow\to I)$ and $(\Rightarrow\to II)$ we must have both $\neg A \in \Delta$ and $B \in \Delta$. In turn, if $(A \to B) \in \Gamma_0$, then in view of the rules $(\to\Rightarrow I)$ and $(\to\Rightarrow II)$ we must have either $A \in \Delta$ or $B \in \Gamma$, and at the same time either $\neg A \in \Gamma$ or $\neg B \in \Delta$.

Let $\Sigma = (\Gamma \Rightarrow \Delta)$. We shall first prove that Σ can be extended to a saturated sequent $\Sigma^* = (\Gamma^* \Rightarrow \Delta^*)$ which is not provable in **IRS**. If Σ is already saturated, we are done. Otherwise we start with the sequent Σ and expand it step by step by closing it under the subsequent rules of **IRS** without losing the non-provability property. More exactly, we define a sequence $\Sigma_0, \Sigma_1, \Sigma_2 \ldots$ such that:

1. $\Sigma_{i-1} \subseteq \Sigma_i$ for each $i \geq 1$;
2. Σ_i is not provable.

We take $\Sigma_0 = \Sigma_1 = \Sigma$; then conditions 1,2 above are satisfied for $i = 1$. Assume now we have constructed sequents $\Sigma_0, \Sigma_1, \ldots, \Sigma_k$ satisfying those conditions, and Σ_k is still not saturated. Then there is a rule $r = \dfrac{\Pi_1 \ \cdots \ \Pi_l}{\Pi}$ in **IRS** such that $\Pi \subseteq \Sigma_k$ but $\Pi_i \not\subseteq \Sigma_k$ for $i = 1, \ldots, l$. Since Σ_k is not provable, there must be a i such that $\Sigma_k \cup \Pi_i$ is not provable. Indeed, if $\Sigma_i \cup \Pi_i$ were provable for all $i, 1 \leq i \leq l$, then we could deduce $\Sigma_k \cup \Pi$ from the provable sequents $\Sigma_k \cup \Pi_i, i = 1, \cdots, l$, using rule r, which in view of $\Sigma_k \cup \Pi = \Sigma_k$ would contradict the fact that Σ_k is not provable. Thus there is an $i_0, 1 \leq i_0 \leq l$, such that $\Sigma_k \cup \Pi_{i_0}$ is not provable, and we take $\Sigma_{k+1} = \Sigma_k \cup \Pi_{i_0}$. Obviously, the sequence $\Sigma_0, \Sigma_1, \ldots, \Sigma_{k+1}$ satisfies conditions 1,2 above.

Since all the rules in **IRS** have the subformula property, it is clear that after a finite number n of such steps we will have added all the possible premises of the rules r in **IRS** whose conclusions are contained in the original sequent Σ or its descendants in the constructed sequence, obtaining finally a saturated extension $\Sigma^* = \Sigma_n$ of Σ which is not provable in **IRS**.

Thus we have:

- $\Sigma^* = (\Gamma^* \Rightarrow \Delta^*)$ is closed under the rules in **IRS** applied backwards
- $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$
- $\not\vdash_{\mathrm{IRS}} \Sigma^*$

We use Σ^* to define a counter-valuation for Σ, i.e. a legal valuation v of $\mathcal{L}^I_{RS}$ under $\mathcal{M}^I_{RS}$ such that $v \not\models \Sigma$. Namely, we put [1]:

- For any propositional symbol $p \in \mathbf{P}$,

$$v(p) = \begin{cases} \mathbf{t} & \text{if } p \in \Gamma^* \\ \mathbf{f} & \text{if } \neg p \in \Gamma^* \\ \mathbf{u} & \text{otherwise} \end{cases} \tag{5}$$

- For any $A, B \in W_I$

$$v(\neg A) = \neg v(A) \tag{6}$$

$$v(A \to B) = \begin{cases} v(A) \to v(B) & \text{if not } (v(A) = v(B) = \mathbf{u}) \\ \mathbf{t} & \text{if } v(A) = v(B) = \mathbf{u} \\ & \text{and } (A \to B) \in \Gamma^* \\ \mathbf{u} & \text{otherwise} \end{cases} \tag{7}$$

[1] For simplicity, we drop the $\tilde{}$ decoration marking the interpretation of connectives.

