Topics in Extremal Combinatorics (0366.4996)- Fall ’21

Instructor: Asaf Shapira

Home Assignment 1

Due date: 2/11/21

Please submit organized and well written solutions!

Problem 1. Show that for every $0 < \alpha, \beta, \gamma < 1$ and $\delta > 0$ there is $\epsilon$ so that if $K$ is a 3-partite graph on vertex sets $A, B, C$ and $d(A, B) = \alpha$, $d(B, C) = \beta$, $d(A, C) = \gamma$ and $(A, C)$ and $(B, C)$ are $\epsilon$ regular, then $K$ contains $|A||B||C|(\alpha \beta \gamma \pm \delta)$ copies of $K_3$.

Problem 2. Suppose $H$ is an $n$-vertex graph of maximum degree $\Delta$. We proved in class that if $G$ has $4n$ vertices and density at least $1 - \frac{1}{8\Delta}$ then $G$ contains a copy of $H$. We also proved that if $G$ has $C(\delta, \Delta) \cdot n$ vertices and $G$ is bi-($c(\delta, \Delta), \delta$)-dense then $G$ has a copy of $H$ (for an appropriate $c(\delta, \Delta)$). Do the proofs of these two results work also when $H$ is only assumed to be $\Delta$-degenerate?

Problem 3. Show that $r(5, n) \leq Cn^4/\log^3 n$ and that there is an $n$-vertex $K_4$-free graph $G$ satisfying $\alpha(G) \leq cn^{2/5} \log n$.

Problem 4. Prove the following “embedding lemmas” in which $G$ is a bipartite graph with vertex sets $X, Y$ (Hint: Just do it!).

- Suppose every collection of $r$ vertices in $X$ have at least $b$ common neighbors in $Y$. Then $G$ contains a copy of every bipartite graph with vertex sets $A, B$ satisfying $|A| < |X|, |B| \leq b$ and where every vertex in $B$ has at most $r$ vertices in $A$.

- Suppose every collection of $r$ vertices in $X$ have at least $n$ common neighbors in $Y$, and every collection of $r$ vertices in $Y$ have at least $n$ common neighbors in $X$. Then $G$ contains a copy of every $n$-vertex $r$-degenerate bipartite graph.