Dynamics on homogeneous spaces \ Barak Weiss

General

Webpage for this course http://www.math.tau.ac.il/~barakw/homogeneous_spaces

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Recommended books:

(1) Hassellblatt and Katok - good introduction but not specifically relevant for our course
(2) Bekka and Meyer - basic recommended introduction.
(3) Einsidler and Ward, Part I - Introduction to ergodic theory with a view tward number theory - very highly recommended
(4) Einsiedler, Ward and Lindenstrauss - a work in progress. You should send an email to receive a draft.

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part 1. In dynamics we consider a space X with a map $T : X \to X$, or a one parameter flow $t \in \mathbb{R}$, $\phi_t : X \to X$ such that $\phi_{t+s} = \phi_s \circ \phi_t$. The setting include many problems from physics (examples...). Mathematicians used two frameworks to answer questions about such a system:

Topological Dynamics - X a compact metrizable Hausdorff topological space.$T : X \to X$ a homeomorphism.

Ergodic Theory - $(X, \mathcal{B}, \mu)$ a measure space. $T : X \to X$ a measurable, measure preserving map, i.e., for any $A \in \mathcal{B}$, $\mu\left(T^{-1}A\right) = \mu(A)$. (the reason we're taking the inverse image is that the intersection of inverse image is the inverse image of the intersection).

We are going to be interested in situations that allows very complicated dynamics.

add picture

Lets have a taste of one fundamental result in each of these settings:

Theorem 1 (Poincare's recurrence theorem). *$(X, \mathcal{B}, \mu)$ a probability space. $T : X \to X$ measurable, measure preserving. Then*

$$\forall A \in \mathcal{B} \quad \mu(A) > 0 \quad \exists n \geq 1 \quad T^{-n}(A) \cap A \neq \varnothing.$$

Proof. $T^{-n}(A) \cap A = \varnothing$ for all n. Suppose $j > i \geq 1$. Then

$$T^{-j} \cap T^{-i}(A) = T^{-i}\left(T^{-(j-i)}(A) \cap A\right) = \varnothing.$$

Therefore the sets $T^{-i}(A)$, $i = 1, 2, ...$ are pairwise disjoint. Now

$$1 \geq \mu(X) \geq \mu\left(\bigcup_{i=1}^{\infty} T^{-i}(A)\right) = \sum_{i=1}^{\infty} \mu\left(T^{-i}(A)\right) = \sum_{i=1}^{\infty} \mu(A) \Longrightarrow \mu(A) = 0$$

is a contradiction. □

Theorem 2 (Folklore). *If X is compact metrizable, $T : X \to X$ a homeomorphism, then there exists $x_0 \in X$ and $n_k \to \infty$ such that $T^{n_k}(x_0) \to x_0$. (x_0 is called a recurrent point).*

Before proving the theorem we introduce a definition and a Lemma.

Definition 3. $X_0 \subset X$ is called minimal if:

(1) X_0 is closed.
(2) $T(X_0) \subset X_0$
(3) For any $Y \subset X_0$ satisfying (1) and (2), $Y = \varnothing$ or $Y = X_0$.

Lemma 4. *Any (X, T) as in Theorem 2 contains a minimal set.*

Proof. (of the Lemma) Denote $\mathcal{K} = \{X_0 \subset X$ satisfying (1) and (2) and $X_0 \neq \varnothing\}$ and order $\mathcal{K}$ by reverse inclusion, i.e.

$$X_1 \succ X_2 \Longleftrightarrow X_1 \subset X_2.$$

$X_0 \in \mathcal{K}$ will satisfy (3) $\Longleftrightarrow$ it is minimal element w.r.t $\succ$. Using Zorn Lemma we're done. □

Proof. (of Theorem 2) Suppose $X_0 \subset X$ minimal, $x_0 \in X_0$. Define

$$Y = \left\{y : \exists n_k \to \infty, T^{n_k} x_0 \to y\right\}.$$

Due to (1), $Y \subset X_0$. X is compact and therefore Y is closed. Y is not empty and therefore by minimality $Y = X_0 \ni x_0$. (Note we got $Y = \overline{\{T^n x_0 : n \in \mathbb{N}\}}$). □

What we realy want to do in science is to predict the future. There is a basic idea in many sciences that one should look on equilibrium states. What we find in dynamics is that there are very complicated equilibrium states. the main objective will be to find/classify all invariant measure. In the case of a contracting fixed point system, the only invariant measure is Dirac's δ measure concentrated in the fixed point. Indeed, by Poincare's reccurence theorem, any positive measure set would have to intersect itself - but it doesn't (if not containing the fixed point).

add Picture.

The idea of ergodic theory is to reduce the study of invariant measures to the study of ergodic invariant measures. A consequence of the ergodic decomposition theorem and the pointwise ergodic theorem is that "if we understand all invariant measures we will understand the asymptotics of all orbits". For examples, sets which assume zero measure under all the ergodic invariant measures - no orbit will eventually spend time in them. The parallel part of ergodic measures in the topolgical dynamics setting is played by minimal sets.s

Part 2. A homogenuous space is a coset space of Lie groups: G is a Lie group. $H \subset G$ is a closed subgroup. $X = G/H = \{gH : g \in G\}$. G acts on X via $(g_1 g_2)H = g_1(g_2 H)$.

A Lie group is a manifold with group laws which are smooth. By smoothness we mean partial derivatives of all orders exists. A manifold is something that locally looks like $\mathbb{R}^n$. The two group operations which are supposed to be smooth are multiplication and inverse:

$$\begin{aligned} G \times G &\to G \\ (g,h) &\mapsto gh \end{aligned}$$

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

Example 5. A few examples of Lie groups:

(1) $(\mathbb{R}, +)$. The map

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto x+y \end{aligned}$$

is obviously smooth.

(2) $\mathbb{S}^1 = \left\{e^{2\pi i\theta} : \theta \in \mathbb{R}\right\}$. Multiplication as complex numbers (the same as addition mod 1). Understand the meaning of smoothness by introducing an atlas on $\mathbb{S}^1$.

(3) Matrix groups, e.g.

$$\begin{aligned} GL_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : det(A) \neq 0\}, \\ SL_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : det(A) = 1\} \end{aligned}$$

with group laws as matrices.

Remark 6. Similar definition, with “variety” instead of “manifold” and “algebraic” instead of “smooth” defines algebraic variety.

We want to put a topology on G/H.

Fact 7. *(will be discussed later) G/H carries the structure of a manifold. The map*

$$\begin{aligned} \pi : G &\to G/H \\ \pi(g) &= gH \end{aligned}$$

is smooth and hence so is the map

$$\begin{aligned} G \times G/H &\to G/H \\ (g_1, g_2 H) &\mapsto (g_1 g_2)H \end{aligned}$$

Definition 8. An action of a group G on a set X is a map

$$\begin{aligned} G \times X &\to X \\ (g,x) &\mapsto gx \end{aligned}$$

that satisfies:

(1) $(g_1 g_2)x = g_1(g_2 x)$. The relevant diagram is commutative.
(2) $ex = x$ for all $x \in X$.

If G is a Lie group and X is a manifold, we can speak of a *Lie group action*, i.e. an action for which $G \times X \to X$ is a smooth map of manifolds. G acts on $X = G/H$ via $g_1(g_2H) = (g_1g_2)H$. The action is well defined: If $g_2H = g_3H$ then there exists $h \in H$ such that

$$g_2 = g_3h \Rightarrow g_1g_2 = g_1g_3h \Rightarrow (g_1g_2)H = (g_1g_3h)H = (g_1g_3)H.$$

Remark. Our definition is for right action. There is point in talking about left actions.

Example 9. If G acts transitively on X (i.e., $\forall x_1, x_2 \in X \quad \exists g \in G$ such that $gx_1 = x_2$) then X is in G-equivariant (see Definition11 below) bijection with G/H where $H = \{g \in G : gx_0 = x_0\}$ for some $x_0 \in X$. Why? Choose $x_0 \in X$, define $H = G_{x_0} = \{g \in G : gx_0 = x_0\}$ (this subgroup of G is called the stablizer of x_0), and a map

$$\begin{array}{rcl} F : G/H & \to & X \\ gH & \mapsto & gx_0 \end{array}$$

Onto: If $x \in X$, $\exists g \in G$ such that $gx_0 = x$ (by transitivity), so $gH \mapsto x$.

Injective: If $g_1H,\ g_2H \mapsto x$ then $x = g_1x_0 = g_2x_0 \Rightarrow g_1^{-1}x = x_0 = g_1^{-1}g_2x_0 \Rightarrow g_1^{-1}g_2 \in H \Rightarrow g_1H = g_2H$ (same argument for well defined).

Exercise 10. Check that the orbit map F is equivariant.

Definition 11. G acts on $X_1,\ X_2$. A map $F : X_1 \to X_2$ is called $G-$ *equiinvariant* if $\forall g \in G \quad \forall x \in X \quad gF(x) = F(gx)$.

Add diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_2 \\ g\downarrow & \circlearrowright & \downarrow g \\ X_1 & \xrightarrow{F} & X_2 \end{array}$$

Example 12. Some examples of homogenuous spaces:

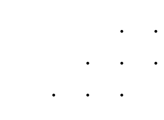
(1) $G = SL_n(\mathbb{R})$, $\Gamma = H = SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : det(A) = 1\}$ (denoted by gamma for historical reasons). To describe G/Γ, it's enough to find an action of G on a space X which is transitive, and for which Γ is the stabilizer of a point.

$$\begin{array}{rcl} X & = & \{\text{lattices in } \mathbb{R}^n \text{of covolume } 1\} \\ & = & \{\mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_n : v_1, ..., v_n \text{ linearily independent with } det(v_1, ..., v_n) = 1\} \end{array}$$

. G acts by

$$g(\mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_n) = \mathbb{Z}gv_1 \oplus ... \oplus \mathbb{Z}gv_n$$

E.g. if $n = 2$, any point of X is a subset of $\mathbb{R}^2$ that looks like



This space will play important role in this course, and infact, most of the difficulties in understanding the dynamics of in the general setting of homogenuous spaces, appear already for this example.

(2) $G = SL_2(\mathbb{R})$, $X = \mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. There is a metric on $\mathbb{H}$ such that $SL_n(\mathbb{R})$ is the group of isometries of $\mathbb{H}$, acting by Mobius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

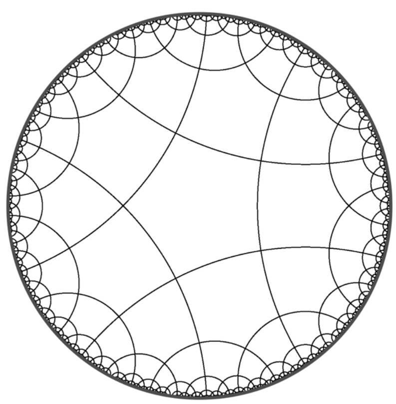
EXERCISE 13. This actions is transitive (appeared in diophantine approximation course last semester)

Therefore $\mathbb{H} = G/(\text{stabilizer of a point})$ (the stabilizer of i is $SO(2,\mathbb{R})$).

(3) $1 \le k \le n$, $Gr_{k,n}(\mathbb{R}) = \{k\text{-dimensional linear subspaces of } \mathbb{R}^n\}$ (Grassmannian). $G = SL_n(\mathbb{R})$ acts on $Gr_{k,n}(\mathbb{R})$. Let $L_0 = span(e_1, ..., e_k)$ where $e_1, ..., e_n$ is the standard basis of $\mathbb{R}^n$.

$$Stab(L_0) = \left\{ \begin{pmatrix} * & * \\ \underbrace{0}_{n-k \times k} & * \end{pmatrix} \right\}$$

Picture of the disc model of the hyperbolic plane tesselated by copies of isometric hyperbolic pentagons. The reason you can do it is because the angles are bigger then what you expect (90 degrees. One actually needs only four pentagons to cover a vertex).



Put a point in the center of each pentagon. There is a famous problem of counting how many points there will be (asymptotically) in a big disc. The Euclidean intuition says that the number of pentagons should be asymptotic to the area of the disc divided by the area of the pentagon. But, looking at the picture of the hyperbolic disc, we see that it is not at all clear - most of the pentagons intersecting the disc are near the boundary of the disc. It is a theorem Margulis proved in his thesis:

THEOREM 14 (Margulis). $N(R) \sim \frac{area(B(v_0,R))}{area(pentagon)}$.

To prove it Margulis looked at $\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\} \curvearrowright SL_2(\mathbb{R})/\Gamma$. His method applies to a much more general setting, but only recently more delicate estimations appeared and it is still a very active research front) (complete open problem...).

21.10

Part 1. We will describe few other applications of the method of proving by translating the problem to a problem about dynamics on homogeneous spaces.

Theorem 15 (Oppenheim conjecture '29 for n>4, '50 n>2. Proved by Margulis '86). *Let Q be an indefinite nondegenerate quadratic form in $n \geq 3$ variables. Then either Q is a multiple of a rational form, or $Q(\mathbb{Z}^n)$ is dense.*

Definition 16. $Q(v) = L(v,v) = \sum_{1\leq i,j\leq n} a_{ij}v_iv_j$ where $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $v = (v_1, ..., v_n)$.

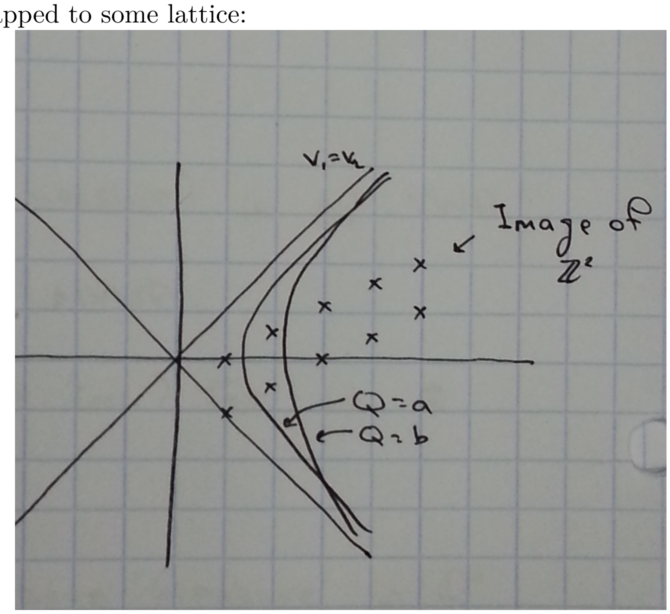
Any form can be put in the following shape by a change of variables

$$Q(v) = v_1^2 + ... + v_p^2 - v_{p-1}^2 - ... - v_{p+q}^2$$

where $1 \leq p \leq p+q \leq n$ and (p,q) is called the signature of the form.

Definition 17. A form is nondegenarate $\iff p+q=n$. A form is indefinite $\iff p \neq 0$ and $q \neq 0$.

Geometric interpratation (take $n=2$) for $Q(v_1,v_2) = v_1^2 - v_2^2 = (v_1+v_2)(v_1-v_2)$. After a change of variables, the level sets of Q are hyperbolas, and $\mathbb{Z}^2$ is mapped to some lattice:



Q is a multiple of a rational form $\iff \exists\lambda \neq 0$ s.t. $\lambda a_{ij} \in \mathbb{Z} \quad \forall i,j$. In this case $Q(\mathbb{Z}^n) \in \lambda^{-1}\mathbb{Z}$, so not dense.

The theorem is false for $n=2$: Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\alpha > 0$, be badly approximable, and assume $\alpha^2 \notin \mathbb{Q}$. $\exists C > 0$ such that all $p,q \in \mathbb{Z},\ q \neq 0$ satisfy

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

Now consider the form $Q(x,y) = x^2 - \alpha^2y^2$. For all $p,q \in \mathbb{Z}$ we have, if $q \neq 0$

$$\left|p^2 - \alpha^2q^2\right| = q^2\left|\frac{p^2}{q^2} - \alpha^2\right| = q^2\left|\frac{p}{q} - \alpha\right|\left|\frac{p}{q} + \alpha\right| \geq q^2\left|\frac{p}{q} - \alpha\right|\alpha \geq q^2\frac{C}{q^2}\alpha = C\alpha > 0,$$

wlog $\frac{p}{q}$ is close to α than $-\alpha$ so $\left|\frac{p}{q}-\alpha\right| \geq \left|\frac{-\alpha-\alpha}{2}\right| = \alpha$. If $q=0$ than $Q(p,q) = p^2 \geq 1$.

(Complete reduction to Ratner's theorem...)

Definition 18. $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is called VWA (very well approximable) if $\exists \varepsilon > 0$ such that there ∞-many solutions $p \in \mathbb{Z}^d$, $0 \neq q \in \mathbb{Z}$ to

$$\left\| x - \frac{1}{q} p \right\| \leq \frac{1}{q^{1+\frac{1}{d}+\varepsilon}}.$$

This is usually not the case.

Theorem 19 (Dirichlet). *$\forall x \in \mathbb{R}^d$, for all $Q > 0$ there are solutions to*

$$\begin{aligned} \left\| x - \frac{1}{q} p \right\| &\leq \frac{1}{qQ}, \\ q &< Q^d \end{aligned}$$

In particular, if $x \notin \mathbb{Q}^d$, there are ∞-many solutions to

$$\left\| x - \frac{1}{q} p \right\| \leq \frac{1}{q^{1+\frac{1}{d}}}.$$

Definition 20. $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is called MVWA (multiplicatively very well approximable) if $\exists \varepsilon > 0$ such that there are ∞-many solutions $p \in \mathbb{Z}^d$, $0 \neq q \in \mathbb{Z}$ to

$$\prod_{i=1}^{d} |q x_i - p_i| \leq \frac{1}{q^{1+\varepsilon}}.$$

Claim. VWA $\Longrightarrow$ MVWA

Proof. Note that we didn't define the norm in the definition of VWA, but it doesn't matter because the constant will be digested by the exponent. Using the supremum norm:

$$\begin{aligned} \max_i \left| x_i - \frac{p_i}{q} \right| &\leq \frac{1}{q^{1+\frac{1}{d}+\varepsilon}} \Rightarrow \\ \max_i |q x_i - p_i| &\leq \frac{1}{q^{\frac{1}{d}+\varepsilon}} \end{aligned}$$

Therefore, multiplying all coordinates we get

$$\prod_{i=1}^{d} |q x_i - p_i| \leq \left(\frac{1}{q^{\frac{1}{d}+\varepsilon}} \right)^d \leq \frac{1}{q^{1+d\varepsilon}}.$$

□

Theorem 21 (Kchinchine, 30's). *A.e. x (w.r.t. Lesbesgue measure) is not VWA.*

One can show more: a.e. x is not $MVWA$.

Conjecture 22 (Mahler, proved by Sprindzhuk). *For a.e. $(x, x^2, ..., x^d)$ is not VWA. (i.e., in Kchinchine's theorem, replace Lebesgue measure on $\mathbb{R}^d$ with the length measure on the curve $\{(x, x^2, ..., x^d) : x \in \mathbb{R}^d\}$).*

Problem 23.
1. Can we replace the curve$\left\{(x,x^2,...,x^d):x\in\mathbb{R}^d\right\}$ by other manifolds?
2. Can we replace VWA by MVWA?

Sprindzhuk and Baker proved that "yes" for both problems.

Theorem 24 (Kleinbock-Margulis). *Let $\mathcal{C}$ be an analytic nondegenerate curve (analytic: $\mathcal{C}=(c_1(t),...,c_d(t))$, $c_i(t)$ is an analytic function. nondegenerate: not contained in a proper affine subspace). Then a.e. $x\in\mathcal{C}$ is not MVWA.*

Definition 25. Let $0<\varepsilon\leq 1$. A vector $x\in\mathbb{R}^d$ is called ε-Dirichlet-improvable ($DI(\varepsilon)$) if $\exists Q_0\ \forall Q>Q_0$ there exists a solution to:

$$\begin{aligned} \left\Vert x-\frac{1}{q}p\right\Vert_\infty &\leq \frac{\varepsilon}{qQ}, \\ q &< \varepsilon Q^d \end{aligned}$$

Theorem 26 (Davenport and Schmidt, 60's). *If $\varepsilon<1$ then the Lebesgue measure of $DI(\varepsilon)$ vectors is zero.*

Conjecture 27 (Baker and Schmidt, 60's, proved by Shah 2010). *If $\varepsilon<1$, then for any nondegenarate analytic curve $\mathcal{C}$, the measure of the set of $DI(\varepsilon)$ w.r.t the length measure is zero.*

All the theorems were proved using the dynamics of

$$\left\{g_t=\begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt}\end{pmatrix}\right\}\curvearrowright {}^{SL_{d+1}(\mathbb{R})}\big/_{SL_{d+1}(\mathbb{Z})}.$$

Part 2. Fix the following notation: $<x>=\operatorname{dist}(x,\mathbb{Z})=\min_{k\in\mathbb{Z}}|x-k|$.

Conjecture 28 (Littlewood 30's). $\forall\alpha,\beta\in\mathbb{R}$

$$\liminf_{n\to\infty} n<n\alpha><n\beta>=0. \tag{1}$$

Remark. α is not $BA\Longleftrightarrow\liminf_{n\to\infty}n<n\alpha>=0$, and therefore$\liminf_{n\to\infty}n<n\alpha><n\beta>=0$.

Theorem 29 (Einsiedler-Katok-Lindenstrauss). $\dim\left(\{(\alpha,\beta):1\ \textit{fails}\ \}\right)=0$

They proved it by using

$$\begin{pmatrix} \star & & \\ & \star & \\ & & \star\end{pmatrix}\curvearrowright {}^{SL_3(\mathbb{R})}\big/_{SL_3(\mathbb{Z})}.$$

Fix $N\in\mathbb{N}$ and consider $\{\sqrt{n}\bmod 1:n=1,...,N\}=\{t_1<...<t_N\}$. Define $J(N)=\{t_{n+1}-t_n:n=1,...,N-1\}$:

Theorem 30 (Elkies-McMullen 99). *There exists analytic $F_1,F_2:\mathbb{R}\to\mathbb{R}$ (given explicitly) such that if we define*

$$F(t)=\begin{cases}\frac{6}{\pi^2} & t\in[0,\frac{1}{2}]\\ F_1(t) & t\in[\frac{1}{2},2]\\ F_2(t) & t\geq 2\end{cases}$$

Then $\frac{\#\{J\in J(N):NJ\in[a,b]\}}{N}\longrightarrow\int_a^b F(t)dt$

REMARK 31. An average gap has length $\frac{1}{N}$ (complete picture...)