It is easy to see that v defined as above is a well-defined mapping of W_I into $\mathcal{T}$. Indeed, as Σ^* is not provable in **IRS** and $A, \neg A \Rightarrow$ is an axiom of **IRS**, then by (5) $v(p)$ is uniquely defined for any propositional symbol $p \in \mathbf{P}$, whence by (6, 7) $v(\varphi)$ is uniquely defined for any $\varphi \in W_I$.

What is more, by (6, 7) v is a legal interpretation of $\mathcal{L}^I_{RS}$ under the Nmatrix $\mathcal{M}^I_{RS}$, for the interpretations of $\neg, \rightarrow$ under v are compliant with the tables (3) of those operations for this Nmatrix.

As Σ^* is an extension of Σ, in order to prove that $v \not\models \Sigma$, is suffices to prove that $v \not\models \Sigma^*$. By Definition 4, we should prove that, for any $\varphi \in W_I$,

$$v \models \gamma \text{ for any } \gamma \in \Gamma^*, \qquad v \not\models \delta \text{ for any } \delta \in \Delta^* \tag{8}$$

Equation (8) is proved by structural induction on the formulas in $S = \Gamma^* \cup \Delta^*$.

We begin with literals in S, having the form either p or $\neg p$, where $p \in \mathbf{P}$. We have the following cases:

- $\varphi = p$. Then by (5) and the fact that Γ^* and Δ^* are disjoint (for otherwise Σ^* would be provable), we have: $v(\varphi) = \mathbf{t}$ if $\varphi \in \Gamma^*$ and $v(\varphi) \neq \mathbf{t}$ if $\varphi \in \Delta^*$
- $\varphi = \neg p$. If $\varphi \in \Gamma^*$, then by (5) $v(p) = \mathbf{f}$, whence $v(\varphi) = \neg\mathbf{f} = \mathbf{t}$ by (6). In turn, if $\varphi \in \Delta^*$, then $\varphi \notin \Gamma^*$, whence $v(p) \neq \mathbf{f}$ and $v(\varphi) = \neg v(p) \neq \mathbf{t}$.

Define the rank ρ of a formula φ by:

$$\rho(p) = 1, \quad \rho(\neg\varphi) = \rho(\varphi) + 1, \quad \rho(\varphi \rightarrow \psi) = \rho(\varphi) + \rho(\psi) + 1$$

Now assume that (8) is satisfied for formulas in S of rank up to n, and suppose that $A, B \in S$ are at most of rank n. We need to prove that (8) holds for $\neg A$ and $A \rightarrow B$

We begin with negation. Let $\varphi = \neg A$. As the case of $A = p \in \mathbf{P}$ has already been considered, it remains to consider the following two cases:

- $A = \neg B$. Then we have $\varphi = \neg\neg B$.
  - If $\varphi \in \Gamma^*$, then by rule $(\neg\neg \Rightarrow)$ we have $B \in \Gamma^*$, since Σ^* is a saturated sequent. Hence by inductive assumption $v(B) = \mathbf{t}$, and by (6) $v(\varphi) = \neg\neg\mathbf{t} = \mathbf{t}$.
  - In turn, if $\varphi \in \Delta^*$, then by rule $(\Rightarrow \neg\neg)$ we have $B \in \Delta^*$, whence by inductive assumption $v(B) \neq \mathbf{t}$, and in consequence $v(\varphi) \neq \mathbf{t}$.
- $A = B \rightarrow C$. We again have two cases:

- If $\varphi \in \Gamma^*$, then by rules $(\neg \to\Rightarrow I)$ and $(\neg \to\Rightarrow II)$ we have $A, \neg B \in \Gamma^*$ since Σ^* is saturated. Hence by inductive assumption $v(A) = \mathbf{t}$ and $v(B) = \mathbf{f}$ (because $v(\neg B) = \mathbf{t}$). Thus by (7) $v(A \to B) = \mathbf{f}$, and $v(\varphi) = \neg\mathbf{f} = \mathbf{t}$.
- If $\varphi \in \Delta^*$, then by rule $(\Rightarrow \neg \to)$ we have either $A \in \Delta^*$ or $\neg B \in \Delta^*$. By the inductive assumption, this yields either $v(A) \neq \mathbf{t}$ or $v(B) \neq \mathbf{f}$. Thus by (7) $v(A \to B) \neq \mathbf{f}$, whence $v(\varphi) \neq \mathbf{t}$.

