Topics in Extremal Combinatorics (0366.4996)- Fall ’21

Instructor: Asaf Shapira

Home Assignment 2

Due date: 7/12/21

Please submit organized and well written solutions!

Problem 1. Let $G$ be an $n$-vertex graph of average degree $t$. Suppose we construct an independent set by repeatedly (i) removing a vertex $v$ of minimum degree (ii) adding $v$ to the independent set (iii) removing $v$’s neighbors. Show that we are guaranteed to get an independent set of size at least $n/(1 + t)$.

Problem 2. Let $T$ be a $k$-partite graph where each of the classes $V_1, \ldots, V_k$ is of size $t^k - 1$.

- Suppose none of the bipartite graphs $(V_1, V_i)$ ($i > 1$) contains an empty $t \times t$ bipartite graph. Conclude that some vertex $v \in V_1$ is adjacent to at least $t^{k-2}$ vertices in each of the sets $V_2, \ldots, V_k$.
- Conclude that if $T$ has no copy of $K_k$ then it contains an empty $t \times t$ bipartite graph.
- Conclude that for any $H$ on $k$ vertices, if $T$ has no induced copy of $H$ with one vertex in each class $V_i$, then it contains either an empty $t \times t$ bipartite graph or a complete one, and hence that every $n$-vertex induced $H$-free graph contains either an empty or complete bipartite graph of size at least $(n/k)^{1-1/k}$. 

Problem 3. Show that if $H$ is a cograph then every $n$-vertex graph without an induced copy of $H$ has a homogenous set of size $n^\epsilon$ for some $\epsilon = \epsilon(H) > 0$.

Problem 4. Show that for every graph $H$ there is a graph $G$ so that every 2-coloring of $G$ contains an induced monochromatic copy of $H$. By this we mean that there is a set $S$ of $|V(H)|$ vertices so that if we keep only the edges inside $S$ of one of the colors, we will get an induced copy of $H$. Note that this is weaker than what is usually referred to as the Induced Ramsey Theorem.

**Hint:** Use the Erdős-Hajnal Theorem together with Erdős's lower bound for Ramsey’s theorem.