(1) If instead of looking at $\{\sqrt{n} \bmod 1\}$ we looked at random points (independent, distributed according to Lebesgue) then almost surely, a similar result holds, with $F(t)$ replaced by e^{-t}.
(2) The "non-standard" frequency for $\{\sqrt{n} \bmod 1\}$ was discovered by numerical experiments (Boshernitzan). Numerically, for all $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, the gap frequencies for $\{n^\alpha \bmod 1\}$ seem to behave like those random sequences, i.e. according to density e^{-t}.
(3) The proof is using

$$\begin{pmatrix} e^t & & \\ & e^{-t} & \\ & & 1 \end{pmatrix} \curvearrowright {}^{SL_2(\mathbb{R})\ltimes\mathbb{R}^2}\!/\!_{SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2}.$$

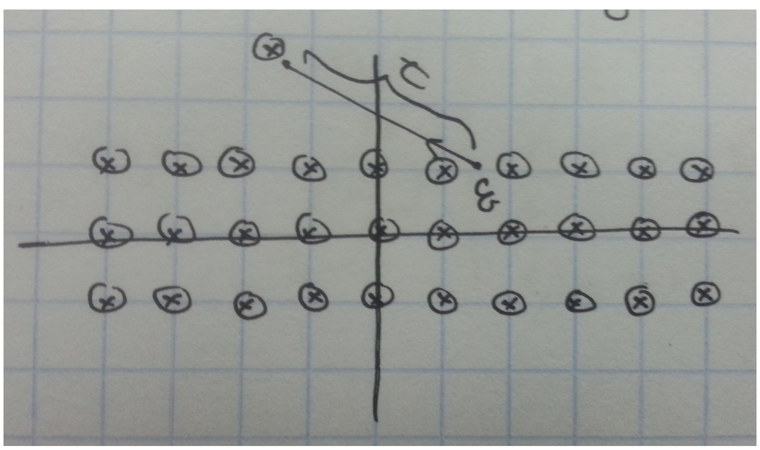
THEOREM 32 (Marklof and Strombergssonn, Visibility in a forest). *$\Lambda \subset \mathbb{R}^d$ is a lattice, $\rho > 0$, $K_\rho = \mathbb{R}^d \setminus (\Lambda + B_\rho(0))$. If $q \in K_\rho$, $v \in S^{d-1}$,*

$$\tau(q, v; \rho) = \inf\{t > 0 : q + tv \notin K_\rho\}.$$

Fix $q \in \mathbb{R}^d \setminus \Lambda$. Then $\exists \Phi = \Phi_{\Lambda,q} : (0,\infty) \to (0,\infty)$, $\int_0^\infty \Phi dt = 1$ such that

$$\lim_{\rho\to\infty} \lambda\left(\left\{v \in S^{d-1} : \tau(q, v; \rho) \geq \frac{\xi}{\rho^{d-1}}\right\}\right) = \int_\xi^\infty \Phi(t)dt.$$

Moreover, $\exists \Phi_0$ such that for all Λ, a.e. q, $\Phi_0 = \Phi_{\Lambda,q}$



The theorem was proved using the homogeneous dynamical system

$$\begin{pmatrix} \star & & & \\ & \ddots & & \\ & & \star & \\ & & & 1 \end{pmatrix} \curvearrowright {}^{SL_d(\mathbb{R})\ltimes\mathbb{R}^d}\!/\!_{SL_d(\mathbb{Z})\ltimes\mathbb{Z}^d}.$$

REMARK 33. For the Poisson forest, the function is different.

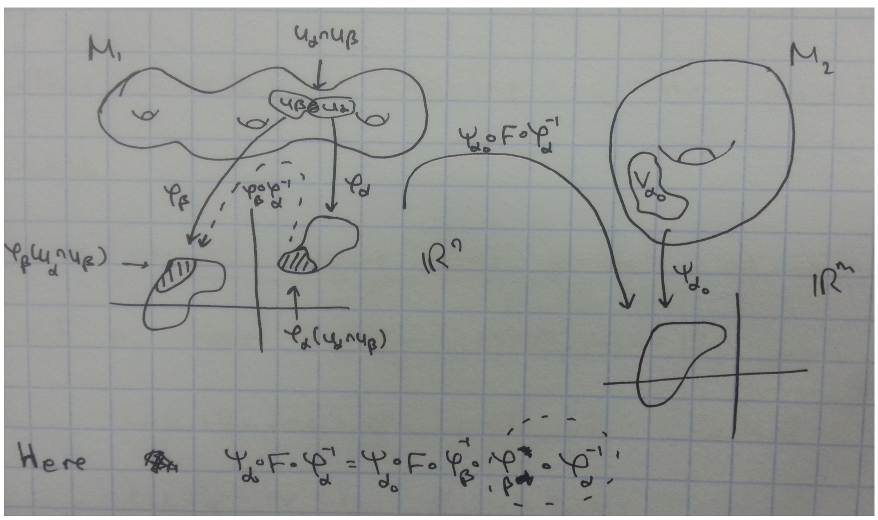
Part 3 - Crash course on manifolds. In all the examples we saw a space G/Γ. We will describe a general context where it is possible to take this factor and get good properties. In our cases usually Γ is discrete and it is a little easier then the general case.

Real algebraic groups$\subset$ Lie groups $\subset$Locally compact second countable (lcsc) topological groups.

It is not hard to define a Lie group which is not real algebraic. Nevertheless, one of the big achievements of the 20th century mathematics (Weil, Chevaly, Killing, Cartan...) was the classification of Lie groups. One of the outcomes of it, is a proof that any semisimple Lie group have models as real algebraic groups! In the last context, it is possible to construct the Haar measure.

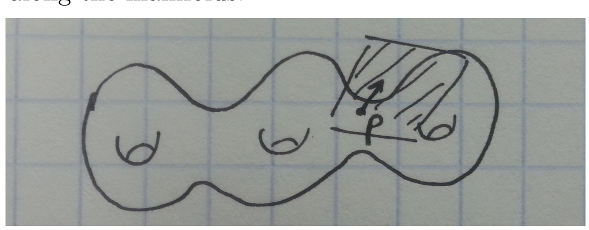
Definition 34. An n-dim manifold M is a separable topological space, with an atlas of charts, i.e. $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ such that

(1) Each $U_\alpha$is open, $M = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$
(2) $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image, which is homeomorphic to a ball.
(3) $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is smooth for any α, β. This is called the transition maps.



Remark 35. In this course and in many other cases, smooth will mean C^∞. One can take continuous, C^k, analytic.

At each point $p \in M$ there is an n-dimensional vector space T_pM, called the tangent space to M at p. The tangent space should tell us something about infinitesimal motions along the manifolds.



We give two definitions:

(1) Geometric - Derivatives of smooth paths: Define an equivalence relation on paths $\gamma : (-\varepsilon, \varepsilon) \to M,\ \gamma(0) = p$ by $\gamma_1 \sim \gamma_2$ if for any (some) α such

that $p \in U_\alpha$,

$$\frac{d}{ds}\bigg|_{s=0} \varphi_\alpha \circ \gamma_1 = \frac{d}{ds}\bigg|_{s=0} \varphi_\alpha \circ \gamma_2.$$

Then $T_p(M) = \{\text{equivalence classes}\}$.

(2) Algebraic: If $f : M \to \mathbb{R}$ and $\gamma : (-\varepsilon, \varepsilon) \to M,\ \gamma(0) = p$ then we have a map: "directional derivative of f at p along γ"

$$\frac{d}{ds}\bigg|_{s=0} f \circ \gamma.$$

Well defined (depends only on equivalence class of γ. It is a derivation: $D : C^\infty(M) \to \mathbb{R}$ linear such that $D(fg) = D(f)g(p) + f(p)D(g)$. The algebraic definition of $T_p(M)$ is just the space of all derivations on $C^\infty(M)$

Theorem 36. *$\gamma \mapsto \frac{d}{ds}\big|_{s=0} f \circ \gamma$ is a bijection. The dimension of the resulting vector space is n.*

Definition 37. (The tangent bundle of M) $T(M) = \{(p, v) : p \in M,\ v \in T_p(M)\} = \coprod_{p\in M} T_p(M)$.

This is the definition as a set.

Theorem 38. *$T(M)$ is a manifold of dimnesion $2n$.*

Proof. (Idea) If (U, α) is a coordinate chart on M,

$$(U_\alpha \times \mathbb{R}^n, (\varphi_\alpha, \text{derivative using } \varphi_\alpha))$$

is a coordinate chart on $T(M)$. □

Definition 39. A vector field on M is a smooth map $X : M \to T(M)$ such that $X(p) = (p, \cdot)$. In other words, if $\pi : T(M) \to M$ is the natural projection, then $\pi \circ X = id$.

Definition 40. Riemannian metric is a smooth varying inner product on each $T_p(M)$. I.e., for each p, we have $(\cdot, \cdot)_p : T_p(M) \times T_p(M) \to \mathbb{R}$ an inner product, such that if X, Y are any two vector fields on M, then

$$p \mapsto (X(p), Y(p))$$

is smooth.

A manifold M, with a fixed Riemannian metric, is called a Riemannian manifold.

Definition 41. Given smooth curve $\gamma : (a, b) \to M$, the length of γ is

$$\int_a^b \|\gamma'(t)\|_p \, dt = \int_a^b (\gamma'(t), \gamma'(t))_{\gamma(t)}^{\frac{1}{2}} \, dt.$$

(This infact only dependens the image of the curve and not the parametrization).

Using this we can define a metric

Theorem 42. *On a riemannian manifold,*

$$d(p, q) = \inf\ \ell(\gamma)$$

where the infimum is over all smooth curves $\gamma : (a, b) \to M$ from p to q and $\ell(\gamma)$ is the length of γ is a metric.

28.10

Part 1. $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$, $p \in U$.

$$D_p f : \mathbb{R}^n \to \mathbb{R}^m$$

is the linear map that best approximate f, in the sense that

$$f(p+h) - f(p) = D_p f(h) + o(\|h\|)$$

(if there is no such linear map we say that f is not differtiable). In a more concrete way, if the partial derivatives exist and smooth then this is equal to

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i=1,\ldots,m,j=1,\ldots,n}.$$

Theorem 43. *(Chain rule) given* $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^k$, *we have*

$$D_p(g \circ f) = D_{f(p)} g D_p f$$

For manifolds, assume $f : X \to Y$ is a map between manifolds, if $p \in X$ then $D_p f : T_p(X) \to T_{f(p)}(Y)$ is defined by $[\gamma] \mapsto [f \circ \gamma]$.

Definition 44. If X is a manifold, $\{U_\alpha\}$ is a cover by open sets, then partition of unity subordinate to $\{U_\alpha\}$ is a collection $\psi_\alpha : U_\alpha \to \mathbb{R}$ of smooth functions, such that $\psi_\alpha(x) \in [0,1]$ for all $x \in X$, $\psi_\alpha|_{X \setminus U_\alpha} = 0$, $\sum_\alpha \psi_\alpha(x) = 1$.

Theorem 45. *Partitions of unity exist.*

A volume element on a Riemannian manifold. Motivation: On $\mathbb{R}^n$, we have the Riemann integral. For $f : U \to \mathbb{R}$, U open in $\mathbb{R}^n$, there exists a change of variable formula: given $g : V \to U$ which is one-to-one,

$$\int_{g(V)} f(z)dz = \int_V f(g(x)) \left|\mathrm{Jac}(g)(x)\right| dx.$$

Other notation: $dx = dx_1 dx_2 ... dx_n$. Note that we have for example $dx_1 dx_2 = dx_2 dx_1$ (this is a consequence of the Fubini's theorem) (remark about that applying Fubini's is only possible when the limits of the integrals are in the right order).

Let g be a Riemannian metric on X, $f : X \to \mathbb{R}$ smooth, $\{\psi_\alpha\}$ is a partition of unity. $f(x) = \sum_\alpha \psi_\alpha(x) f(x)$ so it is enough to know how to integrate each one of the $\psi_\alpha f$, and these vanish outside U_α. So we define,

$$\int_X f(x)dx = \int_{\varphi_\alpha(U_\alpha)} f \circ \varphi_\alpha^{-1}(x) \sqrt{\left|\det g^\alpha\left(\varphi_\alpha^{-1}(x)\right)\right|} dx,$$

where $g^\alpha_{i,j}(p) = g\left(D_{\varphi_\alpha(p)} \varphi_\alpha^{-1}(e_i), D_{\varphi_\alpha(p)} \varphi_\alpha^{-1}(e_j)\right)$ and $\{e_i\}$is the standard basis for $\mathbb{R}^n$ (look at the drawing to understand where this formula comes from). We will discuss two things: well defineness and the sign issue.

Lemma 46. *If g is a bilinear form on $\mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation,*

$$\det\left(g(Ae_i, Ae_j)_{i,j}\right) = (\det A)^2 \det\left(g(e_i, e_j)_{i,j}\right).$$

(the matrix on the right side is called the Grahm matrix of g w.r.t. to a basis $e_1, ..., e_n$.)

Proof. Denote $B=(g(e_i,e_j))\in M_n(\mathbb{R})$. Then $g(u,v)=u^tBv\Rightarrow g(Au,Av)=(Au)^tB(Av)==u^tA^tBAv$ so

$$\det\left(g\left(Ae_i,Ae_j\right)_{i,j}\right)=\det\left(A^tBA\right)=\left(\det A\right)^2\det\left(B\right)=\left(\det A\right)^2\det\left(g\left(e_i,e_j\right)_{i,j}\right).$$

□

We will use this lemma to prove the well defineness of the integral. Assume we are integrating over $U_\alpha\cap U_\beta$, by change of variables, $y=\varphi_\beta\circ\varphi_\alpha^{-1}(x)$, we get

$$\int_{\varphi_\alpha(U_\alpha)} f\circ\varphi_\alpha^{-1}(x)\cdot\sqrt{\det g^\alpha\left(\varphi_\alpha^{-1}(x)\right)}dx=\int_{\varphi_\beta(U_\beta)} f\circ\varphi_\beta^{-1}(y)\cdot\sqrt{\det g^\alpha\left(\varphi_\beta^{-1}(y)\right)}\cdot\left|\mathrm{Jac}\left(\varphi_\alpha\circ\varphi_\beta^{-1}\right)\right|dy.$$

So we have to prove

$$\begin{aligned}\sqrt{|\det(g^\beta)|} &= \left|\det\left(g\left(D\varphi_\beta^{-1}(e_i),D\varphi_\beta^{-1}(e_j)\right)\right)\right|^{\frac12}\\ &= \left|\det\left(g\left(D\varphi_\alpha^{-1}\circ D\varphi_\alpha\circ D\varphi_\beta^{-1}(e_i),D\varphi_\alpha^{-1}\circ D\varphi_\alpha\circ D\varphi_\beta^{-1}(e_j)\right)\right)\right|^{\frac12}\\ &= \left|\mathrm{Jac}\left(\varphi_\alpha\circ\varphi_\beta^{-1}\right)\right|\left|\det\left(g\left(D\varphi_\alpha^{-1}(e_i'),D\varphi_\alpha^{-1}(e_j')\right)\right)\right|^{\frac12}.\end{aligned}$$

The Lebesgue integral is more general than the Riemann integral, but in practice we usually use the Riemann integral and the change of variables to make calculations, rather than using the Lebesgue integral and the Radon-Nikodym derivative. But the truth is that there are some more problems with the Riemann integral. For example, there exist an integrable function f and a continuous function g, such that $f\circ g$ is not integrable. Also in dimension bigger than 1, one need to carry in mind that the change of variables formula is true only for one-to-one maps (one dimensional discussion on the Riemann integral and how does the chage of variables is working also in the non-monotonic case).

This is why one wish to work with a volume form, in order to allow any change of variables in higher dimensional integration. This is done by some algebra that is required after realizing that the following should be true:

$$\int f(x)dx_1\wedge dx_2\wedge...\wedge dx_n=-\int f(x)dx_2\wedge dx_1...\wedge dx_n.$$

The meaning of this in manifolds is that when $\mathrm{Jac}\left(\varphi_\alpha\circ\varphi_\beta^{-1}\right)=\det\left(D\varphi_\alpha\circ D\varphi_\beta^{-1}\right)$ is positive everywhere we say that X is oriented, and we can replace volume element by volume form.

Part 2. Now that we know everything there is to know about Riemannian manifolds...

Theorem 47. *Closed subgroup of a Lie group is a Lie group.*

Proof. (Not so easy... We prove below the case of $SL_n(\mathbb{R})$. see Raghunathan "discrete subgroups of Lie groups" for the general case). □

Fact 48. *(Implicit function theorem)* $n,m>0$, $f:\mathbb{R}^{n+m}\to\mathbb{R}^m$ *smooth,* $p=(a,b)\in\mathbb{R}^{n+m}$. *Assume* $\mathrm{rank}D_pf=m$, *and w.l.o.g.* $\det\left(\frac{\partial f_i}{\partial y_j}(p)\right)\neq0$. *Then* $\exists U$ *open in* $\mathbb{R}^n$,$V\subseteq\mathbb{R}^m$ *open,* $v\in V$, $g(a)=b$, $g:U\to V$ *smooth s.t.*

$$\{q\in U\times V:f(q)=f(p)\}=\{(x,g(x)):x\in U\}.$$

Equivalently, there exists a neighborhood W of p such that $W \cap f^{-1}(c)$ is a manifold where $c = f(p)$, with dimension m.

THEOREM 49. *$SL_n(\mathbb{R})$ is a manifold.*

PROOF. $SL_n(\mathbb{R}) = \det^{-1}(1)$ and $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is smooth. In order to show that $SL_n(\mathbb{R})$ is a manifold, it's enough to show $\operatorname{rank}(D_x(\det)) = 1$ for any $x \in SL_n(\mathbb{R})$. Let $e \in SL_n(\mathbb{R})$ be the identity $e = I_{n\times n}$.

$$D_e det\,(v) = \left.\frac{d}{dt}\right|_{t=0} \det \begin{pmatrix} 1+t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \left.\frac{d}{dt}\right|_{t=0} (1+t) = 1,$$

for the vector $v = \begin{pmatrix} 1+t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$. Since det has image in $\mathbb{R}$, to prove full rank, it's enough to prove $D_e \det \neq 0$ and we just showed that.

For any $g \in SL_n(\mathbb{R})$,

$$\begin{aligned} L_{g^{-1}} : M_n(\mathbb{R}) &\to M_n(\mathbb{R}) \\ h &\mapsto g^{-1}h \end{aligned}$$

is a diffeomorphism of $M_n(\mathbb{R})$, and its inverse is L_g. We have

$$\det\left(L_{g^{-1}}h\right) = \det\left(g^{-1}h\right) = \underbrace{\det\left(g^{-1}\right)}_{\text{fixed number}} \det(h).$$

By the chain rule: $D_g(L_{g^{-1}} \circ \det) = D_e(\det) \cdot D_g\left(L_{g^{-1}}\right)$. Therefore by using the above equality, $D_g(\det) = \det(g)\, D_e(\det) D_g\left(L_{g^{-1}}\right)$ and since $D_e(\det)$ has full rank, and $L_{g^{-1}}$is invertible (so $D_g\left(L_{g^{-1}}\right)$ is invertible), $D_g(\det)$ also has full rank. □

Review - Nattali and Erez (17/12/13).

If G is lcsc group then G has two measures, μ and ν , called left and right Haar measures, respectively. We will state things for lcsc groups but prove for Lie groups.

DEFINITION 50. Let μ be a measure:

(1) μ is called Borel if the σ-algebra is the Borel σ-algebra (the smallest which contains the all open sets).
(2) μ is regular if for any $A \in \mathcal{B}$ we have

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ is compact}\} = \inf\{\mu(V) : V \supseteq A \text{ is open}\}.$$

(3) μ is Radon, if $\mu$is regular and $\mu(K) < \infty$ for any compact K.

THEOREM 51. *(Existence of Haar measure and properties)*

(1) *There exist invariant measures μ and ν invariant under left and right multiplication, respectively.*
(2) *Up to a scalar multiplication, μ and ν are the unique Radon measures satisfying 1.*

(3) *There exists a homomorphism* $\Delta_G : G \to \mathbb{R}^*_{\geq}$ *(multiplicative group of positive reals) so that for any* $A \in \mathcal{B}$, $\forall g \in G$, $\mu\left(gAg^{-1}\right) = \Delta_G(g)\mu(A)$ *and* $\nu\left(gAg^{-1}\right) = \Delta_G(g)\nu(A)$.
(4) $\mu(K) < \infty$ *when* K *is compact,* $\mu(O) > 0$ *if* O *is open.*
(5) $\mu(G) < \infty \iff G$ *is compact.*

Proof. (For Lie groups)

(1) We will define μ. We saw that if we have a Riemannian metric on a manifold, we can use it to define a volume element, and then by partition of unity - a measure (by Riesz representation theorem, a regular Borel measure is uniquely determind by the integral of all compactly supported continuous functions). We will construct a Riemannian metric on $T(G)$ which is invariant under left multiplication, and this will yield a left invariant measure.
Let $(\cdot,\cdot)_e$ denote any inner product on $T_e(G)$. For any $g \in G$, we have $D_g\left(L_{g^{-1}}\right) : T_g(G) \to T_e(G)$ so define

$$(u,v)_g = \left(D_g\left(L_{g^{-1}}\right)(u), D_g\left(L_{g^{-1}}\right)(v)\right)_e$$

for any $u, v \in T_g(G)$ ("Transporting" the inner product to all $T_g(G)$ by left multiplication. A discussion about contra vs co variant objects, measures vs Riemannian metric). By construction and the chain rule it is left invariant: For any $g, g_1 \in G$, $u, v \in T_{g_1}G$, the map $D_{g_1}L_g : T_{g_1}G \to T_{gg_1}G$ satisfy

$$\begin{aligned}
\left(D_{g_1}\left(L_g\right)(u), D_{g_1}\left(L_g\right)(v)\right)_{gg_1} &= \left(D_{(gg_1)^{-1}}\left(L_{(gg_1)^{-1}}\right)D_{g_1}\left(L_g\right)(u), D_{(gg_1)^{-1}}\left(L_{(gg_1)^{-1}}\right)D_{g_1}\left(L_g\right)(v)\right)_e \\
&= \left(D_{g_1}\left(L_{g_1^{-1}}\right)(u), D_{g_1}\left(L_{g_1^{-1}}\right)(v)\right)_e \\
&= (u,v)_{g_1}
\end{aligned}$$

(This proof usually does not appear in textbooks because they deal with the more general case of lcsc groups).

(2) Uniqueness: Let μ_1 and μ_2 be regular Borel left-invariant measures. Let $\mu_3 = \mu_1 + \mu_2$. Then $\mu_1, \mu_2 \ll \mu_3$. Enough to show that μ_1 is a scalar multiple of μ_3. Let h be the Radon-Nikodym derivative $\frac{d\mu_1}{d\mu_3}$. I.e., $\forall f \in C_c(G)$, $\int_G f d\mu_1 = \int_G fh d\mu_3$. To show $\mu_1 = c\mu_3$, it is enough to show $h = const$ μ_3-a.e. Let $z \in G$,

$$\begin{aligned}
\int f(g)h(g)d\mu_3(g) &= \int f(g)d\mu_1(g) \\
&= \int f(zg)d\mu_1(g) \\
&= \int f(zg)h(g)d\mu_3(g) \\
&= \int f(g_1)h(z^{-1}g_1)d\mu_3(g_1)
\end{aligned}$$

(Add reasoning above...). Since this identity holds for all f , we have $h(g) = h(z^{-1}g)$. So h is a.e. constant.