It remains to consider implication. Let $\varphi = A \to B$. We have the following two cases:

- $\varphi \in \Gamma^*$. If $v(A) = v(B) = \mathbf{u}$, then $v(\varphi) = \mathbf{t}$ by (7).
  Assume now that $(v(A), v(B)) \neq (\mathbf{u}, \mathbf{u})$. As Σ^* is saturated, then by rules $(\to\Rightarrow I)$ and $(\to\Rightarrow II)$ we have

  (i) either $A \in \Delta^*$ or $B \in \Gamma^*$, and

  (ii) either $\neg B \Rightarrow\in \Delta^*$ or $\neg A \in \Gamma^*$.

  If $B \in \Gamma^*$, then by the inductive assumption $v(B) = \mathbf{t}$, and $v(\varphi) = v(A \to B) = \mathbf{t}$ by (7). By analogous reasoning, if $\neg A \in \Gamma^*$, then $v(A) = \mathbf{f}$, and $v(\varphi) = \mathbf{t}$ as well.
  It remains to consider the case when $A \in \Delta^*$ and $\neg B \Rightarrow\in \Delta^*$. Then by the inductive assumption we have $v(A) \neq \mathbf{t}, v(\neg B) \neq \mathbf{t}$, with the latter implying $v(B) \neq \mathbf{f}$. As by assumption $(v(A), v(B)) \neq (\mathbf{u}, \mathbf{u})$, then by the foregoing and (7) we have $v(\varphi) \in \{\mathbf{f} \to \mathbf{u}, \mathbf{f} \to \mathbf{t}, \mathbf{u} \to \mathbf{t}\} = \{\mathbf{t}\}$.
- $\varphi \in \Delta^*$. Then, as Σ^* is saturated, by rules $(\Rightarrow\to I)$ and $(\Rightarrow\to II)$ we have $\neg A \in \Delta^*$ and $B \in \Delta^*$. Hence by the inductive assumption we have $v(\neg A) \neq \mathbf{t}$, i.e. $v(A) \neq \mathbf{f}$, and $v(B) \neq \mathbf{t}$. In view of (7) *and the fact that* $\varphi \notin \Gamma^*$, this yields $v(\varphi) = v(A) \to v(B) \in \{\mathbf{u} \to \mathbf{f}, \mathbf{t} \to \mathbf{f}, \mathbf{t} \to \mathbf{u}\} \cup \{\mathbf{u}\} = \{\mathbf{f}, \mathbf{u}\}$, whence $v(\varphi) \neq \mathbf{t}$. [2]

Thus (8) holds and $v \not\models \Sigma$, which ends the completeness proof. ■

It should be noted that results stronger than Theorem 1 can be proved for **IRS**: actually, we could also prove strong completeness and compactness of that calculus (for the natural consequence relation between sequents induced by $\mathcal{M}^I_{RS}$). However, we have refrained from doing it, since such results would be superfluous here in view of the main focus of our paper.

[2] This is the only place in the proof in which the axioms are needed in their general form. Indeed, unlike the calculi for classical logic, the calculus here is *not* complete if the identity axioms $A \Rightarrow A$ are restricted to sequents containing only literals.

Note 2. As disjunction and conjunction can be represented using negation and implication according to the relationships:

$$A \vee B \equiv \neg A \to B, \quad A \wedge B \equiv \neg(A \to \neg B)$$

they can be treated as derived operations in our language. From the above representation and **IRS**, we can derive the following sequent rules for conjunction and disjunction:

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg A}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \qquad \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

The above system, complemented with the negation rules from **IRS**, can be used for reasoning in the rough set logic $\mathcal{L}_{RS}$. Namely, if we first apply an obvious translation τ from the language of $\mathcal{L}_{RS}$ to the language $\mathcal{L}'_{RS}$ of atomic predicate expressions combined with $\neg, \vee, \wedge$ such that

$$\tau((A \cup B)x) = \tau(Ax) \vee \tau(Bx) \qquad \tau((A \cap B)x) = \tau(Ax) \wedge \tau(Bx)$$
$$\tau((-A)x) = \neg\tau(Ax)$$

then the system mentioned above, together with the substitution principle, is complete for $\mathcal{L}'_{RS}$ — and so also for $\mathcal{L}_{RS}$.

8. Relations with Kleene and Łukasiewicz 3-valued Logics

8.1. $\mathcal{L}^I_{RS}$ as the Common Part of Kleene and Łukasiewicz Logics

Our next task is to compare the logic $\mathcal{L}^I_{RS}$ with the well-known propositional three-valued logics proposed by Kleene and Łukasiewicz. Let us denote the latter logics by $\mathcal{L}_K$ and $\mathcal{L}_L$, respectively. Obviously, without any loss of generality we can assume that $\mathcal{L}_K$ and $\mathcal{L}_L$ have the same set of well-formed formulas as $\mathcal{L}^I_{RS}$. From now on, we will denote that set by W.

First, recall that $\mathcal{L}^I_{RS}$ is given by the Nmatrix $\mathcal{M}^I_{RS} = (\mathcal{T}, \mathcal{D}, \{\widetilde{\neg}, \widetilde{\rightarrow}\})$, where

$\widetilde{\neg}$	**f**	**u**	**t**
	t	**u**	**f**

$\widetilde{\rightarrow}$	**f**	**u**	**t**
f	**t**	**t**	**t**
u	**u**	{**u**, **t**}	**t**
t	**f**	**u**	**t**

(9)

Now, using the same notational convention, we can say that the $\{\neg, \rightarrow\}$ versions of Kleene and Łukasiewicz logics are given by ordinary matrices $\mathcal{M}_K$ and $\mathcal{M}_L$ with $\mathcal{T}, \mathcal{D}$ and the common interpretation $\widetilde{\neg}$ of negation as in the Nmatrix $\mathcal{M}^I_{RS}$, but different interpretations of implication, given, respectively, by $\rightarrow^K$ and $\rightarrow^L$ defined below:

$\rightarrow^K$	**f**	**u**	**t**
f	**t**	**t**	**t**
u	**u**	**u**	**t**
t	**f**	**u**	**t**

$\rightarrow^L$	**f**	**u**	**t**
f	**t**	**t**	**t**
u	**u**	**t**	**t**
t	**f**	**u**	**t**

(10)

From (9) and (10) it clearly follows that the matrices $\mathcal{M}_K$ and $\mathcal{M}_L$ are included in the Nmatrix $\mathcal{M}^I_{RS}$, and represent its two different "determinizations". An obvious consequence of this fact is:

LEMMA 2. *The system* **IRS** *is sound for both Kleene and Łukasiewicz logics.*

PROOF. Immediate by the soundness of **IRS** for $\mathcal{L}^I_{RS}$ and the fact that every valuation $v : W \rightarrow \mathcal{T}$ legal under the matrix $\mathcal{M}_K$ or $\mathcal{M}_L$ is also legal under the Nmatrix $\mathcal{M}^I_{RS}$. ■

Now we will show that to make **IRS** also complete for $\mathcal{L}_K$ and $\mathcal{L}_L$ it suffices to add just one sequent rule for each logic:

THEOREM 2. *Let (K) and (L) be the two following sequent rules:*

$$(\mathbf{K})\ \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad (\mathbf{L})\ \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Then:

1. *The system* $\mathbf{IRS}^K$ *obtained by adding rule (K) to* **IRS** *is sound and complete for Kleene logic;*
2. *The system* $\mathbf{IRS}^L$ *obtained by adding rule (L) to* **IRS** *is sound and complete for Łukasiewicz logic.*

Proof. As it is easy to check that rule (K) is sound for $\mathcal{L}_K$ and rule (L) is sound for $\mathcal{L}_L$, then both calculi are sound for the respective logics. The proofs of their completeness are obtained by modifying in an appropriate way the completeness proof for **IRS** in Theorem 1.

The first modification is common for both logics and consist in modifying the definition of the counter-valuation v for complex formulas: namely, since both logics are deterministic, we simply take v to be the unique extension of the valuation of propositional variables defined as in Theorem 1.

Obviously, the valuations obtained in this way are legal for both matrices, so it remains to prove that they are indeed counter-valuations.