(3) Let us first show that for any $A \in \mathcal{B}$ we have that $\frac{\mu(A)}{\mu(gAg^{-1})}$ is independent of A (as long as the denominator $\neq 0$). Define a measure ν_0 by $\nu_0(A) = \mu\left(A^{-1}\right)$. Since μ is left invariant:

$$\nu_0(Ag) = \mu\left((Ag)^{-1}\right) = \mu\left(g^{-1}A^{-1}\right) = \mu\left(A^{-1}\right) = \nu_0(A).$$

Similarly, for any $g_0 \in G$, we have that ν_{g_0} which is defined by $\nu_{g_0}(A) = \mu\left(A^{-1}g_0^{-1}\right)$ is right invariant. By 2, there is a scalar $c(g_0)$ such that $\nu_{g_0} = c(g_0)\nu_0$. Now define

$$\begin{array}{rcl} \Delta_G : G & \to & \mathbb{R}^*_{\geq} \\ \Delta_G(g_0) & \mapsto & c(g_0). \end{array}$$

Then $\forall A \in \mathcal{B}$,

$$\mu\left(A^{-1}g_0^{-1}\right) = \nu_{g_0}(A) = \Delta_G(g_0)\nu_0(A) = \Delta_G(g_0)\mu\left(A^{-1}\right)$$

and by left invariance of μ, for all $A \in \mathcal{B}$ for which $\mu\left(A^{-1}\right) > 0$, we have that $\Delta_G(g_0) = \frac{\mu\left(g_0A^{-1}g_0^{-1}\right)}{\mu(A^{-1})}$ is independent of A. Now it is clear that

$$\Delta_G(gg_0) = \frac{\mu\left(gg_0A^{-1}\left(gg_0\right)^{-1}\right)}{\mu\left(A^{-1}\right)} = \frac{\Delta_G(g)\mu\left(g_0A^{-1}g_0{}^{-1}\right)}{\mu\left(A^{-1}\right)} = \Delta_G(g)\Delta_G(g_0).$$

(4) Clear.
(5) $\Longleftarrow$ is just 4. For $\Longrightarrow$, Let $c = \mu(G) < \infty$. Since μ is regular, there exists a compact K such that $\mu(K) > \frac{c}{2}$. For any $x \in G$, $\mu(K) = \mu(xK)$ so $K \cap xK \neq \varnothing$, therefore $\exists k_1 = xk_2\ k_1, k_2 \in K$. So $x = k_1k_2^{-1} \in KK^{-1}$. Therefore G is the image of $K \times K$ under the map $(g_1, g_2) \mapsto g_1g_2^{-1}$ which is clearly continuous. But a continuous image of a compact is compact, therefore G is compact.

□

4.11

Part 1.

Definition 52. A Lie algebra is a vector space $\mathfrak{g}$ equiped with a bilinear

$$\begin{array}{rcl} \mathfrak{g} \times \mathfrak{g} & \to & \mathfrak{g} \\ (u, v) & \mapsto & [u, v] \end{array}$$

called Lie bracket such that:
(1) $[x, x] = 0$.
(2) $[x, [y, z]] + [z, [x, y]] + [y, [z.x]] = 0$ (Jacobi identity)

The Lie algebra of a Lie group$G$ is $\mathfrak{g} = T_e(G)$. To define the bracket, for any $g \in G$ define

$$\begin{array}{rcl} int_g : G & \to & G \\ h & \mapsto & ghg^{-1}, \end{array}$$

and $Ad(g) = D_e(int_g) : \mathfrak{g} \to \mathfrak{g}$. Conjugation satisfies $int_{g_1} \circ int_{g_2} = int_{g_1g_2}$. This implies that $Ad : G \to GL(\mathfrak{g})$ is a homomorphism. So we can take its derivative: define $ad = D_e(Ad)$. Now $ad : \mathfrak{g} \to T_{Id}(GL(\mathfrak{g})) = gl(\mathfrak{g})$. Define $[x, y] = ad(x)(y) \in G$ (follow the diagram...).

Exercise 53. Check that the Jacobi identity is satisfied.

More concretely, if $g \in G$ what is $Ad(g)$? Let $v \in \mathfrak{g}$ be represented by $\gamma : (-\varepsilon, \varepsilon) \to G$ smooth such that $\gamma(0) = e$. Then,

$$Ad(g)(v) = \frac{d}{dt}\bigg|_{t=0} g\gamma(t)g^{-1}.$$

If $u, v \in g$ are represented by paths $\alpha(t), \gamma(t)$ then

$$[u, v] = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \alpha(t)\gamma(t)\alpha(t)^{-1} = \frac{d}{ds}\bigg|_{s=0} Ad(\alpha(s))(v).$$

It looks scarry but we can actually make calculations using it. If G is abelian then nothing interesting is happening because always $[u, v] = 0$. In general we get a linear model for the Lie group which is very helpful. What is the relation to haar measure?

Exercise 54. $\Delta_G(g) = \frac{\mu(gAg^{-1})}{\mu(A)} = |\det(Ad(g))|$.

This is a very important identity. (complete discussion about the algebraic nature...). There is no analogue for general topological groups.

Finally: the quotient G/Γ. For a topological space X and an equivalence relation on X we can sometimes talk about the quotient topology $Y = {}^{X}\!/\!\sim$ by looking on

$$\begin{aligned} \pi : X &\to Y \\ x &\mapsto [x] \end{aligned}$$

and define open sets in Y as $U \subseteq Y$ such that $\pi^{-1}(U)$ is open in X. Suppose Γ is discrete, i.e., $\forall \gamma \in \Gamma \exists U \subseteq G$ open set such that $\Gamma \cap U = \{\gamma\} \iff \gamma_n \xrightarrow[n\to\infty]{} \gamma_0$ iff $\gamma_n = \gamma$ for all sufficiently lafrge $n \iff \exists U \subseteq G$ open such that $U \cap \Gamma = \{e\}$.

Remark 55. One can form a nice quotient, i.e., a smooth manifold structure on G/Γ which defines the quotient topology, whenever $\Gamma$is closed. We will only carry this out for Γ discrete.

Recall: G is equipped with a right-invariant Riemannian metric therefore the path metric on G is right-invariant: $d_G(g_1, g_2) = \inf \{\ell(\gamma)\}$ where

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

d_G is right-invariant by construction, i.e, $d_G(g_1, g_2) = d_G(g_1 h, g_2 h)$ for all $g_1, g_2, h \in G$. In general d_G is not left invariant. If it were then for all $g, h \in G$,

$$d_G(e, h) = d_G(g, hg) = d(e, g^{-1}hg).$$

This is not the case in general: Take $G = SL_2(\mathbb{R})$, $g = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g^{-1}hg = \begin{pmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{pmatrix}$, $g^{-n}hg^n = \begin{pmatrix} 1 & \frac{1}{4^n} \\ 0 & 1 \end{pmatrix}$. Then we can apply the above equality again and again and get contradiction.

Definition 56. Let $X = G/\Gamma$, $\pi : G \to X$ the natural projection defined by $\pi(g) = g\Gamma$.

$$d_X(\pi(g_1), \pi(g_2)) = \inf_{\gamma_1,\gamma_2\in\Gamma} d_G(g_1\gamma_1, g_2\gamma_2) = \inf_{\gamma\in\Gamma} d_G(g_1, g_2\gamma).$$

Remark 57. Easy to check that this is a metric. The fact that $x_1 \neq x_2 \Rightarrow d_X(x_1, x_2) > 0$ follows from the discreteness of Γ. Also easy to check that $\inf = \min$.

Lemma 58. *$\forall K \subseteq X$ compact $\exists r = r(K) > 0$ such that $\forall x_0 \in K$.*

$$\begin{array}{rcl} B_G(e,r) & \to & B_X(x_0, r) \\ g & \mapsto & gx_0 \end{array} \tag{2}$$

is an isometry. In fact, for $K = \{x_0\}$, can take

$$r = \frac{1}{4} \inf_{\{\gamma\in\Gamma,\gamma\neq e\}} \{d_G(h\gamma, h)\} \tag{3}$$

where $x_0 = \pi(h)$.

Proof. Suffices to prove the lemma when $K = \{x_0\}$, because if r satisfies that 6 is an isometry then 6 is also an isometry for every $x' \in B_X(x_0, \frac{r}{2})$ with $\frac{r}{2}$ instead of r. Now cover K by finitely many balls to get general case. (add diagram of G over the quotient...). Note: r in 3 is positive beacuase Γ is discrete. Let $g_1, g_2 \in B_G(e, r)$. Then

$$d_X(g_1 x, g_2 x_0) \in d_X(g_1\pi(h), g_2\pi(h)) = \inf_{\gamma\in\Gamma} d_G(g_1 h, g_2 h\gamma) = \inf_{\gamma\in\Gamma} d_G\left(g_1, g_2 h\gamma h^{-1}\right).$$

Want to show it is attained for any $\gamma \in \Gamma$. $\gamma = e$ and $\gamma \neq e$. For $\gamma \in \Gamma$, if $d_G\left(g_1, g_2 h\gamma h^{-1}\right) = d_G(g_1, g_2) < 2r$, Then

$$d_G\left(g_2 h\gamma h^{-1}, e\right) \leq d\left(g_2 h\gamma h^{-1}, g_1\right) + d_G(g_1, e) < 3r.$$

Similarly,

$$d_G\left(h\gamma h^{-1}, e\right) = d_G\left(e, h\gamma^{-1}h^{-1}\right) \leq d_G(e, g_2) + d_G\left(g_2, h\gamma h^{-1}\right) \leq r + d_G\left(g_2 h\gamma h^{-1}, e\right) < 4r.$$

So $d_G(h\gamma, h) < 4r$ and therefore $\gamma = e$. □

Definition 59. For fixed $g_0 \in G$, the injectivity radius of π at g_0 is

$$r_0(g) = \left\{ r : B_G(e,r) \underset{g\mapsto g\pi(g_0)}{\to} B_X(\pi(g_0), r) \text{ is and isometry} \right\}.$$

Sometimes we denote $r_0(x) = r_0(g)$ if $x = \pi(g)$ and call it the injectivity radius of X at x.

Corollary 60. *If $K \subseteq X$ is compact then $\exists K' \subseteq G$ compact such that $\pi(K') = K$.*

Proof. Exercise. □

Part 2. Some definitions relevant for the exercise sheet:

DEFINITION 61. G is called simple if $\dim G > 1$ if it contains no proper closed connected normal (Lie) subgroups.

G is called semisimple if G contains simple subgroups $G_1, ..., G_n$, with $G_i \cap G_j$ discrete for $i \neq j$, such that $\forall g_1 \in G_i, g_2 \in G_j,\ g_1 g_2 = g_2 g_1$ and G is generated by $G_1, ..., G_n$.

We will continue to use the last theorem proved at the end of the last part.

DEFINITION 62. A fundamental domain for G/Γ (sometiomes called fundamental domain for right action of Γon G) is a measurable $F \subseteq G$ such that $\forall g \in G$ there exists a unique $\gamma \in \Gamma$such that $g\gamma \in F$. This is true $\iff G = \coprod_{\gamma \in \Gamma} F\gamma$.

EXAMPLE 63. $G = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$. One fundamental domain is $F_1 = [0,1) \times [0,1)$. Other fundamental domains: take any two vectors $v_1, v_2 \in \mathbb{Z}^2$ with $|\det(v_1, v_2)| = 1$,

$$F_2 = \{sv_1 + tv_2 : s, t \in [0,1)\}.$$

For $A = (v_1; v_2)$ we have $Ae_i = v_i$, $i = 1, 2$ so $A(F_1) = F_2$.

EXERCISE 64. Prove that F_2 is a fundamental domain.

LEMMA 65. *If $\Gamma < G$ is discrete then there exists a fundamental domain.*

PROOF. Let $\pi : G \to X$ be the projection. We claim that there exist sets B_n, $n \in \mathbb{N}$ open so that $\pi|_{B_n}$ is one-to-one and $G = \bigcup B_n$, and such that the following diagram commutes: $B_G(h,r) = B_G(e,r)h$. Cover G by $B_G(h,r)$ with $r = r(h)$ and take a countable subcover (we can do it since G is σ-compact), and denote $B_n = B_G(h_n, r_n)$. Now define sets F_n recursively by $F_1 = B_1$, $F_2 = B_2 \setminus \pi^{-1}(\pi(B_1))$, and in general for every $n \in \mathbb{N}$ define $F_n = B_n \setminus \pi^{-1}(\pi(B_1 \cup ... \cup B_{n-1}))$. Now let

$$F = \bigcup_{n=1}^{\infty} F_n.$$

F is a fundamental domain if and only if $\forall g \in G$ there is a unique $\gamma \in \Gamma$ such that $g\gamma \in F$. By construction, there is a unique $f \in F$ such that $\pi(f) = \pi(g)$ so f, g are in the same coset, therefore there exists $\gamma \in \Gamma$ such that $g\gamma = f \in F$. □

REMARK 66. Can similarly prove that if Y_1, Y_2 are measurable subsets of G such that $\pi|_{Y_1}$ is injective and $\pi(Y_2) = X$, then there exists a fundamental domain F such that $Y_1 \subseteq F \subseteq Y_2$.

LEMMA 67. *If F_1, F_2 are fundamental domains for G/Γ and ν is right invariant haar measure on G then $\nu(F_1) = \nu(F_2)$.*

PROOF. $\forall x \in F_1$ there exists a unique $\gamma = \gamma_x$ such that $x\gamma \in F_2$. Let $F_1(\gamma) = \{x \in F_1 : \gamma_x = \gamma\}$. Then $F_1 = \coprod_{\gamma \in \Gamma} F_1(\gamma)$. So $\bigcup F_1(\gamma)\gamma \subset F_2$ is a fundamental domain (intersects every Γ orbit since F_1 intersects every Γ orbit). Therefore, $\bigcup F_1(\gamma)\gamma = F_2$. Now we use right invariance of the measure ν to get

$$\nu(F_1) = \sum_{\gamma \in \Gamma} \nu(F_1(\gamma)) = \sum_{\gamma \in \Gamma} \nu(F_1(\gamma)\gamma) = \nu(F_2).$$

□

Remark 68. More generally, the proof will work the same for ν that is only right invariant under multiplication by all elements of Γ). Note that in this proof we really use the fact that Γ is discrete. Also, a similar argument shows that if $Y \subset G$, $\pi|_{Y_1}$ is injective and F_1, F_2 are two fundamental domains then (complete...).

Proposition 69. *TFAE:*

1. *$\exists$ a G invariant probability measure on G/Γ.*
2. *$\exists F \subset G$ a fundamental domain for G/Γ with $\mu(F) < \infty$, where μ is left haar measure on G, and μ is also right Γ-invariant.*

Proof. We will prove only the direction we'll use. The other direction is a bit technical and is left as an exercise.

$2 \Rightarrow 1$. Define m_x on X as follows: $m_x(A) = \frac{1}{\mu(F)}\mu\left(F \cap \pi^{-1}(A)\right)$ for any $A \subset X$. Clearly, it is aprobability measure. Suppose $g \in G$, $A \subset X$. Then

$$\begin{aligned} \mu(gA) &= \frac{1}{\mu(F)}\mu\left(F \cap g\pi^{-1}(A)\right) \\ &= \frac{1}{\mu\left(g^{-1}F\right)}\mu\left(g^{-1}F \cap \pi^{-1}(A)\right) \\ &= \frac{1}{\mu(F)}\mu\left(F \cap \pi^{-1}(A)\right) \\ &= m_x(A), \end{aligned}$$

where the one before last equality follows from the fact that $g^{-1}F$ is also a fundamental domain (why...). □

Exercise 70. prove the other direction, $1 \Rightarrow 2$.

Definition 71. Γ is called a lattice in G if G/Γ admits a G-invariant probability measure.

Γ is called a uniform lattice (cocompact lattice) if additionaly G/Γ is compact. Otherwise G/Γ is called non-uniform.

Example 72. $\mathbb{Z}^d$in $\mathbb{R}^d$ is a cocompact lattice. $SL_d(\mathbb{Z})$is a non-uniform lattice in $SL_d(\mathbb{R})$.

Definition 73. A sequence (x_n) in X is called divergent (notation: $x_n \xrightarrow[n\to\infty]{} \infty$) if $\forall K \subseteq X$ compact $\exists n_0 \forall n \geq n_0$ $x_n \notin K$. Equivalently, has no convergent subsequence.

Proposition 74. *Let Γ be a lattice in G, $X = G/\Gamma$, (x_n). TFAE:*

1. $x_n \xrightarrow[n\to\infty]{} \infty$.
2. $r(x_n) \xrightarrow[n\to\infty]{} 0$

Proof. $2 \Rightarrow 1$. If $r(x_n) \xrightarrow[n\to\infty]{} 0$ but $x_n \xrightarrow[n\to\infty]{} \infty$ then take a convergent subsequence $x_{n_k} \xrightarrow[k\to\infty]{} y$. We have $r(y) > 0$ so there exists a neighborhood U of y such that for any $x \in U$, $r(x) > \frac{r(y)}{2}$. A contradiction.

$1 \Rightarrow 2$. Suppose $x_n \xrightarrow[n\to\infty]{} \infty$. Assume $\limsup r(x_n) > 0$. To get a contradiction, replace x_n with a subsequence so that $r(x_n) > \varepsilon$ for every n. Each

$$\bar{B}(x, \varepsilon) = \{x_0 \in X : d_X(x, x_0) \leq \varepsilon\}$$

is compact. $x_n \xrightarrow[n\to\infty]{} \infty$ so for each $x_i$only finitely many x_j satisfy $x_j \in \bar{B}(x,\varepsilon)$. By taking another subsequence we can ensure that $\forall i \neq j$, $d_X\left(x_i, \frac{\varepsilon}{2}\right)$ are all disjoint, and have some measure because they are isometric projections of balls in G. Contradiction to finiteness of the measure. □

4.11

Part 1. No lecture next week!

REMARK 75. $\pi(g_n) \to \pi(g)$ iff $\exists \gamma_n \in \Gamma$ such that $g_n\gamma_n \to h$.

We will be discussing actions of subgroups $H \subseteq G$ on $X = G/\Gamma$.

If $x \in X$ then $Hx = \{hx : h \in H\} \subseteq X$. We have a map $H \longrightarrow X$ defined by $h \mapsto hx$, and it descends to the orbit map

$$\begin{array}{ccc} H/H_x & \longrightarrow & X \\ hH_x & \mapsto & hx \end{array}$$

where $H_x = Stab_H(x)$. This is completetely general that this map is bijection:

Onto is clear. Injective -

$$h_1x = h_2x \Rightarrow h_2^{-1}h_1x = x \Rightarrow h_2^{-1}h_1 \in H_x \Rightarrow h_1H_x = h_2H_x.$$

We want to understand also the topology of this map. H_x is definitely closed. If $x = \pi(g_0)$, then

$$\begin{aligned} H_x &= \{h \in H : hx = x\} \\ &= \{h \in H : \exists \gamma \in \Gamma \text{ such that } hg_0 = g_0\gamma\} \\ &= \left\{h \in H : \exists \gamma \in \Gamma \text{ such that } h = g_0\gamma g_0^{-1}\right\} \\ &= H \cap g_0\Gamma g_0^{-1}. \end{aligned}$$

This map is always continuous, it comes from the fact that our action was continuous.

PROPOSITION 76. *Orbit map is a homeomorphism onto its image (i.e. its inverse is continuous)* $\Longleftrightarrow$ *Hx is closed (we omit the proof, uses Baire category theorem?)*

So, it is interesting to find closed orbits Hx. Schematic picture...

Obviously, if H_x is cocompact (i.e. H/H_x is compact) then Hx is compact, hence closed.

PROPOSITION 77. *If $\Gamma$is a lattice in G and H_x is a lattice in H then Hx is closed.*

PROOF. Take $h_nx \longrightarrow y$ and show that $y \in H_x$. Denote $\Lambda = H_x$.

(1) Suppose $\pi_H(h_n)$ has a convergent subsequence, where $\pi_H : H \longrightarrow H/\Lambda$. If (after passing to a subsequence) $\pi_H(h_n) \longrightarrow \pi_H(h_0)$. Therefore there exists $\lambda_n \in \Lambda$ such that $h_n\lambda_n \longrightarrow h_0$. Then $y \longleftarrow h_nx = h_n\lambda_nx \longrightarrow h_0x$, so $y = h_0x$.
(2) Suppose $\pi_H(h_n)$ has no convergent subsequence. Then (by result of precious lecture) the injectivity radius of H/Λ at $\pi_H(h_n)$ is going to zero, i.e. $\exists \lambda_n \in \Lambda \setminus \{0\}$ such that $d_H(h_n, h_n\lambda_n) \xrightarrow[n\to\infty]{} 0$. Therefore

$d_G(h_n, h_n\lambda_n) \xrightarrow[n\to\infty]{} 0$. Write $\lambda_n = g_0\gamma_n g_0^{-1}$, $\gamma_n \neq e$. Then since the metric is left invariant,

$$d_G(h_n g_0, h_n g_0 \gamma_n) = d_G\left(h_n, h_n g_0 \gamma_n g_0^{-1}\right) \xrightarrow[n\to\infty]{} 0.$$

So the injectivity radius of G/Γ at $\pi_G(h_n g_0)$ is going to 0. $\Rightarrow \pi_G(h_n g_0) = h_n x$, because $x = \pi_G(g_0)$.$\Rightarrow$ $(h_n x)$ has no convergent subsequence, contradicting $h_n \xrightarrow[n\to\infty]{} y$.

□

Discussion on Ratner theorems (complete...).

Examples: $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$**,...,** $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$**.** First of all, it is only interesting to understand orbits of subgroups up to conjugacy. This is because

$$gHg^{-1}(gx) = gHx.$$

In $SL_2(\mathbb{R})$ there are only four closed connected such nontrivial subgroups up to conjugacy:

(1) $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

(2) $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

(3) $h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

(4) $h_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Each element is conjugate to one of these by using Jordan form (argument?...). These are the geodesic flow and the horocycle flow.

If M is a Riemannian manifold, then a geodesic arc on M is a smooth curve $\gamma : (a,b) \longrightarrow M$ such that

$$\forall s \in (a,b) \exists \varepsilon \forall s_0 \in (s-\varepsilon, s+\varepsilon) \cap (a,b) \ \mathrm{dist}_M(\gamma(s), \gamma(s_0)) = |s - s_0|$$

i.e. γ is locally distance minimizing. The unit tangent bundle of M is

$$UM = \{(x,v) : x \in M, v \in T_x(M), \|v\|_x = 1\} \subseteq TM.$$

M is called complete (geodessically complete) if for all $(x,v) \in UM$ there exists a unique geodesic $\gamma : \mathbb{R} \longrightarrow M$ such that $\gamma(0) = x$, $\gamma'(0) = v$.