Kleene logic. It can be easily seen that the proof of the fact that $v \not\models \Sigma^*$ given in Theorem 1 remains unchanged save for the case when $\varphi = (A \to B) \in \Gamma^*$ and $v(A) = v(B) = \mathbf{u}$, for we can no longer make use of the fact that in this case $v(A \to B) = \mathbf{t}$. However, if $\varphi = (A \to B) \in \Gamma^*$, then from rule (K) it follows that either $\neg A \in \Gamma^*$ or $B \in \Gamma^*$. Yet by the inductive assumption $\neg A \in \Gamma^*$ implies $v(A) = \mathbf{f} \neq \mathbf{u}$, while $B \in \Gamma^*$ implies $v(B) = \mathbf{t} \neq \mathbf{u}$, which means that we cannot have $v(A) = v(B) = \mathbf{u}$.

Łukasiewicz logic. For Łukasiewicz logic, if $\varphi = (A \to B) \in \Gamma^*$, and $v(A) = v(B) = \mathbf{u}$, then $v(A \to B) = \mathbf{u} \to^L \mathbf{u} = \mathbf{t}$, so we have no problem in this case. However, we must search the proof of Theorem 1 for the places where we use the fact that $(\mathbf{u} \to \mathbf{u}) \neq \mathbf{t}$, and try to modify them to hold true for Łukasiewicz logic.

The only such place is the case when $\varphi = (A \to B) \in \Delta^*$. Then, as Σ^* is saturated, by rules (L), $(\Rightarrow\to I)$ and $(\Rightarrow\to II)$, we have

(i) either $A \in \Gamma^*$, or $\neg B \in \Gamma^*$

(ii) $\neg A, B \in \Delta^*$.

By the inductive assumption, this implies:

(i) either $v(A) = \mathbf{t}$ or $v(\neg B) = \mathbf{t}$ (whence $v(B) = \mathbf{f}$)

(ii) $v(\neg A) \neq \mathbf{t}$ (whence $v(A) \neq \mathbf{f}$), and $v(B) \neq \mathbf{t}$.

If in (i) we assume that $v(A) = \mathbf{t}$, then from the fact that $v(B) \neq \mathbf{t}$ and the table of implication for Łukasiewicz logic it clearly follows that $v(A \to B) \neq \mathbf{t}$. In turn, as $v(A) \neq \mathbf{f}$ by (ii), $v(B) = \mathbf{f}$ implies $v(A \to B) \neq \mathbf{t}$.

The above reasoning clearly shows that the completeness proof of the calculus **IRS** given in Theorem 1 can indeed be modified to yield completeness proof for both $\mathbf{IRS}^K$ and $\mathbf{IRS}^L$, which ends the proof of our theorem. ∎

It can be easily seen that the sequent calculus $\mathbf{IRS}^K$ is very similar to the known calculi for Kleene logic given in [24, 1, 6], while $\mathbf{IRS}^L$ is simpler than the previous sequent calculi for Łukasiewicz logics, given e.g. in [1, 5, 2]. Of course, since the hitherto existing calculi are also complete for those logics, then $\mathbf{IRS}^K$ and $\mathbf{IRS}^L$ must necessarily be equivalent to them. This equivalence can also be proved directly — but we leave that as an exercise to the reader.

8.2. Common Conservative Extension of the Three Logics

In the foregoing, we have shown how we can extend the complete sequent calculus for $\mathcal{L}^I_{RS}$ to obtain complete sequent calculi for Kleene and Łukasiewicz logics. Now we will show how to embed those logics in a natural extension of the above rough set logic by adding to its language a special unary operator well known in 3-valued logics, which is usually used to represent "definedness" [15] or "classicality" [11] or "normality" [7], but here denotes "crispness" in the sense of rough sets. Namely, let $\mathbf{C} : \mathcal{T} \to \mathcal{T}$ be given by:

$$\mathbf{C}(\mathbf{f}) = \mathbf{C}(\mathbf{t}) = \mathbf{t}, \quad \mathbf{C}(\mathbf{u}) = \mathbf{f}$$

and let $L^C_{RS}, \mathcal{M}^C_{RS}$ be the language L^I_{RS} and the Nmatrix $\mathcal{M}^I_{RS}$ extended with the connective $\mathbf{C}$ and its interpretation given above, respectively. Then it is easy to see that Kleene and Łukasiewicz implications can be defined in the logic L^C_{RS} generated by $\mathcal{M}^C_{RS}$ as follows:

$$A \to_L B \equiv (\neg \mathbf{C}A \to \mathbf{C}B) \to (A \to B)$$
$$A \to_K B \equiv (\neg A \to_L B) \to_L B$$

Accordingly, Kleene and Łukasiewicz logics can be treated as sublogics of the logic $\mathcal{L}^C_{RS}$.