The geodesic flow on UM is defined as follows: $g_t(x,v) = (y,w)$ where $y = \gamma(t)$, $w = \gamma'(t)$ where γ is the unique geodesic through x in direction v. This is actually an action of $\mathbb{R}$ on UM.

Part 2. Why $g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$ is the geodesic flow? Let $\mathbb{H} = \{x + iy : y > 0\}$ be the upper half plane. G acts on $\mathbb{H}$ by Mobius transformations on the right (this is not the usuall notation. Whenever you have a right action you can get a left action by applying an anti-isomorphism. In our case: transpose) according to the rule

$$z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az + c}{bz + d}.$$

This will be consistent with our notation in the rest of the course.

Claim 78. G acts transitively.

Proof. For any $y>0$,

$$i\cdot\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ xy^{-\frac{1}{2}} & y^{-\frac{1}{2}} \end{pmatrix}=\frac{y^{\frac{1}{2}}i+xy^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}=x+iy.$$

□

Claim 79. Stabilizer of i is $SO\left(2,\mathbb{R}\right)=\left\{h_\theta=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}:\theta\in\mathbb{R}\right\}$.

Proof. $i\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}=i\iff$

$$\begin{aligned} i &= \frac{ai+c}{bi+d}=\frac{\left(c+ai\right)\left(d-bi\right)}{b^2+d^2}= \\ &= \frac{ab+cd}{b^2+d^2}+i\frac{ad-bc}{b^2+d^2}=\frac{ab+cd}{b^2+d^2}+\frac{1}{b^2+d^2}i \end{aligned}$$

$\iff ab+cd=0$, $b^2+d^2=1$. □

We have established $\mathbb{H}=SO_2(\mathbb{R})\backslash SL_2(\mathbb{R})$ (G equivariant for the right action). To define a Riemannian metric on $\mathbb{H}$ which is G invariant, we start with an inner product $(\cdot,\cdot)_i$ on $T_i\mathbb{H}$ which is $SO_2(\mathbb{R})$ invariant, and then when $i\cdot g=z$, define

$$(u,v)_z=\left(D_{g^{-1}}(z)(u),D_{g^{-1}}(z)(v)\right)_i$$

and it will be G invariant by construction.

There is only one possible $SO_2(\mathbb{R})$invariant for us to choose up to scaling. So let $(u,v)_i$ be the standard inner product

$$(u,v)=u_1v_1+u_2v_2.$$

Using g from previous claim, if $z=x+iy$, $g=\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ xy^{-\frac{1}{2}} & y^{-\frac{1}{2}} \end{pmatrix}$. We have

$$g^{-1}=\begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ -xy^{-\frac{1}{2}} & y^{\frac{1}{2}} \end{pmatrix}.$$

Then

$$\left(z+h\right)g^{-1}=\frac{y^{-\frac{1}{2}}\left(z+h\right)-xy^{-\frac{1}{2}}}{y^{\frac{1}{2}}}=\frac{z+h}{y}-\frac{x}{y}$$

So $D_{g^{-1}}$ is scalar multiplication by $\frac{1}{y}$, therefore

$$(u,v)_z=\frac{(u,v)_i}{y^2}.$$

(Other notation: $ds=\frac{\sqrt{dx^2+dy^2}}{y}$).

We have a way for passing from a Riemannian metric to a volume form

Remark 80. G invariant volume element is $\frac{dxdy}{y^2}$ (we'll skip the proof. We had the relevant formulas for the general case).

$SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ by isometries (w.r.t Riemannian metric we constructed). Therefore $SL_2(\mathbb{R})$ acts on $T\mathbb{H}$ by

$$(z,u)\cdot g = (z\cdot g, D_g(z)(u))\,.$$

Since G preserves the Riemannian metric, $U\mathbb{H}$ is mapped to itself.

Claim 81. $G = SL_2(\mathbb{R})$ transitive on $U\mathbb{H}$, and the stabilizer is $\{\pm Id\}$ (using dimension consideration we can hope that it is the case)

Proof. (Idea of...) First check that for any $u_1, u_2 \in T_i\mathbb{H}$ there exists $g \in SO_2(\mathbb{R})$ such that $D_g(i)u_1 = u_2$. From the computation we'll get that for $u \in T_i\mathbb{H}$, $u = (\cos\alpha, \sin\alpha)$, $\|u\|_i = 1$, $g \in SO_2(\mathbb{R})$

$$D_g(i)(u) = (\cos(\alpha+2\theta), \sin(\alpha+2\theta)).$$

In particular, for any $u \in T_i\mathbb{H}$, $Stab_G(i,u) = \{\pm Id\}$. Now, if $(z_1,u_1),(z_2,u_2) \in U\mathbb{H}$ then by transitivity of the G action on $\mathbb{H}$ there exist g_1, g_2 such that $z_1 g_1 = i = z_2 g_2$. Also there exists $g_3 \in SO_2(\mathbb{R})$ such that

$$D_{g_3}(i)\left(D_{g_1}(z_1)(u_1)\right) = D_{g_2}(z_2)(u_2)$$

Then $g_1 g_3 g_2^{-1}$ maps (z_1,u_1) to (z_2,u_2). □

Corollary 82. $U\mathbb{H} = \{\pm Id\}\backslash G = PSL_2(\mathbb{R})$. *In other words,* $PSL_2(\mathbb{R})$ *acts simply transitively on* $U\mathbb{H}$.

This situation is very rare. Actually it is a very big thesis that every Lie group has exactly one symmetric space on which it acts this way.

Claim 83. The length minimizing path from i to $e^t i$ is on the line $\gamma(s) = e^s i = i\cdot\begin{pmatrix} e^{\frac{s}{2}} & \\ & e^{-\frac{s}{2}}\end{pmatrix}$, and the distance is $\mathrm{dist}(i, e^t i) = t$.

Proof. (Just a calculation...) □

Denote (temporarily) the geodesic flow on $U\mathbb{H}$ by G_t. Previous computation shows that

$$G_t(i,i) = (i\cdot g_t, ie^t).$$

There is an issue about the normalization but whats important is that it still pointing upwards. Since $SL_2(\mathbb{R})$ acts by isometries, it maps geodesics to geodesics. So

$$G_t(z,u) = G_t(i\cdot g, D_g(i)(\text{upward pointing vector}) = Image\left(G_t(i,i)\right)$$

So

$$G_t(z,u) = G_t\left(i\cdot g_t\cdot g, D_g(i\cdot g_t)(i)\right).$$

This is the same as the action of g_t on $PSL_2(\mathbb{R})$ on the left.

Part 3. Check some movies at `www.josleys.com`

Applying a mobius transformation to the straight line upward at i gives a semicircle which is perpendicular to the line $y = 0$. The geodesic flow is just moving along a geodesic.

The horocycle flow is just

There is an intersting interplay between these two flows, that is demonstrated by the equality

$$g_t h_s^- g_{-t} = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-t}s & 1 \end{pmatrix} = h_{e^{-t}s}^-$$

By rewriting $h_{e^{-t}s} g_t = g_t h_s^-$. So, if $x = h_{s_0}^- y$ then $g_t y = g_t h_{s_0}^- x = h_{e^{-t}s_0} g_t x$. Therefore,

$$\text{dist}\left(g_t y, g_t x\right) \le \text{ length of path } s \mapsto h_{e^{-t}s}^- g_t x \to 0.$$

Corollary 84. *$\{h_s^- x : s \in \mathbb{R}\} = \{y : dist(g_t y, g_t x) \to 0\}$(we proved $\subseteq$).*

Next step, suppose $\Gamma \subseteq G$ is discrete. Then

$$U\mathbb{H}/\Gamma = \{\pm \text{Id}\} \setminus PSL_2(\mathbb{R})/\Gamma$$

In general when one is dividing twice he get in to trouble, unless he does it once from each time. In this case, $\{\pm Id\}$ is a central subgroup so up to some technicalities,

$$U\mathbb{H}/\Gamma = U\left(\mathbb{H}/\Gamma\right)$$

Example 85. D is a regular n-gon with geodesic edges, right angles, embedded in $\mathbb{H}$, $n \ge 5$. (Such n-gons exist). For each side of D let γ be the rotation around its midpoint. This will give rise to a tiling of $\mathbb{H}$, invariant under the group Γ that is generated by those rotations. D is compact so $\mathbb{H}/\Gamma$is compact. $\mathbb{H} = SO_2(\mathbb{R})/SL_2(\mathbb{R})$ so Γ is compact in $SL_2(\mathbb{R})$.

$$UD = U\left(\mathbb{H}/\Gamma\right) = PSL_2\left(\mathbb{R}\right)/\Gamma$$

Example 86. I said that there might be some technicalities. This happens when your subgroup fixes points. This happens for example in $\mathbb{H}/PSL_2(\mathbb{R})$. This space is called the modular surface.

$$g_t \curvearrowright SL_2(\mathbb{R})/SL_2(\mathbb{Z})$$

is the geodesic flow. Notice: this surface is not a manifold. It has three punctures at the edges of the fundamental domain.

25.11

Part 1. For a general Riemanninan manifold which is geodesically complete (i.e. we can discuss geodesic flow), for each $x \in UM$ the set

$$\{y \in UM : d\left(g_t x, g_t y\right) \to 0\}$$

is called the stable horospherical leaf of x. In our setup, the horospherical leaves are orbits of h_s^- (h_s^+).

Pictures complete...

The space of lattices $X_d = SL_d(\mathbb{R})/SL_d(\mathbb{Z})$. A lattice in $\mathbb{R}^d$ is a disrecte cocompact subgroup ($\Longleftrightarrow$ discrete subgroup so that the quotient has finite volume), i.e., a group of the form

$$\Lambda = \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_d,$$

where $v_1, \ldots, v_d \in \mathbb{R}^d$ linearily independent (a fundamental domain is $\left\{\sum_{i=1}^d t_i v_i : t_i \in [0,1)\right\} = g\left([0,1)^d\right)$.

A lattice is called unimodular if the volume of the fundamental domain is 1 (This has a different meaning from the use we had to this word earlier). This happens $\iff |\det(g)| = 1$.

If Λ is a lattice in $\mathbb{R}^d$ and if $v_1, \ldots, v_d$ generate Λ (as a group), then the v_i are called a basis for Λ. Any two bases for unimodular lattices can be obtained from each other by applying a linear transformation of $\det = \pm 1$. Since replacing v_i by $-v_i$ does not change the lattice, we can assume $\det = 1$. This means that the space of unimodular lattices has a transitive action of $SL_d(\mathbb{R})$. The stabilizer of $\mathbb{Z}^d$ is $SL_d(\mathbb{Z})$. Therefore $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ parametrizes all unimodular lattices in $\mathbb{R}^d$. This is not just an abstract correspondence - the coset $gSL_d(\mathbb{Z})$ corresponds to the lattice $g\mathbb{Z}^d$ which is the image of $\mathbb{Z}^d$ under the linear map g acting from the left. This is just $\mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_d$ where $g = (v_1, \ldots, v_d)$ are the columns of g.

Example 87. $X_2 = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. We can easily interpret the standard fundamental domain Ω in this language is as follows: Let Λ be a lattice. By choosing unimodular lattices, we were taking one lattice from each equivalence class $\Lambda \sim t\Lambda$, $t \neq 0$ (homothety). Instead of using this convention, choose a representative whose shortest non zero vector has length 1. Applying a rotation by θ which maps v to $e_1 = (1, 0)$ maps the shortest vector v_2 such that v_1, v_2 are linearly independent, to Ω. Indeed, we can assume y coordinate of v_2 is positive. So $v_2 \in \mathbb{H}$. If the x coordinate of v_2 is not in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we can add an integer multiple of v_1 to v_2 to make it shorter, therefore $v_2 = (x, y)$, $x \leq \frac{1}{2}$. Since $\|v_2\| \geq \|v_1\| = 1$, necessarily $v_2 \in \Omega$. We have identified each lattice in $\mathbb{R}^2$ up to homothety with a pair (v_2, θ), $v_2 \in \Omega$. Can think of such a pair as an elemnt of $U\mathbb{H}$.

Next we will show that $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ has a finite $SL_d(\mathbb{R})$-invariant measure, i.e. $SL_d(\mathbb{Z})$ is a lattice in $SL_d(\mathbb{R})$. This can be done in many ways including by explicit calculation (find a fundamental domain and compute its Haar measure), but we will take an indirect approach.

Remark 88. Suppose $\Lambda_n \xrightarrow[n\to\infty]{} \Lambda_0$ and denote $\Lambda_n = \pi(g_n)$. This happens $\iff \exists \gamma_n \in SL_d(\mathbb{Z})$ such that $g_n\gamma_n \xrightarrow[n\to\infty]{} g_0 \iff$ for every $n \in \mathbb{N}_0$ $\exists$ a basis $v_1^n, ..., v_d^n$ of Λ_n such that $v_i^n \xrightarrow[n\to\infty]{} v_i^0 \iff \Lambda_0 = \{\lim v_n : v_n \in \Lambda_n \text{ a convergent sequence } \}$ is a lattice (in general this is not the case).

Reduction theory (= choice of fundamental domain).

Theorem 89. *Let $\Lambda \subseteq \mathbb{R}^d$ be a (not necessarily unimodular) lattice. Define*

$$\lambda_k(\Lambda) = \inf\{r > 0 : \Lambda \cap B(0, r) \text{ contains klinearily independent vector}\}$$

(i.e. $\lambda_k(\Lambda) = r \iff \bar{B}(0, r)$ contains k linearily independent vectors of Λ, but $\bar{B}(0, r')$ does not, for any $r' < r$). Then $\lambda_1(\Lambda) \leq ... \leq \lambda_d(\Lambda)$ and there are constants c_1, c_2 (depending only on d) and a basis $v_1, \ldots, v_d$ of Λ such that $c_1\lambda_i(\Lambda) \leq \|v_i\| \leq c_2\lambda_i(\Lambda)$ and

$$c_1 \leq \frac{\lambda_1(\Lambda)\cdots\lambda_d(\Lambda)}{covol(\Lambda)} \leq c_2.$$

Here, covol(Λ) is the measure of the fundamental domain for Λ in $\mathbb{R}^d$. These λ_ks are called the successive minima of Λ.

Remark 90. Given such a basis $v_1, ..., v_d$ if we let g be the matrix with colomuns v_i then $g\mathbb{Z}^d = \Lambda$, $\Lambda \cong gSL_d(\mathbb{Z})$. The set of such g's is a set of coset representatives, i.e., a choice of fundamental domain.

Remark 91. In general, if one choose v_1 to be shortest nonzero vector in Λ and v_{i+1} to be the shortest vector linearily independent of $v_1, ..., v_i$ (by induction), the resulting $v_1, \ldots, v_d$ will be linearily independent but not necessarily a basis:

Example 92. In $\mathbb{R}^5$, let $\Lambda = \text{span}_{\mathbb{Z}} \left\{ e_1, e_2, e_3, e_4, e_5, u = \frac{1}{2} \sum_{i=1}^{5} e_i \right\}$. Then, $\|u\| = \frac{1}{2}\sqrt{5} > 1$, but shortest vectors selected in the straightforward way will be $e_1, ..., e_5$.

Remark 93. There are many reduction algorithms Korkine-Zolotarev, Minkowski, LLL (the last is very useful in computer science). Our proof will involve Korkine-Zolotarev method for selecting the v_is.

Proof. $\lambda_1(\Lambda) \leq ... \leq \lambda_d(\Lambda)$ is clear by definition. Continue by induction. $d = 1$ - clear. Suppose we have proved the theorem for $d-1$ and $d-2$. Let v_1 be a shortest nonzero vector in Λ. Let $W = v_1^{\perp}$ be the space perpendicular to v_1, let $\pi : \mathbb{R}^d \to W$ be the orthogonal projection, and denote $\Lambda' = \pi(\Lambda)$. Then

$$\text{covol}\,(\Lambda') = \frac{\text{covol}\,(\Lambda)}{\|v_1\|}.$$

This is because a fundamental domain Ω' of Λ' can be enlarged to a fundamental domain for Λ by

$$\Omega = \left\{ w + tv_1 : w \in \Omega', t \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}.$$

Indeed, suppose you have a vector $x \in \mathbb{R}^d$, then by projecting it to W we find a representative $x' + w \in \Omega'$. Now $w = \pi(v)$ so there exist $t \in \mathbb{R}$ such that $x+v-w = tv_1$ and we can let n be the integer closest to t. We will use the successive minima of Λ' to approximate those of Λ. First claim that $\lambda_1(\Lambda') \geq \frac{\sqrt{3}}{2}\lambda_1(\Lambda)$. Suppose (by contradiction) that $w \in \Lambda'$ and $\|w\| < \frac{\sqrt{3}}{2}\|v_1\|$. Then $\exists v \in \Lambda$ such that $w = \pi(v)$. Replace v by $v + nv_1$, $n \in \mathbb{Z}$ to get $v = w + tv_1$, $|t| \leq \frac{1}{2}$. v_1 is perpendicular to w so by pythagoras $\|v\|^2 = \|w\|^2 + t^2\|v_1\|^2$. Therefore

$$\|w\|^2 = \|v\|^2 - t^2\|v_1\|^2 \geq \left(1 - t^2\right)\|v_1\|^2 \geq \frac{3}{4}\|v_1\|^2,$$

and taking square roots gives contradiction. Now claim that for $k = 1, \ldots, d-1$

$$\lambda_k(\Lambda') \leq \lambda_{k+1}(\Lambda) \leq c\lambda_k(\Lambda').$$

If $v_1, \ldots, v_{k+1}$ are linearily independent vectors in Λ with lengths at most $\lambda_{k+1}(\Lambda)$, then $w_i = \pi(v_i)$ contain at least k independent vectors of Λ', whose length is not larger than $\lambda_{k+1}(\Lambda)$. This proves the left hand side inequality. To prove the right side inequality, let $w_1 = \pi(v_2), ..., w_k = \pi(v_{k+1})$. Each w_i has length at most $\lambda_k(\Lambda')$. As before write $v_{i+1} = w_i + tv_1$,$|t| \leq \frac{1}{2}$.

$$\|v_{i+1}\| \leq \lambda_k(\Lambda')$$

....complete

By induction, $c_1(d-1) \leq \frac{\lambda_1(\Lambda')\cdots\lambda_d(\Lambda')}{\mathrm{covol}(\Lambda')} \leq c_2(d-1)$ this means that

$$c_1(d-1) \leq \frac{\lambda_1(\Lambda)\lambda_1(\Lambda')\cdots\lambda_d(\Lambda')}{\mathrm{covol}(\Lambda)} \leq c_2(d-1)$$

Changing lensts gives $c_1(d) \leq \frac{\lambda_1(\Lambda)\cdots\lambda_d(\Lambda)}{\mathrm{covol}(\Lambda)} \leq c_2(d)$.

The basis $v_1,\ldots,v_d$ in the last assertion is constructed by induction. v_1 is the shortest vector, and $v_2,...,v_d$ are such that $w_i = \pi(v_{i+1})$ are the basis chosen for Λ' such that $v_{i+1} = w_i + tv_1$, $|t| \leq \frac{1}{2}$. Write $a_i \asymp b_i$ if $\exists c_1, c_2$ such that

$$c_1 a_i \leq b_i \leq c_2 a_i$$

By induction hypothesis, $\|w_i\| \asymp \lambda_i(\Lambda') \asymp \lambda_{i+1}(\Lambda)$.

$$\|w_i\| \leq \|v_{i+1}\| \leq \|w_i\| + \frac{1}{2}\|v_1\| = O(\lambda_{i+1}(\Lambda)) + O(\lambda_1(\Lambda))$$

so $\|v_{i+1}\| \asymp \lambda_{i+1}(\Lambda)$. The v_i chosen in this way span Λ (ex...). □

2.12

Part 1.

THEOREM 94. *(Mahler's compactness criterion)*$A \subseteq G = SL_d(\mathbb{R})$, $B = \pi(A)$. *Then the following are equivalent:*

(1) $\overline{B}$ *is compact.*
(2) $\exists\delta > 0 \forall \Lambda \in B$, $\lambda_1(\Lambda) \geq \delta$.
(3) $\exists\delta > 0 \forall g \in A \forall v \in \mathbb{Z}^d \setminus \{0\}$, $\|gv\| \geq \delta$.

Equivalently, $\pi(g_n) = \Lambda_n \to \infty$ in $X_d \iff \lambda_1(\Lambda_n) \to 0 \iff \exists v_n \in \mathbb{Z}^d \setminus \{0\}$ $\|g_n v_n\| \to 0$.

PROOF. $2 \iff 3$ is obvious.

$1 \Rightarrow 2$: Suppose $\overline{B}$ is compact, and (by contradiction) there exists a sequence $\Lambda_n \in B_n$ such that $\lambda_1(\Lambda_n) \to 0$. There exists a convergent subsequence, denote it by Λ_n, $\Lambda_n \to \Lambda_0$. Let $0 < \varepsilon < \lambda_1(\Lambda_0)$. For all suffieciently large n, Λ_n contains a vector v_n of length less than ε. Multiplying v_n by an integer, we can assume $\|v_n\| \geq \frac{\varepsilon}{2}$. $v_n \in \Lambda_n$, $v_n \in B(0,\varepsilon)$. Passing to a subsequence, $v_n \to v$, $\frac{\varepsilon}{2} \leq \|v\| \leq \varepsilon$. But $v \in \Lambda_0$, contradiction.