To obtain a complete proof system for $\mathcal{L}^C_{RS}$, it suffices to extend the sequent calculus **IRS** we have developed for $\mathcal{L}^I_{RS}$ with the following rules for the connective **C**:

$(\Rightarrow \mathbf{C}\ I)\qquad \dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathbf{C}A} \qquad (\Rightarrow \mathbf{C}\ II)\qquad \dfrac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \mathbf{C}A}$

$(\mathbf{C} \Rightarrow)\qquad \dfrac{\Gamma, A \Rightarrow \Delta \quad \Gamma, \neg A \Rightarrow \Delta}{\Gamma, \mathbf{C}A \Rightarrow \Delta} \qquad (\Rightarrow \neg\mathbf{C})\qquad \dfrac{\Gamma, A \Rightarrow \Delta \quad \Gamma, \neg A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\mathbf{C}A}$

$(\neg\mathbf{C} \Rightarrow I)\qquad \dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg\mathbf{C}A \Rightarrow \Delta} \qquad (\neg\mathbf{C} \Rightarrow\ II)\qquad \dfrac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma, \neg\mathbf{C}A \Rightarrow \Delta}$

It is easy to see that the proof of Theorem 1 can be adapted in a straightforward way to incorporate **C**, thus providing a completeness proof for the above-mentioned sequent calculus for $\mathcal{L}^C_{RS}$.

9. Weak Logics

Having examined the strong version of the logic $\mathcal{L}^I_{RS}$ and its connections with well-known 3-valued logics, we shall now turn to the weak version of that logic. Denote by $\mathcal{M}^I_{RS,w}$ the Nmatrix differing from $\mathcal{M}^I_{RS}$ just by having $\{\mathbf{u}, \mathbf{t}\}$ as its set of designated values, and by $\mathcal{L}^I_{RS,w}$ the logic generated by that Nmatrix. Obviously, both the logics have the same sets of well-formed formulas, and both Nmatrices — the same sets of legal valuations.

If we denote

$$\overline{A} = \begin{cases} B & \text{if } A = \neg B, \\ \neg A & \text{otherwise} \end{cases}$$

then the duality between the strong logic $\mathcal{L}^I_{RS}$ and the weak logic $\mathcal{L}^I_{RS,w}$ mentioned in Section 3 can be expressed as follows:

Proposition 1. *For any valuation v legal under $\mathcal{M}^I_{RS}$ and $\mathcal{M}^I_{RS,w}$, and any sequent $\Gamma \Rightarrow \Delta$,*

$$v \models_w (\overline{\Delta} \Rightarrow \overline{\Gamma}) \text{ iff } v \models_s (\Gamma \Rightarrow \Delta) \qquad (11)$$

where the subscripts s, w denote satisfaction in $\mathcal{M}^I_{RS}$ and $\mathcal{M}^I_{RS,w}$, respectively, while $\overline{S} = \{\overline{A} \mid A \in S\}$ for any set of formulas S.

Proof. By Definition 4, we have $v \models_w (\overline{\Delta} \Rightarrow \overline{\Gamma})$ iff

$$\text{either } v(\overline{\delta}) = \mathbf{f} \text{ for some } \delta \in \Delta \text{ or } v(\overline{\gamma}) \neq \mathbf{f} \text{ for some } \gamma \in \Gamma \qquad (12)$$

Considering the truth table of negation and the definition of $\overline{A}$, we have:

$$v(\overline{\delta}) = \mathbf{f} \text{ iff } v(\delta) = \mathbf{t} \text{ and } v(\overline{\gamma}) \neq \mathbf{f} \text{ iff } v(\gamma) \neq \mathbf{t}$$

Hence (12) is equivalent to

$$\text{either } v(\gamma) \neq \mathbf{t} \text{ for some } \gamma \in \Gamma \text{ or } v(\delta) = \mathbf{t} \text{ for some } \delta \in \Delta$$

As the latter condition reduces to $v \models_s (\Gamma \Rightarrow \Delta)$, we have thus proved the desired equivalence (11). ■

The above proposition shows that any result for the strong logic $\mathcal{L}^I_{RS}$ can be translated to a corresponding result for the weak logic logic $\mathcal{L}^I_{RS,w}$. Thus it is easy to see that we can obtain a sound and complete sequent calculus for $\mathcal{L}^I_{RS,w}$ out of the calculus **IRS** for $\mathcal{L}^I_{RS}$ by replacing the rules $(\to\Rightarrow I), (\to\Rightarrow II)$ and $(\Rightarrow \neg \to)$ of **IRS** with the rules given below:

$$(\Rightarrow \neg \to I) \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \to B)}$$

$$(\Rightarrow \neg \to II) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(A \to B)}$$

$$(\to\Rightarrow) \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

The fact that the resulting system is sound and complete for $\mathcal{L}^I_{RS,w}$ can be either deduced from the duality principle and the soundness and completeness of **IRS**, or proved directly by appropriately modifying the proof of Theorem 1.

Using a similar method, we can also obtain a sound and complete sequent calculus for the logic $\mathcal{L}^C_{RS,w}$ (a weak version of the logic $\mathcal{L}^C_{RS}$, featuring the operator C), as well as for the two possible determinizations of $\mathcal{L}^I_{RS,w}$.

NOTE 3. This time, the two possible determinizations of the Nmatrix $\mathcal{M}^I_{RS,w}$ lead to two 3-valued *paraconsistent* logics[3]. Though Łukasiewicz implication itself has not been used in the literature on paraconsistent logics, the language of Łukasiewicz 3-valued logic is equivalent with regard to its expressive power to the language of the 3-valued paraconsistent logic J_3 [10, 11], which is the strongest logic considered by the Brazilian school of paraconsistent logics [8, 7, 9]. Accordingly, the logic we obtain from the "Łukasiewicz" determinization of $\mathcal{M}^I_{RS,w}$ is equivalent to J_3, and the sequent calculus we obtain for it is a new cut-free, sound and complete proof system for J_3.

[3] A paraconsistent logic is a logic where a single contradiction does not imply every formula.

In turn, the "Kleene" determinization leads to the $\{\neg, \vee, \wedge\}$-fragment of J_3, also known as *Pac* [1] (and the sequent calculus we obtain for it is the well-known one [1]).

10. Conclusions and Further Research

We have shown that the use of non-deterministic matrices is helpful in describing rough sets using 3-valued logics. What is more, it has turned out that the strong version of the propositional logic generated by the non-deterministic matrix underlying the basic rough set logic has close relations to Kleene and Łukasiewicz 3-valued logics, while the weak version of that logic is closely related to two major 3-valued paraconsistent logics.

The rough set logic itself that we have considered here is very simple, so the natural next step is to consider a more complex language for describing rough sets. The first extension of the language can consist in treating the membership formulas of the form Ax used here as atomic ones, and building over them a richer language using suitably chosen, three-valued logical connectives. In addition, we could also use some kind of a two-valued operator(s) (like e.g. the definedness-crispness operator $\mathbf{C}$ employed here, or the J_k operators of Rosser-Turquette logic, acting as selectors of the individual logical values) to translate the formulas of the many-valued rough set logic to two-valued ones, and then combine them using the classical logical connectives. In this way, we could obtain a two-level logic of the type considered in [13], combining the expressive power of 3-valued and classical logics.

A major extension in another direction would be to consider formulas representing membership in rough sets according to more than just one equivalence relation in the knowledge base. Then we could obtain a 3-valued analogue of the two-valued, many-modal logic developed in [14].

Needless to say, the rich body of research in rough set theory suggests further possible lines of research in rough set logics, like e.g. rough relations, dependence on attributes, reducts, etc.

Finally, all the results obtained here or in any further extensions suggested above can be presented in the framework of rough sets based on information systems rather than knowledge bases (in the sense of Definition 1), which is more suitable for many practical applications.

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