$2 \Rightarrow 1$: Let $\Lambda_n \in B$ be a sequence, we need to show it has a convergent subsequence. Since Λ_n are unimodular, we have by Theorem 89

$$\delta^{d-1}\lambda_d(\Lambda_n) \leq \lambda_1(\Lambda_n)^{d-1}\lambda_d(\Lambda_n) \leq \lambda_1(\Lambda_n)\cdots\lambda_d(\Lambda_n) \leq c_2.$$

Therefore $\lambda_d(\Lambda_n) \leq \frac{c_2}{\delta^{d-1}}$. Let $v_1^n,\ldots,v_d^n$ be a basis to Λ_n with $\|v_i^n\| \in [c_1\lambda_i(\Lambda_n), c_2\lambda_i(\Lambda_n)]$ as in the Theorem 89, and $g_n = (v_1^n,\ldots,v_d^n) \in G$. $\det g_n = \pm 1$, by changing v_i^n with $-v_i^n$ we can assume that $\det g_n = 1$. all columns of g_n are bounded in norm by $c_2\lambda_d(\Lambda_n) \leq \frac{c_2^2}{\delta^{d-1}}$. So all matrices g_n have bounded entries, so by passing to a subsequence assume $g_n \to g$. Then $\Lambda_n \to \pi(g)$. □

Denote $K_\varepsilon = \{\Lambda \in X_d : \lambda_1(\Lambda) \geq \varepsilon\}$. Clearly $\bigcup_{\varepsilon>0} K_\varepsilon = X_d$. By Mahler's compactness theorem this is an exhaustion of X_d by compact sets, i.e. all $K_\varepsilon$are compact and any bounded subset of X_d is contained in some K_ε. ... (complete)

Part 2. $X \in \mathbb{R}^d$ is very well approximable (VWA) if $\exists \varepsilon > 0$ and sequences $p_n \in \mathbb{Z}^d$, $q_n \in \mathbb{N}$ such that

$$\left\| x - \frac{1}{q_n}\mathbf{p}_n \right\| < \frac{1}{q_n^{1+\frac{1}{d}+\varepsilon}}.$$

$X \in \mathbb{R}^d$ is very well multiplicatively approximable (VWMA) if $\exists \varepsilon > 0$ and $\mathbf{p}_n = (p_1^n, ..., p_d^n) \in \mathbb{Z}^d$, $q_n \in \mathbb{N}$ such that

$$\prod_{i=1}^{d} |q_n x_i - p_i^n| < \frac{1}{q_n^{1+\varepsilon}}.$$

X is badly approximable (BA) if $\exists c > 0$ such that for all $\mathbf{p}_n \in \mathbb{Z}^d$, $q_n \in \mathbb{N}$

$$\left\| x - \frac{1}{q_n}\mathbf{p}_n \right\| > \frac{c}{q_n^{1+\frac{1}{d}}}.$$

Both VWA and BA have lebesgue measure zero.

Remark 95. If one wish to approximate X by

$$\left\| x - \left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n} \right) \right\| < \frac{1}{\max q_i}$$

then one gets the one-dimensional problem.

Proposition 96. *VWA$\Rightarrow$ VWMA*

Proof. WLOG, we use the sup-norm. We have $\max_{i=1,\ldots,d} |q_n x_i - p_i^n| < \frac{1}{q_n^{\frac{1}{d}+\varepsilon}}$. Then $\prod_{i=1}^{d} |q_n x_i - p_i^n| < \frac{1}{q_n q_n^{d\varepsilon}}$.

Some notation:

$$g_t = \begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix}, \ , g_{\mathbf{t}} = \begin{pmatrix} e^{t_1} & & & \\ & \ddots & & \\ & & e^{t_d} & \\ & & & e^{-[\mathbf{t}]} \end{pmatrix} \ \tau(x) = \begin{pmatrix} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_d \\ & & & 1 \end{pmatrix}$$

where $[\mathbf{t}] = \sum_{i=1}^{d} t_i$, and

$$\Lambda_x = \pi\left(\tau(x)\right)$$

where $\pi : SL_{d+1}(\mathbb{R}) \to X_{d+1}$. □

Theorem 97 (Dani '86). *$x \in \mathbb{R}^d$ is BA_d $\iff$ $\{g_t \Lambda_x : t \geq 0\}$ is bounded in X_{d+1}.*

Proof. vectors in $g_t \Lambda_x$ are of the form $g_t \tau(x) v$, where $v \in \mathbb{Z}^{d+1}$. Write $v = (-\mathbf{p}, q)$ with $\mathbf{p} \in \mathbb{Z}^d$. Then

$$g_t \tau(x) v = \left(e^t \left(q x_1 - p_1\right), \ldots, e^t \left(q x_d - p_d\right), e^{-dt} q \right).$$

First show $\Leftarrow$: By Mahler's compactness criterion, $\exists \delta > 0$ such that $v \in \mathbb{Z}^{d+1} \setminus \{0\}$, $\|g_t \tau(x) v\| \geq \delta$. Let's use sup-norm and specialize to $v = (-p, q)$. WLOG $\delta < 1$. Choose t so that $e^{-dt} q = \frac{\delta}{2}$ $(t > 0)$. So $\max_{i=1,\ldots,d} \left| e^t \left(q_i x_i - p_i\right) \right| \geq \delta$. Divide by $e^t q$ and we get

$$\left\| x - \frac{1}{q}\mathbf{p} \right\|_\infty \geq \frac{\delta}{q e^t} = \frac{\delta}{q} \cdot \frac{\delta^{\frac{1}{d}}}{2^{\frac{1}{d}} q^{\frac{1}{d}}} = \frac{c}{q^{1+\frac{1}{d}}}.$$

$\Rightarrow$: $x \in BA_d \Rightarrow \exists c > 0$ such that $\left\| x - \frac{1}{q}\mathbf{p} \right\| \geq \frac{c}{q^{1+\frac{1}{d}}}$ for all $\mathbf{p} \in \mathbb{Z}^d$, $q \in \mathbb{N}$. Write $v = (-p, q)$. Suppose first that $q = 0$. Then

$$\left\| g_t \tau(x) v \right\| = e^t \left\| p \right\| \geq e^t \geq 1.$$

If $q \neq 0$ we can assume $q \in \mathbb{N}$, $t \geq 0$. If $e^{-dt} q \geq 1$, then ofcourse $\left\| g_t \tau(x) v \right\| \geq 1$. Else, $\Rightarrow e^{-dt} q \leq 1 \Rightarrow q^{-\frac{1}{d}} \geq e^{-t}$. We have

$$\left\| x - \frac{1}{q}\mathbf{p} \right\|_\infty \geq \frac{c}{q^{1+\frac{1}{d}}} \Rightarrow \exists i \ \left| q x_i - p_i \right| \geq \frac{c}{q^{1+\frac{1}{d}}} \geq c e^{-t}$$

hence $\exists i$ such that $e^t \left| q x_i - p_i \right| \geq c$. $\square$

Part 3. Dani's theorem is almost a trivial calculation, but it will be very profound. Suppose $d = 1$. $g_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \curvearrowright X_2 = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, $\Lambda_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \pi\left(h_x^+\right)$. $\{h_x^+ : x \in \mathbb{R}\}$ is a closed loop in the space of lattices. Dani's theorem says that the geodesic flow starting from Λ_x in direction i. The basic philosophy will be to track the flow of the whole closed loop. At each step we'll get a much longer loop, recall the relation $g_t h_x^+ g_{-t} = h_{e^{2t}x}^+$.

For general d,

$$\begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix} \begin{pmatrix} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_d \\ & & & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & & & \\ & \ddots & & \\ & & e^{-t} & \\ & & & e^{dt} \end{pmatrix} = \begin{pmatrix} \text{nothing} & \text{multiplied} \\ \text{multiplied by } e^{-(d+1)t} & \text{noth} \end{pmatrix}$$

THEOREM 98. *(Kleinbock-Margulis thm - "Dani's correspondence") $X \in VWA \Longleftrightarrow \exists \delta > 0$, $t_n \nearrow \infty$ such that $\lambda_1\left(g_{t_n}\Lambda_x\right) \leq e^{-\delta t_n} \Longleftrightarrow \exists \delta > 0$, $t_n \nearrow \infty$ such that $g_{t_n}\Lambda_x \notin K_{e^{-\delta t_n}}$. (infinitely many excursions at time t, outside $K_{e^{-\delta t_n}}$. Equivalent also to $d\left(x_0, g_{t_n}\Lambda_x\right) \geq c t_n$ for some $c > 0$ - a faster than linear escape)*

$X \in VWMA \Longleftrightarrow \exists \delta > 0$, $t_n \in \mathbb{Z}^d_{\geq 0}$, $\left\| t_n \right\| \to \infty$ such that $g_{t_n}\Lambda_x \notin K_{e^{-\delta[t_n]}} \Longleftrightarrow \exists \delta > 0$, $t_n \in \mathbb{R}^d_{\geq 0}$, $\left\| t_n \right\| \to \infty$ such that $g_{t_n}\Lambda_x \notin K_{e^{-\delta[t_n]}}$.

PROOF. Exercise (very similar to the proof of the original Dani's theorem) $\square$

THEOREM 99. *(Kleinbock-Margulis '98) Let $\mathcal{C} = \{\varphi(s) : s \in I\}$, where $I \subseteq \mathbb{R}$ is an interval and $\varphi : I \to \mathbb{R}^d$ is an analytic map such that $\mathcal{C}$ is not contained in an affine hyperplane. Then for a.e. s, $\varphi(s)$ is not $VWMA$.*

REMARK 100. In 1930's Mahler conjectured that for a.e. s, $\left(s, s^2, \ldots, s^d\right) \in \mathbb{R}^d$ is not VWA. Proved by Spirdzhuk in the 60's. Theorem of KM (resolving conjectures of Spirdzhuk and Alan Baker) extended to arbitrary curves and replaced VWA by VWMA.

REMARK 101. KM also discuss higher dimensional manifolds and relax the condition that φ is analytic. Assumption that $\mathcal{C}$ is not contained in an hyperplane is replaced by the assumption that at a.e. point the image of Taylor approximation of φ is not contained in an affine hyperplane for large enough degree. Such curves are called non-degenerate. The statement fails if $\mathcal{C}$ is contained in an affine hyperplane defined over $\mathbb{Q}$.

To prove KM theorem we will have to prove three theorems:

Theorem 102. *$\varphi : I \to \mathbb{R}^d$ as in Theorem 99. Then $\exists \tilde{C}, \alpha > 0$ such that for any $\varepsilon > 0$ and $t \in \mathbb{Z}^d_{>0}$ (this is true of course for any real vector but we need it this way),*

$$\left|\left\{s \in I : g_t \Lambda_{\varphi(s)} \notin K_\varepsilon\right\}\right| \leq |I| \cdot \tilde{C} \varepsilon^\alpha .$$

Reminder of the Borell-Cantelli lemma: μ is a finite measure, $A_1, A_2, \ldots$ measureable such that $\sum_{n=1}^\infty \mu(A_n) < \infty$. Then for a.e. x, $\#\{i : x \in A_i\} < \infty$.

Proof. (Of Theorem 102$\Rightarrow$Theorem 99) Fix $\delta > 0$, $\{\mathbf{t}_1, \mathbf{t}_2, \ldots\} \subseteq \mathbb{Z}^d_{\geq 0}$.Let $A_n = \left\{s \in I : g_{\mathbf{t}_n} \Lambda_{\varphi(s)} \notin K_{e^{-\delta[\mathbf{t}_n]}}\right\}$. By Theorem 102 we have $|A_n| \leq |I| \, \tilde{C} e^{-\delta[\mathbf{t}_n]}$. So

$$\sum_{n=1}^\infty \mu(A_n) \leq |I| \, \tilde{C} \sum_{n=1}^\infty e^{-\delta[\mathbf{t}_n]} \leq O\left(\sum_{t=1}^\infty t^{d-1} e^{-\alpha\delta t}\right) < \infty$$

where the last inequality is because there are $O\left(t^{d-1}\right)$ vector $\mathbf{t}_n$ with $[\mathbf{t}_n] = t$. Therefore by Borell-Canteli lemma $|\limsup A_n| = 0$. Taking $\delta_k \to 0$, get that

$$0 = \left|\left\{s : \exists k \exists^\infty n \; g_{\mathbf{t}_n} \Lambda_{\varphi(s)} \notin K_{e^{-\delta_k[\mathbf{t}_n]}}\right\}\right| = \left|\{s : \varphi(s) \in VWMA\}\right| .$$

$\square$

Definition 103. $f : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$ an interval, $C, \alpha > 0$. f is called (C, α)-good on J if $\forall \varepsilon > 0$

$$\frac{|\{s \in J : |f(s) \leq \varepsilon|\}|}{|J|} \leq C \left(\frac{\varepsilon}{\|f\|_J}\right)^\alpha$$

where $\|f\|_J = \sup_{s \in J} |f(s)|$. Add picture...

dDenote by $\mathcal{W}$ the space of rational linear subspaces of $\mathbb{R}^{d+1}$. So if $V \in \mathcal{W}$, $V \cap \mathbb{Z}^{d+1}$ is a lattice in V. Define $\ell_V : G \to \mathbb{R}$ by

$$\ell_V(g) = \operatorname{covol}\left(gV / g\left(V \cap \mathbb{Z}^{d+1}\right)\right)$$

i.e., $\ell_V(g)$ is the $\dim V$-dimensional volume of the image by g of the fundamental domain for $V / V \cap \mathbb{Z}^{d+1}$.

Theorem 104. *$\forall C, \alpha > 0 \exists C' > 0$ such that if $I \subseteq R$ is an interval, $h : I \to SL_{d+1}(\mathbb{R})$, $0 < \rho \leq 1$, $s_0 \in I$, I_0 is an interval $(s_0 - r, s_0 + r)$ so that $3I_0 \subseteq I$ and the following for all $V \in \mathcal{W}$:*

(1) *$\ell_V \circ h$ is (C, α)-good on $3I_0$.*
(2) *$\|\ell_V \circ h\|_{I_0} \geq \rho$.*

Then $\left|\left\{s \in I_0 : \Lambda_{h(s)} \notin K_\varepsilon\right\}\right| \leq C' \left(\frac{\varepsilon}{\rho}\right)^\alpha |I_0|$.

We will apply Theorem 104 $h(s) = g_t(\tau(\varphi(s)))$.

9/12

Part 1. Add picture and explantion about a curve not spending to much time in the cusp. Theorem 104 has many applications, among them the non-escape of mass. We first prove it implies Theorem 102.

Theorem 105. *Theorem 104 $\Longrightarrow$ Theorem 102*

Proof. Define $h(s) = g_t\tau(\varphi(s))$ (t is a vector). The conclusion of Theorem 104 implies the conclusion of Theorem 102 (expand the analytic map to $3I_0$). Therefore it is enough to verify the conditions of Theorem 104. Recall

$$\ell_V(g) = \mathrm{covol}\left(gV/g\left(V\cap\mathbb{Z}^{d+1}\right)\right)$$

and we compute it using the exterior algebra $\bigwedge_{i=1}^{k}\mathbb{R}^{d+1}$. This is a vector space of dimension $\begin{pmatrix} d+1 \\ k \end{pmatrix}$. If $e_1,\ldots,e_{d+1}$ is the standard basis for $\mathbb{R}^{d+1}$. $J = (j_1 < \ldots < j_k)$, $j_i \in \{1,\ldots,d+1\}$. The tensors $e_J = e_{j_1}\wedge\ldots\wedge e_{j_k}$ form a basis for $\bigwedge_{i=1}^{k}\mathbb{R}^{d+1}$. Extend $\wedge$ to $\bigoplus_{k=0}^{d+1}\bigwedge_{i=1}^{k}\mathbb{R}^{d+1}$ by demending:

1. $u\wedge v = -v\wedge u$ for $u,v\in\mathbb{R}^{d+1}$.
2. If $u_1,\ldots,u_\ell$ limearily independent then $u_1\wedge\ldots\wedge u_\ell = 0$.
3. $\forall a_1,a_2,u_1,u_2,v\ (a_1u_1+a_2v_2)\wedge v = a_1(u_1\wedge v)+a_2(v_2\wedge v)$.
4. If $g\in GL_{d+1}(\mathbb{R})$ then $g(u\wedge v) = gu\wedge gv$.
5. If $g\in GL_{d+1}(\mathbb{R})$ then $g(e_J) = ge_{j_1}\wedge\ldots\wedge ge_{j_k}$.
6. Choose an inner product which makes $\{e_J\}$ an orthonormal basis and take the resulting norm. Then if $v_1,\ldots,v_k$ generate $V\cap\mathbb{Z}^{d+1}$ then
$$\mathrm{covol}\left(gV/g\left(V\cap\mathbb{Z}^{d+1}\right)\right) = \left\|g\left(v_1\wedge\ldots\wedge v_k\right)\right\|.$$
7. A standard fact that the covolume of a lattice is the determinant of a matrix.

So back to our proof, if V is a rational subspace in $\mathbb{R}^{d+1}$ of dimension k such that $v_1,\ldots,v_k$ generate $V\cap\mathbb{Z}^{d+1}$. Then$\ell_V(g) = \left\|g\left(v_1\wedge\ldots\wedge v_k\right)\right\|$ Goal: find C,α,ρ such that 1 and 2 are satisfied.

Lemma 106. *If $f_1,\ldots,f_k$ are all (C,α)-good , $\lambda\in\mathbb{R}$. Then*

1. $\max|f_i|$ *is (C,α)-good.*
2. λf_i *is (C,α)-good.*
3. $\sqrt{f_1^2+\ldots+f_k^2}$ *is $(k^{\frac{\alpha}{2}}C,\alpha)$-good.*

Proof. 1 and 2 are trivial. For 3:

$$\begin{aligned}\left|\left\{s\in I:\sqrt{\sum f_i^2}<\varepsilon\right\}\right| &\leq \left|\left\{s\in I:\max|f_i|<\varepsilon\right\}\right| \\ &\leq C\left(\frac{\varepsilon}{\left\|\max|f_i|\right\|_I}\right)^{\alpha} \\ &\leq Ck^{\frac{\alpha}{2}}\left(\frac{\varepsilon}{\left\|\sqrt{\sum f_i^2}\right\|_I}\right)^{\alpha}\end{aligned}$$

where we have used $\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \sqrt{k}\,\|\cdot\|_\infty$. □

The following Lemma is called Remez Lemma from the 30's. KM proved it independently but then there was a question about the most efficient constant, and it appeared to be done by Remez.

Lemma 107. *$\forall d\exists C,\alpha$ such that for any interval $I\subseteq\mathbb{R}$, any polynomial $P\in\mathbb{R}[x]$ of $\deg P\leq d$ is $(C,\alpha)-$good on I.*

Proof. Let $\alpha = \frac{1}{d}$, $C = 2d(d+1)^{\frac{1}{d}}$. Write $b = |I|$, $\sigma = \frac{|\{s\in I:|P(s)|<\varepsilon\}|}{b}$. We want to show $\sigma \leq C\left(\frac{\varepsilon}{\|P\|_I}\right)^{\alpha}$. Choose $s_1, \ldots s_{d+1}$ with $s_{i+1}-s_i \geq \frac{\sigma b}{2d}$ and such that $|P(s)| \leq \varepsilon$. Then by Lagrange interpolation formula $P(s) = \sum_{i=1}^{d} P(s_i) \prod_{j\neq i} \frac{s-s_j}{s_i-s_j}$. By the triangle inequality

$$\begin{aligned} |P(s)| &\leq d\varepsilon \frac{b^d}{\left(\frac{\sigma b}{2d}\right)^d} = \varepsilon \frac{2^d d^{d+1}}{\sigma^d} \Longrightarrow \\ \sigma &\leq 2d^{1+\frac{1}{d}}\left(\frac{\varepsilon}{|P(s)|}\right)^{\frac{1}{d}} \leq C\left(\frac{\varepsilon}{\|P\|_I}\right)^{\frac{1}{d}}. \end{aligned}$$

To choose $s_1, \ldots s_{d+1}$ let $s_1 = \inf\{s : |P(s)| \leq \varepsilon\}$. Inductively $s_{i+1} = \inf\left\{s \geq s_i + \frac{\sigma b}{2d} : |P(s)| \leq \varepsilon\right\}$. If this procedure stops before we have $d+1$ points, we will have a cover of $\{s : |P(s)| \leq \varepsilon\}$ by less then d intervals of length $\frac{\sigma b}{2d}$ - contradiction to the definition of σ. □

We will continue the proof assuming h is a polynomial map instead of analytic. Remez Lemma have a nice extension which might be used here to analytic maps. Verifying condition 1: $\tau \circ \varphi$ is a matrix valued polynomial in 1 variable. $h(s) = g_t\tau(\varphi(s))$ is a matrix valued polynomial of degree independent of t. We have

$$\ell_V(h(s)) = \left\| g_t\tau(s)\left(\sum a_J e_J\right)\right\| = \left\|\sum a_J g_t \tau(\varphi(s)) e_{j_1\wedge\ldots\wedge j_k}\right\|$$

where $\sum a_J e_J$ is element of representing $V \cap \mathbb{Z}^{d+1}$. This is a (complete...) So the only thing we had to care about is that the degree of the polynomial is independent of t.

Verifying 2: Choose $\rho > 0$ such that for every affine hyperplane $\mathcal{L} \subseteq \mathbb{R}^d$, $\|d_{\mathcal{L}}(\varphi(s))\| \geq \rho$ where $d_{\mathcal{L}}(x) = \text{dist}(x, \mathcal{L})$. This exists by a compactness argument (the collection of subspaces is compact, but although the collection of affine hyperplanes is not, the collection of relevant affine hyperplanes, i.e., that intersect the curve, is compact). Let $V \in \mathcal{W}$, $V \cap \mathbb{Z}^{d+1}$ generated by (p_i, q_i), $i = 1, ..., k = \dim V$, $q_i \in \mathbb{Z}$, $p_i \in \mathbb{Z}^d$. WLOG we can assume $q_2 = \ldots = q_k = 0$.

Case: 1: $q_1 = 0$, i.e., $V \subseteq V_0 = \{(x_1, \ldots, x_d, 0)\}$. $g_t\tau(\varphi(s)$ uniformly expends all vectors in V_0. Thereore

$$\ell\left(g_t\tau(\varphi(s))\right) \geq \ell(\text{id}) \geq 1$$

V is represented by an integer in exterior $\bigwedge_{i=1}^{k} \mathbb{R}^{d+1}$.

Case: 2: $q_1 \neq 0$. $\|g_t\tau(\varphi(s))(v_1 \wedge \ldots \wedge v_k)\|$ (will appear on webpage). The idea is to decompose $\bigwedge_{i=1}^{k} \mathbb{R}^{d+1} = V^+ \oplus V^0 \oplus V^-$ to invariant subspaces for g_t. g_t expands all vectors in V^+. compute $\tau(\varphi(s))(v_1 \wedge \ldots \wedge v_k)$. It has a nonzero component in V^+. The size of this component can be computed explicitly, and it is $\geq d_{\mathcal{L}}(\varphi(s)) \geq \rho > 0$ for some affine hyperplane $\mathcal{L}$ (depending on V and $v_1, \ldots, v_k$). Since g_t expands V^+,

$$\|g_t\tau(\varphi(s))(v_1 \wedge \ldots \wedge v_k)\| \geq \rho$$

for all t.

□

Part 2. More applications of Theorem 104.

THEOREM 108. *(folklore)*$G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, $h_s^+ = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$. *Then there are no divergent trajectories for* $h_s^+ \curvearrowright G/\Gamma$.

THEOREM 109. *(Margulis '71)*$G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$ $\{u_s\}$ *is a one parameter unipotent subgroup of* G. *Then there are no divergent trajectories for* $u_s \curvearrowright G/\Gamma$.

This was a big theorem when proved, answering a question of Piastetsky-Shapiro. Margulis proved this theorem while he was less then 20! This was a key part in his proof of the arithmeticity theorem (and will play a role in our future proof that $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$) that says that any lattice comes from a certain algebraic construction, extension of scalars. By the time he was 21 he had more theorems than most of us will ever have. The proof of Margulis is using a crazy induction. For a long time no one understood why it works. One of the first was Dani who was able to repeat it in the more accurate following two theorem:

THEOREM 110. *(Dani '79) Same notation as in previous theorem. Then* $\forall x \in G/\Gamma \forall \delta > 0 \exists$ *a compact set* $K \subseteq G/\Gamma$ *such that*

$$\limsup \frac{1}{T} |\{s \in [0,T] : u_s x \in K\}| \geq 1 - \delta.$$

THEOREM 111. *(Dani 86')*$\forall n \exists \tilde{C}, \alpha \forall x \in SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \exists \rho > 0 \forall \{u_s\}$ *unipotent and all* $0 < \varepsilon < \rho$ $\forall T$

$$\frac{1}{T} |\{s \in [0,T] : u_s x \notin K_\varepsilon\}| \leq \tilde{C} \left(\frac{\varepsilon}{\rho}\right)^\alpha$$

This is an extremely uniform estimate. The only thing that depends on the starting point can be ρ. Also, it applies equally well to any unipotent flow.

Clearly Theorem111$\Longrightarrow$Theorem110$\Longrightarrow$Theorem 109 and we are going to prove

THEOREM 112. *Theorem 104*$\Longrightarrow$*Theorem 111*

PROOF. Let $x = \pi(g_0)$, $g_0 \in SL_n(\mathbb{R})$. Let $h(s) = u_s g_0$. We only need to verify the hypotheses for this choice.

$$\ell_V \circ h(s) = \|h(s)v_1 \wedge \ldots \wedge v_k\|$$

where $V \cap \mathbb{Z}^n$ generated by $v_1, \ldots, v_k$. $\{u_s\}$ = unipotent group = polynomial map in matrix entries, of degree $\leq n-1$. To see this, just think of the Jordan form. Since it is a one parameter group, we can put all the matrices in Jordan form simultanuously. Each block will have 1 on the diagonal. The exponent map will be a polynomial since the matrices are unipotent.

Each entry of $h(s)v_1 \wedge \ldots \wedge v_k \in \bigwedge_{i=1}^k \mathbb{R}^n$ is a polynomial of degree $\leq k(n-1)$. By Lemma , 1 is verified.

Now take $\rho = \rho(x)$, $\rho = \inf_{V \in \mathcal{W}} \ell_V(g_0)$. complete... □

Independent proof of Theorem 108 and Theorem 110 in dimension 2.

Add picture...

Think of $SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \cong U\mathbb{H}/\Gamma$. There are two kinds of nilpotent trajectories. One is periodic ("closes up") - sure it is not going to ∞. Otherwise, we can push the trajectory to the fundamental domain, and evetually it will return to $\Omega \cap \{|z| = 1\}$. If (by contradition) $h_x^+ \to \infty$. $\forall s \geq s_0$ projection of $h_s^+ x$ to $\mathbb{H}$

would miss $\Omega \cap \{|z| = 1\}$. So $\forall s \geq s_0$ h_s^+ is a horizontal line, therefore periodic - contradiction.

A theorem of hedlund from the 30's is that any trajectory is either periodic or... complete

Let draw $\ell_v(s) = \|h_s^+ g_0 v\|_\infty$,$v \in \mathbb{Z}^2$, for $v \neq 0$, v primitive (not equal to multiple of another vector in $\mathbb{Z}^2$)

$$\|h_s^+ g_0 v\|_\infty = \left\| \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_v + s y_v \\ y_v \end{pmatrix} \right\| = \max\{|x_v + s y_v|, |y_v|\}$$

Claim: $\nexists v_1, v_2$ primitive, $v_1 \neq \pm v_2$, and s_0 such that

$$\|h_{s_0}^+ g_0 v_1\|_\infty, \|h_{s_0}^+ g_0 v_2\|_\infty < \frac{1}{\sqrt{2}}.$$

If we did have such v_1, v_2 then $\|h_{s_0}^+ g_0 v_1\|_2, \|h_{s_0}^+ g_0 v_2\|_2 < 1$. Then the area of the parallelogram spanned by $h_{s_0}^+ g_0 v_1, h_{s_0}^+ g_0 v_2$ is less than 1. But $\det(h_s^+ g_0) = 1$ and the parallelogram spanned by v_1, v_2 has integer vertices so the area must be bigger than 1 - contradiction.

Proof of Theorem 3 (for $n = 2$): Draw a dotted line $y = \rho$ where $\rho > 0$, $\rho < \frac{1}{\sqrt{2}}$, all horizontal in are of height $\geq \rho$. At every fixed s, the set of values $\{\ell_v(s) : v \text{ vector as above}\}$ is discrete. So ρ exists. Let T_0 be a number such that $\ell_v(s) \geq \rho$ for $s = T_0$. (By claim). Let $v_1, \ldots, v_n$ be the set of v as above for which $\exists s \in [0, T]$ such that $\ell_v(s) < \varepsilon$.

$$\begin{aligned} \{s \in [0,T] : u_s x \notin K_\varepsilon\} &= \{s \in [0,T] : \exists V \ \ell_V(s) < \varepsilon\} \\ &\subseteq [0, T_0] \cup \bigcup_{i=1}^{n-1} \{s \in [0,T] : \ell_{v_i}(s) < \varepsilon\} \cup \{s \in [0,T] : \ell_{v_n}(s) < \varepsilon\}. \end{aligned}$$

Therefore,

$$|\{s \in [0,T] : u_s x \notin K_\varepsilon\}| \leq T_0 + \frac{\varepsilon}{\rho} \sum_{i=1}^{n-1} |\{s \in [0,T] : \ell_{v_i}(s) < \rho\}| + \frac{2\varepsilon}{\rho} |\{s \in [0,T] : \ell_{v_n}(s) < \rho\}| \leq T_0 + \frac{2\varepsilon}{\rho} T$$

Diving by T and taking $T \to \infty$ gives the result.

15.12

Part 1 - Proof of nondivergence.

Part 2 - Ergodicity and mixing. We spent time to get a very strong form of nondivergence. The proof of a much weak statement is not simpler.

G is lcsc, $(X, \mathcal{B}, \mu)$ is a probability space. $G \curvearrowright X$ preseving μ.

Definition 113. The action is ergodic if any set $A \in \mathcal{B}$ which is invariant (i.e. $\forall g \in G$ $gA = A$), $\mu(A) = 0$ or $\mu(A) = 1$.

Remark 114. Ergodicity is the natural notion of "indecomposability" in the measurable actions category. If not ergodic we can decompose $X = A \cap (X \setminus A)$, $\mu(A) > 0$, $\mu(X \setminus A) > 0$ and normalize the measure on each piece and study seperately.

Proposition 115. *Suppose X is second countable, $\mathcal{B}$ Borel σ algebra and suppose supp$\mu = X$ (any open subset of X has positive measure). If the action is ergodic, then almost every orbit is dense.*

Proof. Let $O_1, O_2, \ldots$ be open sets generate the topology. For each i let $U_i = GO_i = \bigcup_{g\in G} gO_i$. $\mu(U_i) \geq \mu(O_i) > 0$. Each U_i is invariant so $\mu(U_i) = 1$. $\Omega = \bigcap_{i=1}^{\infty} U_i$, $\mu(\Omega) = 1$. If $x \in \Omega$ then for each i, $x \in U_i = GO_i$ so $\exists g \in G$ such that $x \in gO_i$, i.e. $g^{-1}x \in O_i$. So orbit of x intersects every O_i, so is dense. □

For $G = \mathbb{R}$ or $G = \mathbb{Z}$ a stronger stetment is:

Theorem 116 (Birkhoff pointwise ergodic theorem). *If $\mathbb{R} \curvearrowright (X, \mathcal{B}, \mu)$, the action is ergodic and $f \in L^1(X)$, then for a.e. $x \in X$*

$$\frac{1}{T}\int_0^T f(t.x)dt \to \int_X f d\mu \tag{4}$$

In particular, if X is lcsc then a.e. $x \in X$ is generic, i.e. $\forall f \in C_c(X)$, 4 is satisfied.

A one page proof is in Hasselblat-Katok

Definition 117. The action is mixing if $\forall A, B \in \mathcal{B}$ and any $g_n \to \infty$

$$\mu(g_n A \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

Note that this is only interesting when G is noncompact.

Theorem 118. *If G is noncompact then mixing$\Rightarrow$ergodic*

Proof. If $A \in \mathcal{B}$ is invariant, $g_n \xrightarrow[n\to\infty]{} \infty$ in G . Then

$$\begin{aligned} \mu(A) &= \mu(A\cap A) = \mu(g_n A \cap A) \to \mu(A)\mu(A) = \mu(A)^2 \Rightarrow \\ \mu(A) &= \mu(A)^2 \Rightarrow \\ \mu(A) = 1 \quad &\text{or} \quad \mu(A) = 0. \end{aligned}$$

□

Example 119. $Tx = x + \alpha$ is adition mod 1, where $x \in \mathbb{R}/\mathbb{Z}$, $\alpha \notin \mathbb{Q}$. T is invertible map $\mathbb{S}^1 \to \mathbb{S}^1$ so we get an action of $\mathbb{Z}$:

$$k.x = T^k(x).$$

This action preserve Lebesgue measure.

Claim 120. T is ergodic but not mixing.

Proof. Note that this is an isometric action: For any $k \in \mathbb{Z}$ if $I \subseteq \mathbb{S}^1$ an interval then $k.I = T^k(I)$ is an interval of same length. Since α is irrational, orbit of 0 is dense (exercise. Hint: A closed subgroup of a Lie group is a Lie group, and in particular, the only infinite closed subgroup of $\mathbb{R}$ which is not discrete is $\mathbb{R}$). Hence (recall T is an isometry) any orbit is dense (when $T^{k_\ell}(0) \to x_0$, $T^{k_\ell}(y) \to x_0 + y$).

Let $A = B = (0, \frac{1}{3})$ (projected to $\mathbb{S}^1$). Let k_ℓ be a sequence such that $T^{k_\ell}(0) \to \frac{1}{2}$. $T^{k_\ell}(A)$ very close to $(\frac{1}{2}, \frac{5}{6})$. In particular for large ℓ , $T^{k_\ell}(A) \cap A = \varnothing$ contradicting mixing.

Suppose (by contradiction) that A is T-invariant $0 < \mu(A) < 1$. Let B be $\mathbb{S}^1 \setminus A$, and x, y be density points for A, B respectively, i.e. for any $\varepsilon > 0$ $\exists r$ such that for any interval I of length $< r$ containing a (resp. b), $\frac{\mu(I\cap A)}{\mu(I)} \geq 1 - \varepsilon$ (resp.

$\frac{\mu(I\cap B)}{\mu(I)} \geq 1-\varepsilon$). Aplly with $\varepsilon = \frac{1}{3}$. Let I be as in definition of density pt for a . Let k_ℓ be a sequence such that $T^{k_\ell}(a) \to b$. For large ℓ $T^{k_\ell}(I)$ is an interval containing b, also of length $< r$.

$$\frac{2}{3}\mu(I) \leq \mu(I\cap A) = \mu\left(T^{k_\ell}(I) \cap T^{k_\ell}(A)\right) = \mu\left(T^{k_\ell}(I) \cap A\right) = 1-\mu\left(T^{k_\ell}(I) \cap B\right) \leq \frac{1}{3}\mu(I)$$

contradiciton. Add picture... □

Goal: If $X = G/\Gamma$ where Γ is a lattice in G μ is G invariant

THEOREM 121 (Howe-Moore). *Suppose G is a simple connected Lie group with finite center. Γ is a lattice in G. $H \curvearrowright G/\Gamma$ is mixing (and ergodic) if $\bar{H}$ is not compact.*

Note: This is almost nescessary, minus the case that $G = \bar{H}$ is compact.

23.12

Part 1. Mixing for G-action $\Rightarrow$ Mixing for any subgroup $H \subseteq G \Rightarrow$ Ergodicity of any unbounded subgroup H. So mixing has a property that it passes to subgroups.

This is certainly not true for ergodicity. Take any $\mathbb{R} \curvearrowright X$ and any homomorphism $\mathbb{R}^2 \to \mathbb{R}$.

THEOREM 122 (Howe-Moore 1). *Suppose G is a simple connected Lie group with finite center. Then any ergodic action of G on a propabiblity space is mixing.*

COROLLARY 123. *Γ is a lattice in G. $H \curvearrowright G/\Gamma$ is mixing (and ergodic) if H is unbounded.*

PROOF. Let $X = G/\Gamma$, μ G-inv probability measure. G is transitive on X, hence ergodic . Now apply Theorem 122 to get that the G action is mixing. □

COROLLARY 124. *If $\Gamma < SL_2(\mathbb{R})$ is a lattice then the action of g_t and h_s^+ are ergodic.*

Associated with any action of G on a measure space is the Koopman representation: $G \curvearrowright L^2(X,\mu) = \left\{f : X \to \mathbb{C} : f \text{ is measureable}, \int_X |f(x)|^2 d\mu(x) < \infty\right\}$ by $(gf(x)) = f\left(g^{-1}x\right)$ (use the inverse to get a left action. Use $\mathbb{C}$ and not $\mathbb{R}$ to get ...)

This is a unitary representation, i.e. $\forall g \in G$ g is unitary. Indeed, $\forall f_1, f_2 \in L^2(X,\mu)$,

$$< gf_1, gf_2 >= \int_X f_1\left(g^{-1}x\right) f_2\left(g^{-1}x\right) d\mu(x) = \int_X f_1(x) f_2(x) d\mu(x) =< f_1, f_2 > .$$

Continuuity: complete...

Remark: we did not use finiteness of μ.

DEFINITION 125. Let $G \to U(\mathcal{H})$ be a unitary representation on a Hilbert space. A matrix coefficient is a map $G \to \mathbb{C}$ of the form $g \mapsto < gv_1, v_2 >$, $v_1, v_2 \in \mathcal{H}$.

THEOREM 126 (Howe-Moore 2). *Suppose G is a simple connected Lie group with finite center. Then any ergodic action of G on a Hilbert space $\mathcal{H}$ by unitary operators, without non-zero G-invariant vector. Then all matrix coefficient tend to zero as $g \to \infty$.*

Theorem 127 (The Mautner phenomenon). *$G, \mathcal{H}$ as above. suppose $g \in G$ not belonging to a conpact subgroup. Then any $v \in \mathcal{H}$ which is fixed by g is fixed by G.*

Remark 128. We will deal only with the case G is simple. There are analoguous theorems with weaker hypothesis on G. In particular, for semisimple Lie groups.

Reinterpretation of ergodicity and mixing in terms of the Koopman representation:

Exercise 129. G is ergodic on $(X, \mathcal{B}, \mu) \iff f \in L^2(X, \mathcal{B}, \mu)$ invariant $\Rightarrow f$ is a.e. constant.

G is mixing on $(X, \mathcal{B}, \mu) \iff \forall f_1, f_2 \in L^2(X, \mathcal{B}, \mu), < gf_1, f_2 > \rightarrow_{g \to \infty} \int f_1 d\mu \int f_2 d\mu$.

Note that $gA = A \Rightarrow \mu(gA \triangle A) = 0$ but not the converse. But it is the same in the definition of ergodicity. On the other hand, indicator functions are dense in L^2.

Howe-Moore 2 implies Howe-Moore 1. Let $L^2(X\mu)$ and take $\mathcal{H} = 1^{\perp} = L_0^2(X, \mu)$ to be the zero mean functions, where 1 is the constant function $f \equiv 1$. The orthogonal projection is $f \to f - \int_X f d\mu$. By ergodicity $\mathcal{H}$ has no nonzero G-fixed vectors. Now we can apply Theorem126 to get $\forall f_1, f_2 \in \mathcal{H} < gf_1, f_2 > \rightarrow_{g \to \infty} 0$. If $f_1, f_2 \in L^2(X, \mu)$, $\bar{f}_1, \bar{f}_2$ are the orthogonal projections on $\mathcal{H}$ then

$$\begin{aligned} < gf_1, f_2 > &= < g\left(\bar{f}_1 + \int f_1 d\mu\right), \bar{f}_2 + \int f_2 d\mu > \\ &= < g\bar{f}_1, \bar{f}_2 > + < g\int f_1 d\mu, \bar{f}_2 > + < g\bar{f}_1, \int f_2 d\mu > + < g\int f_1 d\mu, \int f_2 d\mu > \\ &= \int f_1 d\mu \int f_2 d\mu < g1, 1 >= \int f_1 d\mu \int f_2 d\mu \end{aligned}$$

□

Lemma 130 (Cartan KA^+K decomposition). *Any $g \in SL_d(\mathbb{R})$ can be written as $k_1 a k_2$ where $k_i \in SO_d(\mathbb{R})$ and $a \in A^+ = \{diag(e^{t_1}, \ldots, e^{t_d}) : \sum t_i = 0,\ t_1 \geq \ldots \geq t_d\}$ (This is a Weyl chamber. Finite center is used here).*

Proof. $g \cdot g^t$ is symmetric positive definite $((g \cdot g^t)^t = g \cdot g^t$. If $g \cdot g^t v = \lambda v$ then $\lambda < v, v >=< g \cdot g^t v, v >=< gv, gv >$. $\exists k \in SO_d(\mathbb{R})$ such that $kgg^t k^{-1}$ is diagonal, with eigenvalues arranged in decreasing order. $a = a_0^2$, $a_0 \in A^+$. Now we want to show $\exists \ell \in K$ such that $g = ka_0\ell$. Let $\ell = a_0^{-1}k^{-1}g$ and check:

$$\ell\ell^t = a_0^{-1}k^{-1}gg^t k a_0^{-1} = a_0^{-1}k^{-1}kak^{-1}ka_0^{-1} = a_0^{-1}aa_0^{-1} = e.$$

□

Two facts about Hilbert spaces:

- For every linear functional $f \in \mathcal{H}^* \ \exists! v = v_f$ such that $\forall v \in \mathcal{H}\ f(u) =< v, u >$.
- Weak-$*$ topology on H^*: $f_n, f \in \mathcal{H}^*$ say that $f_n \to f$ if $\forall v \in \mathcal{H}\ f_n(v) \to f(v)$. Banach-Alaoglu: The unit ball in $\mathcal{H}^*$ is compact with respect to the weak-$*$ topology.

Mautner implies Howe-Moore 2. Fix $u, v \in \mathcal{H}$ and let $g_n \in G$ $g_n \to \infty$. Using Cartan decomposition write $g_n = k_n a_n \ell_n$, $k_n, \ell_n \in K$, $a_n \in A^+$. Since $g_n \to \infty$ and K is compact, we have $a_n \to \infty$. Reduce the problem to the case $g_n = a_n$. Indeed, enough to show that for any subsequence there exists a subsequence such that the matrix coefficients tend to zero. So assume $k_n \to k$ and $\ell_n \to \ell$.

$$\begin{aligned} \left|< g_n v_1, v_2 > - < a_n \ell v_1, k^{-1} v_2 >\right| &= \left|< k_n a_n \ell_n v_1, v_2 > - < a_n \ell v_1, k^{-1} v_2 >\right| \\ &\leq \left|< a_n \ell_n v_1, k_n^{-1} v_2 > - < a_n \ell_n v_1, k^{-1} v_2 >\right| + \left|< a_n \ell_n v_1, k^{-1} v_2 > - < a_\end{aligned}$$

$$\begin{aligned} &\leq \|a_n \ell_n v_1\| \left\|k_n^{-1} v_2 - k^{-1} v_2\right\| + \|a_n \ell_n v_1 - a_n \ell v_1\| \left\|k^{-1} v_2\right\| \\ &= \|v_1\| \left\|k_n^{-1} v_2 - k^{-1} v_2\right\| + \|\ell_n v_1 - \ell v_1\| \|v_2\| \to 0. \end{aligned}$$

So now $g_n = a_n$ and we're looking on $< a_n u, v >$. Vectors $a_n u$ are vectors of fixed norm in $\mathcal{H}$. By facts 1 and 2 , after passing to a subsequence, exists $w \in \mathcal{H}$ such that $a_n u \to w$ in the weak-$*$, i.e. for all $v \in \mathcal{H}$ $< a_n u, v > \to < w, v >$. Enough to show $w = 0$. $\exists h \in SL_d(\mathbb{R})$ unipotent upper triangular $\neq e$ such that $a_n^{-1} h a_n \to e$.

$$< hw, v >=< w, h^{-1} v >= \lim < a_n u, h^{-1} v >= \lim < a_n^{-1} h a_n u, a_n^{-1} v >= \lim < u, a_n^{-1} v >= \lim < a_n u, v >=< u$$

This holds for all v, so w is fixed by h, i.e. $hw = w$. By Mauntner, w is fixed by G. By assumption , there are no nonzero G-invariant vectors so $w = 0$. $\square$

Lemma 131. *G is lcsc, $\mathcal{H}$ a Hilbert space, $\pi : G \to U\mathcal{H}$ a unitary representation.$v_0 \in \mathcal{H}$ is fixed by a subgroup L of G. Then v_0 is also fixed by any $h \in G$ such that for all $\delta > 0$ $B_G(h, \delta) \cap L B_G(e, \delta) L \neq \varnothing$.*

Remark 132. The hypothesis means (taking $\delta = \frac{1}{n}$) $\exists g_n \to h$, $\ell_n, \ell_n' \in L$ $\ell_n g_n \ell_n' \to e$. This is the same as $\exists g_n \to e$, $\ell_n, \ell_n' \in L$ $\ell_n g_n \ell_n' \to h$

Proof. $\|h v_0 - v_0\| \leftarrow \|\ell_n g_n \ell_n' v_0 - v_0\| = \left\|g_n \ell_n' v_0 - \ell_n^{-1} v_0\right\| = \|g_n v_0 - v_0\| \to \|v_0 - v_0\| = 0$ $\square$

Example 133. $G = SL_2(\mathbb{R})$, $L = \{h_s^+\}$, $h = g_t$, $t > 0$. Want to show that $\exists g_n \to e$, $\ell_n, \ell_n' \in L$ $\ell_n g_n \ell_n' \to h$. Let $g_n = \begin{pmatrix} \cos\left(\frac{1}{n}\right) & -\sin\left(\frac{1}{n}\right) \\ \sin\left(\frac{1}{n}\right) & \cos\left(\frac{1}{n}\right) \end{pmatrix}$. To prove that $\exists \ell_n, \ell_n' \in L$ $\ell_n g_n \ell_n' \to h$, enough to show that $\pi(h)$ is in the closure $\{L\pi(g_n) : n = 1, 2, \ldots\}$ in G/L.

of Mautner phenomenon. (Mautner actually proved a special case, we prove for $SL_n(\mathbb{R})$). For $G = SL_2(\mathbb{R})$ We want to show that for any $g \in G$ with $\{g^n\}$ unbounded, if g fixes v_0 then so does G.

Claim 134. Enough to prove for a conjugate of g. I.e., if $h = g_0 g g_0^{-1}$, g fixes v_0 then h fixes $g_0 v_0$. Since h has the property, G fixes $g_0 v_0$, in particular

Fact 135. *Any matrix $SL_2(\mathbb{R})$ conjugate to one of the following:*

1. $a = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t > 0$ *(hyperbolic)*
2. $u = \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *(parabolic)*
3. $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Therefore we only need to prove it for $g = a$ and $g = u$ (k does not satisfy the condition of the lemma). We have $a^{-n}ua^n \to e$. By the lemma, if a fixes v_0 so does u. We have also that $a^n u^- a^{-n} \to e$ so again by the lemma u^- fixes v_0.

EXERCISE 136. $\{u, u^-\}$ generate G.

Assume u fixes v_0 . Then a fixes h (by the lemma and the example).

The upshot is that G fixes v_0.

For $G = SL_n(\mathbb{R})$, let $g \in SL_n(\mathbb{R})$ and assume v_0 is fixed by g. Define two groups

$$U^{\pm} = \left\{ h \in G : g^{\mp n} h g^{\pm n} \to e \right\}.$$

Suppose first that g has at least one eigenvalue of abssulute value $\neq 1$. By $\det g = 1$ there are eigenvalues $|\lambda_1| > 1 > |\lambda_2|$. In the Jordan form of g look in the places correspond to these eigenvalues. Therefore $U^{\pm}$ are non-trivial.

EXERCISE 137. U^+, U^- generate a normal subgroup of G and hence generate G.

By the Lemma, any element of $U^{\pm}$ fixes v_0 so we are done in this case.

If g has all eigenvalues with absolute value $= 1$ and does not belong to a compact group. Again look on the Jordan form to find a block of the from...

FACT 138 (Jacobson-Morosov). *(Easy for $SL_n(\mathbb{R})$) There is a copy of $SL_2(\mathbb{R}) \hookrightarrow SL_n(\mathbb{R})$ which contains g.*

By proof for $SL_2(\mathbb{R})$, $SL_2(\mathbb{R})$ fixes v_0. $SL_2(\mathbb{R})$ contains an element g with eigenvalue with absolute value$\neq 1$. □

30.12.13

Part 1. Applications of Howe-Moore + Mautner.

$SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. Gauss, Hermite 1905, Siegel 1931, Borel-Harishchandra 1961. Proofs were made by finding a superset of the fundamental domain and prove that it has finite volume. The superset nowadays is called a Siegel set. To compute the volume exactly is much harder (also done by Siegel). We will not do any explicit calculation at all, we use Mautner phenomenon and Howe-Moore to prove:

THEOREM 139. *G is a semisimple real algebraic group defined over $\mathbb{Q}$, $\Gamma = G(\mathbb{Z})$. Then Γ is a lattice in G. More generally, Γ commensurable to $G(\mathbb{Z})$ i.e. $\Gamma \cap G(\mathbb{Z})$ is of finite index in Γ and in $G(\mathbb{Z})$. These lattices are called arithmetic lattices.*

PROOF. First assume $G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$. Let $u(t)$ be a unipotent one-parameter subgroup of G. $\mathcal{H} = L^2(G/\Gamma, \mu)$. μ is the measure induced on G/Γ by Haar measure on G. Since μ is invariant, $G \curvearrowright \mathcal{H}$ is a unitary action. By Mautner, any function in $\mathcal{H}$ which is $\{u(t)\}$ invariant is G invariant, i.e. a.e. constant. Let B be a ball in G/Γ , so $\bar{B}$ compact and $0 < \mu(B) < \infty$. Recall the notation: $\mathcal{W}$=\{rational subspaces of $\mathbb{R}^n$\} , for $V \in \mathcal{W}$, $g \in G$, $\ell_V(g) = \mathrm{covol}(gV/g(V \cap \mathbb{Z}^n))$. For any V , $g \mapsto \ell_V(g)$ is continuous and for any g, the set of values $\{\ell_V(g) : V \in \mathcal{W}\}$ is discrete. The set of values $\{\ell_V(g) : V \in \mathcal{W}\}$ depends only on $\pi(g)$, i.e. if $g_1 = \gamma g_2$, $\gamma \in \Gamma$ then $\{\ell_V(g_1) : V \in \mathcal{W}\} = \{\ell_V(g_2) : V \in \mathcal{W}\}$.

Therefore (since $\bar{B}$ is compact) $\exists\rho>0$ s.t. for every $g \in B$, $\inf\{\ell_V(g) : V \in \mathcal{W}\} > \rho$
By Kleinbock-Margulis $\exists\varepsilon > 0$ such that for all $T > 0$ and all $x \in B$

$$\frac{1}{T}\left|\{t \in [0,T] : u(t)x \notin K_\varepsilon\}\right| < \frac{1}{2}.$$

Let $f = \chi_{K_\varepsilon} \in \mathcal{H}$. Define $\underline{f}(y) = \liminf \frac{1}{n}\int_0^n f(u(t)y)dt$. If $y \in B$ then $\underline{f}(y) \geq \frac{1}{2}$. $\underline{f}$ is $\{u(t)\}$ invariant, because if $y_0 = u(s)y$

$$\begin{aligned}\left|\frac{1}{n}\int_0^n f(u(t)y)dt - \frac{1}{n}\int_0^n f(u(t)y_0)dt\right| &\leq \frac{1}{n}\int_0^n |f(u(t)y)dt - f(u(t+s)y)|\,dt \\ &\leq \frac{1}{n}\left[\int_0^s |f(u(t)y)|\,dt + \int_n^{n+s} |f(u(t)y)|\,dt\right] \\ &\leq \frac{2s}{n} \to 0.\end{aligned}$$

Claim 140. $\underline{f} \in \mathcal{H}$.

Proof. By Cauchy-Schwartz

$$\begin{aligned}\left\|\frac{1}{n}\int_0^n f(u(t)x)dt\right\|_2^2 &= \frac{1}{n^2}\int_0^n\int_0^n\int_{G/\Gamma} f(u(t_1)y)f(u(t_2)y)d\mu(y)dt_1dt_2 \\ &\leq \frac{1}{n^2}\int_0^n\int_0^n \|f\|_2^2\,dt_1dt_2 = \|f\|_2^2\end{aligned}$$

By Fatou:

$$\|\underline{f}\|_2^2 = \int_{G/\Gamma}\liminf\left[\frac{1}{n}\int_0^n f(u(t)y)dt\right]^2 d\mu(y) \leq \liminf\int_{G/\Gamma}\left[\frac{1}{n}\int_0^n f(u(t)y)dt\right]^2 d\mu(y) \leq \|f\|_2^2 < \infty.$$

$\underline{f} \in \mathcal{H}$, $\{u(t)\}$ invariant $\Rightarrow$ by Mautner, $\underline{f}$ is G invariant $\Rightarrow$ $\underline{f}$ is a.e. constant, $\underline{f}(y) \equiv c > 0 \Rightarrow \|f\|_2 = c\mu(G/\Gamma) \Rightarrow \mu(G/\Gamma) < \infty$. □

Sketch of the argument for the generalization: Need a Mautner theorem for a unipotent $\{u(t)\} \subseteq G$. Need a Kleinbock-Margulis theorem for G/Γ . There is an embedding $G \hookrightarrow SL_N(\mathbb{R})$ such that $\Gamma = G(\mathbb{Z}) = G \cap SL_N(\mathbb{Z})$. If we had $G/\Gamma \hookrightarrow SL_N(\mathbb{R})/SL_N(\mathbb{Z})$, then since Kleinbock-Margulis theorem is valid in $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ for all orbits of all unipotent groups, we have an estimate for the amount of time trajectories for $\{u(t)\}$ spend outside compact subsets of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. We will see that G/Γ is a closed orbit in $SL_N(\mathbb{R})/SL_N(\mathbb{Z})$. By general principles, orbit map of a closed orbit is proper, i.e. a pre-image of a compact subset of $SL_N(\mathbb{R})/SL_N(\mathbb{Z})$ in G/Γ is compact . Sketch of the proof: Recall that $GSL_N(\mathbb{Z})$ is closed in $SL_N(\mathbb{R})/SL_N(\mathbb{Z})$. The orbit map is when $G \curvearrowright X$ $x_0 \in X$

$$\begin{aligned} G/\mathrm{Stab}_G(x_0) &\to X \\ [g] &\mapsto gx_0\end{aligned}$$

In our case, $X = SL_N(\mathbb{R})/SL_N(\mathbb{Z})$, $x_0 = SL_N(\mathbb{Z})$, $G/\Gamma \to SL_N(\mathbb{R})/SL_N(\mathbb{Z})$ is the orbit map. Duality principle (complete diagram...). $Hx\Gamma$ is closed as an H-orbit in $G/\Gamma \iff Hx\Gamma$ is closed as a subset in $G/\Gamma \iff Hx\Gamma$ is closed as a Γ-orbit in $H \setminus G$. This is trivail since we're using the quotient topology. Applying it to our case, in order to prove $GSL_N(\mathbb{Z})$ is closed (as a G-orbit in $SL_N(\mathbb{R})/SL_N(\mathbb{Z})$) it suffices to show it is closed as a Γ-orbit.

FACT 141 (From algebraic groups). *Assumption that G is real algebraic semisimple defined over $\mathbb{Q}$ implies $G \setminus SL_N(\mathbb{R})$ is a variety defined over $\mathbb{Q}$, $SL_N(\mathbb{Z})$ action is via polynomials defined over $\mathbb{Q}$. Hence discrete $\Rightarrow$ closed.*

$\square$

Part 2. Recall the lattice point counting question answered by Margulis in his thesis (a translation to english appeared three years ago!). For $R > 0$, Γ a lattice in $SL_2(\mathbb{R})$, $\ell, p \in \mathbb{H}$, define

$$N(R) = \#\left(B(p,R) \cap \ell\Gamma\right).$$

Compute asymptotics of $N(R)$ as $R \to \infty$. We'll see that analoguously to the euclidean case, $N(R) \sim \frac{\text{area}(B(p,R))}{\text{area}(\mathbb{H}/\Gamma)}$.

Let $\Lambda \subseteq \mathbb{R}^2$ a lattice , $p, \ell \in \mathbb{R}^2$

$$\#\left(\Lambda + \ell \cap B(p,R)\right) \asymp \frac{\pi R^2}{\text{covol}\Lambda}$$

this is proved by Gauss. (complete...)

For hyperbolic metric and volume

$$\begin{aligned} \text{area}(B(p,R)) &= 4\pi\left(\sinh^2\left(\frac{R}{2}\right)\right) \asymp \pi e^R \\ \text{area}\left(d \text{ neighborhood of } B(p,R)\right) &\asymp \pi\left(e^{R+d} - e^{R-d}\right) = \pi e^R\left(e^d - e^{-d}\right) \end{aligned}$$

Therefore, from the same argument as in the euclidean case, for hyperbolic lattice counting problem, we get $\exists C_1, C_2$ such that $C_1 e^R \le N(R) \le C_2 e^R$. (Add drawing of on the canceletion argument used by Margulis...).

THEOREM 142. *Let Γ be a lattice in $SL_2(\mathbb{R})$, $\ell, p \in \mathbb{H}$. Then $N(R) \asymp \frac{area(B(p,R))}{covol(\Gamma)} \asymp \frac{\pi e^R}{covol(\Gamma)}$*

THEOREM 143. $g_t = \begin{pmatrix} e^{\frac{t}{2}} & \\ & e^{-\frac{t}{2}} \end{pmatrix}$, $K = \left\{ r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi] \right\}$, *$\Sigma = \mathbb{H}/\Gamma$, $p \in \mathbb{H}$ is any point $S(p,t) = \pi_\Sigma\left(\{z \in \mathbb{H} : d(p,z) = t\}\right)$ the sphere around p of radius t projected to Σ. ν_t is the uniform measure on $S(p,t)$, i.e.,*

$$\int_\Sigma f d\nu_t = \frac{1}{\pi} \int f\left(g_t r_\theta \pi_\Sigma(p)\right) d\theta.$$

Then $\nu_t \to vol$ weak- convergence, i.e., $\forall f \in C_c(\Sigma)$,*

$$\frac{1}{\pi} \int f\left(g_t r_\theta \pi_\Sigma(p)\right) d\theta \to \int_\Sigma f d\text{vol},$$

where vol is a normalized measure.

Part 3.

PROOF. (Theorem 143 $\Rightarrow$ Theorem 142) Let $\varepsilon > 0$, $f \ge 0$ with $\int_\Sigma f d\text{vol} = 1$. f is supported on the ball of radius ε around $\pi_\Sigma(\ell)$. $\tilde{f} = f \circ \pi_\Sigma$ (Bump function in $\mathbb{H}$, supported on ε-neighborhood of points). For any $x \in \Sigma$ let

$$\beta_R(x) = \#\text{ geodesics of length } \le R \text{ joining x to } \pi_\Sigma(p)$$

Equivalently, $\beta_R(x)$ is the cardinality of the fiber over x for $\pi_\Sigma|_{B(p,R)}$. Since $N(R) = \#\{\ell\Gamma \cap B(p,R)\} = \sum_{x\in\ell\Gamma\cap B(p,R)} 1$ we have

$$N(R-\varepsilon) \leq \int_{B(p,R)} \tilde{f}(x)d\mathrm{vol}(x) = \int_\Sigma f(x)\beta_R(x)d\mathrm{vol}(x) \leq N(R+\varepsilon).$$

Alternatively:

$$\int_\Sigma f(x)\beta_{R-\varepsilon}(x)d\mathrm{vol}(x) \leq N(R) \leq \int_\Sigma f(x)\beta_{R+\varepsilon}(x)d\mathrm{vol}(x). \tag{5}$$

Think of $\beta_R(x)d\mathrm{vol}(x)$ as continuous convex combination of integrals along spheres of radius t, where $0 \leq t \leq R$. I.e., $\beta_R(x)d\mathrm{vol}(x)$ is a convex combination of integrals w.r.t. ν_t . As $t\to\infty$, $\int fd\nu_t \to \int fd\mathrm{vol}$, by Theorem 143. So, as $R\to\infty$ RHS of 5 will be asymptotic to

$$\int_\Sigma fd\mathrm{vol}\cdot B(p,R+\varepsilon) = \int_\Sigma fd\mathrm{vol}\cdot \pi e^R.$$

LHS is asymptotic to $\int_\Sigma fd\mathrm{vol}\cdot B(p,R-\varepsilon)$. Any limit point of $\frac{N(R)\cdot\mathrm{area}(\Sigma)}{\mathrm{area}(B(p,R))} \in \left[\frac{\pi e^{R-\varepsilon}}{e^R}, \frac{\pi e^{R+\varepsilon}}{e^R}\right]$. □

Proof. (of Theorem 143)(complete heuristics...) Let $f \in C_c(\Sigma)$, $\tilde{f} = f\circ\pi_\Sigma$ as before, fix $\varepsilon > 0$. Let U be a neighborhood of e in G (complete from the course webpage...) □

Lemma 144 (Main point (waveront Lemma)). *For any neighborhood of e in G $\exists V \subseteq G$ such that $g_tVK \subseteq Ug_tK$ for all $t \geq 0$.*

This is important because: Think of VK as a small fatening of the infinitesimal circle.I.e., VK is the support of a bump function around infinitesimal circle. g_tK is a circle of radius t. The lemma says that a bump function around the infinitesimal circle, when pushed by g_t stays in a uniformly bounded neighborhood of $S(p,t)$.

6/1/14

Part 1. Additional lecture on 20.1.14 9-12 at schriber 08. Exercise sheet is completed (after some corrctions).

Today we will prove some elementary private case of Ratner's theorem.

Classification of invariant measures and orbit-closures for horospherical group actions.

Start with $G = SL_2(\mathbb{R})$, Γ a lattice in G. U the positive unipotent groups. $U \curvearrowright U\,(\mathbb{H}/\Gamma)$.

Theorem 145 (Hedlund '36). *If Γ is cocompact, then all U orbits on G/Γ are dense (i.e., the action is minimal). If Γ is non-uniform, then any U orbit is either dense or compact.*

Remark: This is not the case for the geodesic flow, where one can have very complicated orbit closures (add drawing...). (see exercise sheet).

Reminder: If $\Gamma = SL_2(\mathbb{R})$ (add drawing...)

Applications: Let Γ cocmpact lattice act on $\mathbb{R}^2$ by linear transformations. Then, for any $v \in \mathbb{R}^2 \setminus \{0\}$ its orbit closure is dense in $\mathbb{R}^2$. If Γ is non-uniform, any orbit on $\mathbb{R}^2$ is either dense or discrete. This becomes an application of Hedlund's Theorem if we use the duality principle. Note that $\mathbb{R}^2 \setminus \{0\} = U \setminus G$. U orbit of

$\pi(x) = x\Gamma$ is dense $\iff Ux\Gamma$ is dense in $G \iff \Gamma$ orbit of $Ux = p(x)$ is dense. Also, we need to show that a closed orbit for a countable group is discrete. By contradiction, assume $v\gamma_n \to v_0$. $\Rightarrow v_0 = v\gamma_0$. For any $v\gamma_1 \in v\Gamma_1$, $v\gamma_1 = v_0\gamma_0^{-1}\gamma_1 = \lim v\gamma_n\gamma_0^{-1}\gamma_1$. So any point is an accumulation point. Therefore this set is perfect and cannot be countable.

The following is a more difficult theorem which implies Theorem 145.

THEOREM 146 (Furstunburg '72). *If Γ is cocmpact lattice in $SL_2(\mathbb{R})$, then U action on $X = G/\Gamma$ is uniquely ergodic, i.e., the only U invariant probability measure on X is μ_X (coming from the Haar measure).*

Two arguments for deriving Hedlund's theorem (in the cocompact case) from Furstungburg.

(1) For any $x_0 \in G/\Gamma$ and any $f \in C(X)$,

$$\frac{1}{T}\int_0^T f\left(h_s^+ x_0\right) ds \to \int_X f dx \tag{6}$$

i.e., every point is generic. To this end, define a measure ν_T on X by

$$\int_X f d\nu_T = \frac{1}{T}\int_0^T f\left(h_s^+ x_0\right) ds.$$

Equation 6 is equivalent to the convergence $\nu_T \to \mu_X$ in the weak-* topology. Since the collection of all probability measures on a compact space is weak-* compact, ν_T has convergent subsequences. Enough to show (by unique ergodicity) that any convergent subsequence of ν_T converges to a U invariant measure. So, if $\nu_{T_n} \to \mu$, let $s_0 \in \mathbb{R}$. Need to show that μ is h_s^+ invariant. Indeed,

$$\left|\int_X f(x) d\mu(x) - \int_X f\left(h_s^+ x\right) d\mu(x)\right| = \left|\lim_{n\to\infty} \frac{1}{T}\int_0^T f\left(h_s^+ x_0\right) ds\right|.$$

Since $\mu_X(O) > 0$ for any open $O \subseteq X$, $\forall O$ can find $f \in C(X)$ which has support in O with $\int_X f d\mu_X > 0$. Since every orbit is generic, every orbit visits O (even with the right frequency).

(2) Show U action on X is minimal. Assume by contradiction that $\exists X_0 \subset X$ closed invariant non-empty. Take $x_0 \in X_0$, define ν_T as before. This is a probability measure supported on X_0 and X_0 is compact, so we can take a convergent subsequence $\nu_{T_n} \to \mu$ with μ supported on X_0. μ is U invariant therefore $\mu = \mu_X$ - contradiction. (Somehow, this is a softer argument which one can use even when not havong one parameter group, only amenable group).

THEOREM 147 (Dani '81). *If Γ is non-uniform, then U ergodic, U invariant measure on X is either μ_X or is the length measure (normalized) on a compact U orbit, i.e., if Ux_0 is compact, $Ux_0 = \{h_s^+ x_0 : s \in \mathbb{R}\} = \{h_s^+ x_0 : s \in [o, T]\}$ (justification by closed orbit iff orbit map is a homeomorphism onto its image) with $T > 0$, $x_0 = h_T^+ x_0$, T smallest positive such number, normalized measure ν on Ux_0 is*

$$\int_X f\nu = \frac{1}{T}\int_0^T f\left(h_s^+ x_0\right) ds.$$

THEOREM 148. *Γ non-uniform, any orbit Ux is generic for one of the U invariant measures as in previous theorem.*

Part 2. We will prove a generalization of Furstenberg's unique ergodicity theorem.

THEOREM 149 (Dani '86 - proof by Margulis using ideas in his thesis). Let G be a Lie group with finite center, let $a \in G$ be Ad-diagonalizable, let $H^+ = \{g \in G : a^{-n} g a^n \to e\}$. Let Γ be a cocompact lattice in G. Then the only H^+invariant probabillity measure on $X = G/\Gamma$ is μ_X (the G invariant measure).

Why does this implies Theorem 146? Take $G = SL_2(\mathbb{R})$, $a = \begin{pmatrix} \lambda & \\ & \frac{1}{\lambda} \end{pmatrix}$, $\lambda > 1$.

$$a^{-n} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} a^n = \begin{pmatrix} \lambda^{-n} & \\ & \lambda^n \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda^n & \\ & \lambda^{-n} \end{pmatrix} \cdots$$

i.e., in this case $H^+ = U$.

Terminology: $Ad : G \to Aut(\mathfrak{g})$ is the derivative of conjugation by g at the identity. a is Ad-diagonalizable if $Ad(a)$ is a diagonalizable linear operator on $\mathfrak{g}$. The group H^+ is called the expanding horospherical group for a.

EXAMPLE 150. $G = SL_3(\mathbb{R})$, $a = \begin{pmatrix} \lambda & & \\ & 1 & \\ & & \frac{1}{\lambda} \end{pmatrix}$, $\lambda > 1$,

$$a^{-1}(x_{ij})\, a = \begin{pmatrix} x_{11} & \lambda^{-1} x_{12} & \lambda^{-2} x_{13} \\ & x_{22} & \lambda^{-1} x_{23} \\ \lambda^2 x_{31} & & x_{33} \end{pmatrix}$$

therefore $H^+ = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$. The only possible other options in $SL_3(\mathbb{R})$ (up to what?) are $\begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}$.

Before we get to the proof, some

$$P^- = \left\{ g \in G : \left\{ a^n g a^{-n} \right\} \text{ is bounded } \right\}.$$

FACT 151. *a*

1. H^+, P^- *are groups.*
2. *They are complementary, in the sense that* $\dim H^+ + \dim P^- = \dim G$ *and* $H^+ \cap P^- = \{e\}$.
3. *There are neighborhoods* V_1, V_2 *of* e *in* P^-, H^+ *respectively such that*

$$\begin{aligned} V_1 \times V_2 &\to G \\ (p,h) &\mapsto ph \end{aligned}$$

   is a homeomorphism onto its image.
4. *Let* m_{H^+}, m_{P^-} *denote right (respectively left) Haar measure on* H^+, P^-. *Then the image of* $m_{H^+} \times m_{P^-}$ *under that map is* m_G.
5. $\forall x \in X$,

$$\begin{aligned} H^+ &\to X \\ h &\mapsto hx \end{aligned}$$

   is injective.

Proof. Ideas of... for 1, just note that $Ad(a)$ is an automorphism. For 2, note that

$$\begin{aligned} Lie\left(H^{+}\right) &= \bigoplus_{\lambda>0}\mathfrak{g}_{\lambda} \\ Lie\left(P^{-}\right) &= \bigoplus_{\lambda\leq0}\mathfrak{g}_{\lambda} \end{aligned}$$

where

$$\mathfrak{g}_{\lambda}=\left\{Y\in Lie(G):Ad(a)Y=e^{\lambda}Y\right\}.$$

For 3, compute rank of the map $P^{-}\times H^{+}$ to G at (e,e). By 2 this has a full rank. Then it follows from the implicit inverse function theorem. 4 is just a computation we omit. For 5, if $hx=x$, $h\neq e$ then since $a^{-n}ha^{n}\to e$ we have

$$z\leftarrow a^{-n}=a^{-n}ha^{n}a^{-n}x\to z$$

along a subsequence since X is compact. Therefore there does not exist a neighborhood of z such that the map

$$\begin{aligned} G &\to X \\ g &\mapsto gz \end{aligned}$$

is injective. This implies that the injectivity radius at z is zero which is a contradiction. □

In the context of 4, note that during this course we haven't done any computation using the Haar measure and the reason is that this is somewhat tricky. Here it is easier because the group has a lattice in it and therefore is unimodular.

Proof. (of Theorem 149) Let V_1,V_2 be as in 3. Let $B_0\subseteq H^{+}$, B_0 a neighborhood at e such that $m_{H^{+}}\left(\partial B_0\right)=0$, B_0 bounded. Define $B_n=a^{n}B_0a^{-n}$. We will show that for every $f\in C(X)$, for every $x\in X$:

$$\frac{1}{m_{H^{+}}\left(B_n\right)}\int_{B_n}f\left(ux_0\right)dm_{H^{+}}(u)\to\int_X fd\mu_X. \tag{7}$$

Why this is enough? Let μ be an H^{+} invariant measure. Then by Fubini and the dominated convergence theorem

$$\begin{aligned} \int_X fd\mu &= \frac{1}{m_{H^{+}}\left(B_n\right)}\int_{B_n}\int_X f\left(ux\right)d\mu(x)dm_{H^{+}}(u) \\ &= \frac{1}{m_{H^{+}}\left(B_n\right)}\int_X\int_{B_n}f\left(ux\right)dm_{H^{+}}(u)d\mu(x) \\ &\to \int_X\int_X fd\mu_X d\mu=\int_X fd\mu_X. \end{aligned}$$

Therefore $\int_X fd\mu=\int_X fd\mu_X$ for every $f\in C(X)$ so $\mu=\mu_X$.

To prove it, let $\varepsilon>0$. Enough to show that anny convergent subsequence of the LHS has a subsequence which converges to a limit which is within ε of the RHS. since X is compact , $f\in C(X)$, f is uniformly continuous. So there is $\delta>0$ such that $\forall y\in X\forall h\in G$ with $d_G(h,e)<\delta$,

$$|f(hy)-f(h)|<\varepsilon$$

(We may have to make δ even smaller). Let $V \subseteq P^{-}$ be small enough neighborhood of e so that $m_{P^{-}}(\partial V) = 0$. $d(a^n h a^{-n}) < \delta$ for any $n \in \mathbb{N}$, $h \in V$. Then

$$\frac{1}{m_{H^{+}}(B_n)\, m_{P^{-}}(a^n V a^{-n})} \int_{B_n} \int_{a^n V a^{-n}} f(hux)\, dm_{P^{-}}(h) dm_{H^{+}}(u) \tag{8}$$

is within ε of the LHS of 7. Since $B_n = a^n B_0 a^{-n}$

$$\frac{1}{m_G(B_0 V)} \int_{B_0 V} f\left(a^n g a^{-n} x\right) dm_G(x) \tag{9}$$

□

Since $H^{+} \ni h \mapsto hx$ is injective, making δ smaller we can assume by compactness that $B_0 V \ni g \mapsto gx$ is injective. Take a subsequence so that $a^{-n_k} x \to z$.

$$\left\| \chi_{B_0 V a^{-n_k} x} - \chi_{B_0 V z} \right\|_2 \to 0$$

(Since these sets have boundaries of measure zero).

$$\begin{aligned}
\frac{1}{m_G(B_0 V)} \int_X \chi_{B_0 V}(x) f\left(a^n g a^{-n} x\right) dm_G(x) &= \frac{1}{m_G(B_0 V)} \int_X f(a^n y)\, \chi_{B_0 V a^{-n} x}(y) dm_G(y) \\
&= \frac{1}{m_G(B_0 V)} < a^n f, \chi_{B_0 V a^{-n} x} > \\
&\asymp \frac{1}{m_G(B_0 V)} < a^n f, \chi_{B_0 V z} > \to \int_X f d\mu
\end{aligned}$$

where the last move is by mixing...

Remark 152. We used compactness of G/Γ few times during the proof. The most crucial point we needed it wasd to have a subsequence $a^{-n_k} x \to z$. The upshot is that the ptoof carries thorough unless x satsfies $a^{-n} x \to \infty$. It also carries thorough if we have $x' = hx$, $h \in H^{+}$ is small and $a^{-n} x'$ is bounded. Other places where we used compactness can be avoided by being more carefull. This is done very nicely in the Einsiedler-Ward book.

Same ideas prove the two mentioned Dani's theorems. To prove both theorems simultanuously. Idea: first prove that Equation7 is enough to show on a full measure set (in particular, only for x such that $a^{-n} x$ not bounded). Then prove that if $a^{-n} x \to \infty$ then Ux is compact. Now if μ is some measure not supported on a compact orbit, ergodic, then for a.e. x w.r.t μ.

Applicatinos to number theory: If Ux compact, $U = H^{+} \subseteq SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ then $a^{-n} x \to \infty$ (injectivity radius argument as before). But, $a^n x$ is bounded (possibly replacing x with x' near by x in Ux). Similar argument as before: $a^{-n} Ux$ gets equidistributed. So

$$\frac{1}{m_G(a^{-n} Ux)} \int_{a^{-n} Ux} f\left(a^{-n} ux\right) dm_U(u) - \int_X f d\mu \to 0$$

Theorem 153 (Sarnak '81). *Riemann hypothesis* $\iff \forall f \in C_c^{\infty}(G/\Gamma)\ \forall \varepsilon > 0$ *the rate of convergence is* $O\left(n^{-\frac{1}{2}+\varepsilon}\right)$.

13.1.14

Part 1. Upcoming events about the diagonal group:
Today 14:30 - Pankaj Vishe
Tomorrow 14:00 - Uri Shapira
Next week, same time, schriber 008.
Two applications of Ratner theorem:

Theorem 154 (Ratner's Theorems). *Let G be a connected Lie group, $\Gamma < G$ a lattice, H a connected subgroup generated by an Ad-unipotent elements. Then:*

(1) *Orbit closure theorem: $\forall x \in G/\Gamma$ there exists $L = L(H, x)$ connected subgroup of G with $H \subseteq L \subseteq G$ such that $\overline{Hx} = Lx$ and Lx has finite volume, i.e. L_x is a lattice in L. An orbit Lx as above is called a homogenuous subspace.*
(2) *Measure classification theorem: For any H-invariant, H-ergodic finite measure μ there exists $x \in G/\Gamma$, $L < G$ such that μ is the pushforaward of L-invariant measure on L/L_x under the orbit map. Such a measure is called a homogenuous measure.*
(3) *Genericity theorem: Suppose $H = \{h_t\}$ is a one-parameter Ad-unipotent group. $\forall x \in G/\Gamma$ there exists $\mu = \mu(H, x)$ homogenuous such that x is equidistributed with respect to μ, i.e., $\forall f \in C_c(G/\Gamma)$,*

$$\frac{1}{T}\int_0^T f(h_t x)\, dt \to \int_{G/\Gamma} f d\mu.$$

Definition 155. A matrix (linear transformation) is called unipotent if all its eigenvalues are equal to 1. An element g of a Lie group G is called Ad-unipotent if $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ is unipotent.

Remark 156. This definition depends on G.

Example 157. $g_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$, $G = SL_2(\mathbb{R})$. Eigenspaces for $\mathrm{Ad}(g)$ are $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and the corresponding eigenvalues are e^{2t}, e^{-2t}, 1 . For $g = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, with respect to the same basis - $\mathrm{Ad}(g) = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{pmatrix}$, therefore Ad-unipotent. Note that if we take G as $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, g_t is unipotent!

This cannot happen if G satisfies some condition, in particular:

Proposition 158. *If G is a semisimple linear algebraic group, $Ad(g)$ is unipotent $\iff$ g is a unipotent matrix.*

$L \subseteq G \curvearrowright G/\Gamma$, $x = \pi(g) \in G/\Gamma$. We have a map

$$\begin{array}{ccc} L & \to & G/\Gamma \\ \ell & \mapsto & \ell x \Gamma \end{array}$$

that factors through the stabilizer

$$L_x = \{\ell \in L : \ell x = x\},$$

i.e.,

$$\begin{array}{rcl} L/L_x & \hookrightarrow & G/\Gamma \\ \ell L_x & \mapsto & \ell x\Gamma. \end{array}$$

This is called the orbit map and it is always injective, and

Proposition 159. *The orbit is closed $\iff$ the orbit map is proper, i.e., a homeomorphism onto its image.*

Proposition 160. *Lx has finite volume (with respect to inner product on G/Γ induced by a right invariant Riemannian metric on G) $\iff$ L_x is a lattice in L.*

Note: $L_x = L \cap g\Gamma g^{-1}$. (In leterature, ratner theorem may be phrased analogu-osly using these propositions.

Example 161. $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, the only relevant one-parameter subgroup of G (for G itself the consequences of the theorems are not interesting) is $h_t^+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The theorem says that there exist L such that $H \subseteq L \subseteq G$ and the only options are $L \in \left\{H, P = \begin{pmatrix} * & * \\ & * \end{pmatrix}, G\right\}$. But as in the exercise sheet, P is not unimodular so cannot satisfy conditions (since by the theorem L must contain a lattice), therefore we have $L \in \{H, G\}$. Consequence:

(1) Any H-orbit is either closed or dense (Hedlund's theorem).
(2) Any H-invariant ergodic measure is either G-invariant on the entire space or length measure on a closed orbit (Dani's theorem).
(3) Any H-orbit is either periodic or equidistributed in G/Γ.

The correct strategy to prove the Ratner's theorem is $2 \to 3 \to 1$. The wanted result is the orbit closure theorem (and there is a current project of LMM to prove it directly withour using measure classification and benefit effectivness?).

Extensions:

(1) If H "contains many unipotents" but is not generated by them, analoguous of Ratner's theorem have been established: For example if $H = P$ by a theorem of Shah and Mozes.
(2) If H is discrete, but Zariski dense then analoges of 1,2 were established by Benoist-Quint very recently. Their work studied random walk on these spaces. Example: $A, B \in SL_2(\mathbb{Z})$ are large generic matrices (don't com-mute). Then generically, the group $< A, B >$ is a free Zariski dense in $SL_2(\mathbb{R})$ and one can apply Benoist-Quint.
(3) $H = A \subseteq SL_n(\mathbb{R})$, $n \geq 3$ is the most challenging remaining case. Con-jectures and partial results about analogues of Ratner's theorems for this case (EKL).

Part 2. Oppenheim conjecture15. The state of the art result before Margulis proved it was a result by Devenport-Ridout on '59 for $n \geq 21$. We will prove the stronger restatement of the oppenheim conjecture:

Theorem 162 (Dani-Margulis '89). *$\mathcal{P}$ = Primitive vectors in $\mathbb{Z}^n$. If Q is in-definite nondegenerate quadratic form which is not a scalar of a rational form then $Q(\mathcal{P})$ is dense in $\mathbb{R}$.*

Take $n = 3$, $Q(x) = x^2 + y^2 - z^2$ of signature $(p,q) = (2,1)$. The level set $Q = 0$ is just the standard cone. $Q = -1$ we get two sheets hyperbolid and for $Q = 1$ we get one-sheet hyperboloid. In this context (after a change of variables) think of $\mathbb{Z}^3$ as any lattice in $\mathbb{R}^3$. So the statement of the theorem is equivalent of having lattice points between any two level sets $Q^{-1}(a)$, $Q^{-1}(b)$ $a < b$. The group that is going to play a role is, let H be the component of e in

$$SO(Q) = \{g \in SL_n(\mathbb{R}) : \forall v \in \mathbb{R}^n,\ Q(v) = Q(gv)\} = \left\{g \in SL_n(\mathbb{R}) : g^{tr} B g = B\right\}.$$

Fact 163. *$SO(Q)$ acts transitively on each level set $Q^{-1}(c)$.*

Since some level sets of Q are disconnected, $SO(Q)$ is not connected.

Fact 164. *Some algebraic facts:*

(1) *H is semisimple (simple unless $(p,q) = (2,2)$.*
(2) *Generated by unipotents when $n \geq 3$, H is maximal i.e., $H \subseteq L \subseteq SL_n(\mathbb{R})$, L connected implies $H = L$ or $L = PSL_n(\mathbb{R})$.*
(3) *The centralizer of H in $GL_n(\mathbb{R})$ consists scalar matrices, i.e., if $A \in GL_n(\mathbb{R})$, $Ah = hA$ for every $h \in H$ then $A = \lambda I$ for some $\lambda \in \mathbb{R}^*$.*

This fact have geometric meaning also (think by yourself).
We will show two statements:

(1) For any $x \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ Hx is either closed or dense.
(2) If Hx_0 is closed, where $x_0 = \pi(e)$, $\pi : SL_n(\mathbb{R}) \to SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ then Q is a multiple of a rational form.

Proof of the theorem assuming these statements. By the assumptions, Q is not a multiple of a rational form, so by 2, Hx_0 is not closed therefore by statement 1 Hx_0 is dense, i.e., $H \cdot SL_n(\mathbb{Z})$ is dense in $SL_n(\mathbb{R})$. Given $z \in \mathbb{R}$, let $v \in \mathbb{R}^n$ such that $Q(v) = z$. Let $g \in G$ such that $ge_1 = v$. Let $h_k \in H$, $\gamma_k \in \Gamma$ such that $h_k \gamma_k \to g$.

$$z = Q(v) = Q\left(ge_1\right) = Q\left(\lim h_k \gamma_k e_1\right) = \lim Q\left(h_k \gamma_k e_1\right) = \lim Q\left(\gamma_k e_1\right)$$

and $\gamma_k e_1 \in \mathcal{P}$ since $e_1 \in \mathcal{P}$. ☐

Proof of 1: By fact 6 we can apply Ratner's orbit closure to get $\overline{Hx} = Lx$, $H \subseteq L \subseteq G$. Since H is maximal, $L = H$ or $L = G$ and we get that Hx is either closed or dense.

Proof of 2: Hx_0 is closed (since H is a connected component of $SO(Q)$). Let $\Delta = H \cap \Gamma$, let Q' be an indefinite nondegenerate form of the same signature. Claim that if $\Delta \subseteq SO(Q')$ then Q' is a multiple of Q. Indeed, let $\{u(t)\}$ be a one-parameter unipotent subgroup of H, and let $p \in \mathbb{R}^n$. Define

$$\begin{aligned} f_p : H &\to \mathbb{R} \\ f_p(h) &= Q'\left(h^{-1}p\right). \end{aligned}$$

Since $\Delta \subseteq SO\left(Q'\right)$ the map f_p factors through the quotient by Δ. Define $q(t) = f_p(u(t))$. $q : \mathbb{R} \to \mathbb{R}$ is a polynomial. By Margulis non-diveregence there is $K \subseteq G/\Gamma$ compact such that $\{t \in \mathbb{R} : u(t)x_0 \in K\}$ is unbounded. The orbit map $H/\Delta \hookrightarrow G/\Gamma$ (here we use Hx_0 is closed?) is proper, let K' be the preimage of K under the orbit map. Then K' is compact. $q(t) = f_p(u(t)) = \overline{f_p}\left(u(t)x_0\right)$, $u(t)x_0 \in K'$ for an unbounded set of t's. Therefore q is constant! $Q'(u(t)p) = Q'(p) \Rightarrow$ since this is true for all p we get $\{u(t)\} \subseteq SO(Q') \Rightarrow H \subseteq SO(Q')$. Now let B, B' be the matrices representing Q, Q' respectively, let and $h \in H$. Since $h \in SO(Q')$ we have

$$hB'B^{-1}h^{-1} = hB'h^{tr}h^{-tr}B^{-1}h^{-1} = B'B^{-1}$$

$\Rightarrow B'B^{-1}$ centrelizes any element of H. By fact 6 $B'B^{-1}$ is a scalar matrix $\Rightarrow B'$ is proportional to B. This proves the claim. Remains to prove that Q is a multiple of a rational form, i.e., B is a multiple of a rational matrix. Define

$$V = \left\{S \in M_n(\mathbb{R}) : \forall \gamma \in \Delta\ \gamma^{tr} S \gamma = S\right\}.$$

By the claim, all elements in V are multiples of B. V is defined over $\mathbb{Q}$ becuase each $\gamma \in \Delta$ imposes a $\mathbb{Q}$-linear condition. So V contains a rational matrix, concluding.

These are useful tricks for onee who is going to work in this field. We give a more conceptual idea of why these ideas of applying Ratner's theorem be present.

Geometrical explanation:

$\mathcal{Q}_{p,q} = SO(p,q) \setminus SL_n(\mathbb{R})$. We should think of this space as the space of quadratic forms of signature (p,q) up to scaling. We are intrested in the "set valued map" which maps $Q \in \mathcal{Q}_{p,q}$ to $Q(\mathbb{Z}^n)$. This map is from $\mathcal{Q}_{p,q}$ to a strange space (subsets of $\mathbb{R}$) factors through $SL_n(\mathbb{Z})$, i.e., to

$$\mathcal{Q}_{p,q}/SL_n(\mathbb{Z}) = SO(p,q) \setminus SL_n(\mathbb{R})/SL_n(\mathbb{Z}).$$

To understand this space we look at $SO(p,q)$ orbit on $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ and apply the duality principle.

Part 3.

Conjecture 165 (Littlewood's conjecturee '20s). *$\forall \alpha, \beta \in \mathbb{R}$, $\liminf n\langle n\alpha\rangle\langle n\beta\rangle = 0$ where $\langle x\rangle = dist(x, \mathbb{Z})$.*

Recall, $\alpha \notin BA$ means that $\forall \varepsilon > 0 \exists p, n$ such that $\left|\alpha - \frac{p}{n}\right| < \frac{\varepsilon}{n^2} \iff n\langle n\alpha\rangle \leq n|n\alpha - p| < \varepsilon$. Since $\langle x\rangle \leq \frac{1}{2}$ for every $x \in \mathbb{R}$, for any (α, β) with $\alpha \notin BA$ conjecture is true. So the conjecture is true for a.e. (α, β). The state of the art of results about this conjecture is:

Theorem 166 (Einsiedler-Katok-Lindenstrauss '05). *The pairs of (α, β) for which Littlewood's fails has Hausdorf dimension 0.*

Relation to homogeneous dynamics (will be introduced vaguley): Write

$$n\langle n\alpha\rangle\langle n\beta\rangle = n|n\alpha - p_1||n\beta - p_2|$$

and think of it as a special case of functions of the form $F = |L_1||L_2||L_3|$ where each L_i is a linear functional on $\mathbb{R}^3$. We are asking a question about values taken by such functions F on $\mathbb{Z}^3$. $G = SL_3(\mathbb{R})$ acts transitively on the space of products of 3 linear functionals up to scaling. This space is $A \setminus SL_3(\mathbb{R})$ where A is essentialy the diagonal group ($A \subseteq \mathrm{Stab}_G(F)$ where $F = |xyz|$. Infact there are additional finite index permutations of things you can do).

Conjecture 167 (Margulis). *Let $A \curvearrowright X = SL_3(\mathbb{R})/SL_3(\mathbb{Z})$.*

(1) *Any A-orbit which is bounded is compact.*
(2) *Any A-invariant measure is homogenuous.*

Proposition 168 (Margulis, following Cassels Swinnerton-Dyer in the '50s). *$1 \Rightarrow 2 \Rightarrow$ Littlewood's conjecture.*

Theorem 169 (EKL). *Any A-ergodic invariant measure on X which has positive entropy with respect to some $a \in A$ is homogenuous.*

20.1.14

Part 1. Gap distributions in $\{\sqrt{n} \bmod 1\}$. This sequence is equidistributed in $[0,1]$. This is since $\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\xrightarrow[n\to\infty]{}0$ and for any $k^2\le N<(k+1)^2$, $\frac{k^2}{N}\xrightarrow[N\to\infty]{}0$. Let $a_n=\left\{\sqrt{1}\right\}$, order it in this way $\{a_1,\ldots,a_n\}=\{b_1,\ldots,b_n\}$, $0\le b_1<\ldots<b_N$. A gap is $b_{i+1}-b_i$ for some $0\le i<N$. The average gap is of order $\frac{1}{N}$.

Problem 170. For each N we have N numbers $N\left(b_{i+1}-b_i\right)$ of average $\frac{1}{N}$. How are they distributed?

Fact 171. *If a_n are chosen randomly independently with uniform distribution in $[0,1]$ then $\frac{1}{N}\#\left\{i:N\left|b_{i+1}-b_i\right|\in(\alpha,\beta)\right\}\xrightarrow[N\to\infty]{}\int_\alpha^\beta e^{-t}dt$.*

Theorem 172 (Elkies and McMullen '02). *There are explicit analytic dunctions F_2,F_3 such that*

$$F(t)=\begin{cases}\frac{6}{\pi^2} & 0\le t\le\frac{1}{2}\\ F_2(t) & \frac{1}{2}\le t\le 2\\ F_3(t) & 2\le t\end{cases}$$

is continuous, positive, $\int_0^\infty F(t)dt=1$ and $\frac{1}{N}\#\left\{i:N\left|b_{i+1}-b_i\right|\in(\alpha,\beta)\right\}\xrightarrow[N\to\infty]{}\int_\alpha^\beta F(t)dt$.

Boshernitzan made computer experiment and predicted that the situation for $\sqrt{}$ is different than other roots. $F(t)$ is charachterized by $\int_\alpha^\beta F(t)dt=\mu\left(\{\mathcal{L}\in Y_2:\mathcal{L}\cap T(\sigma)\neq\varnothing\}\right)$ where $Y_2=SL_2(\mathbb{R})\ltimes\mathbb{R}^2/SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$ the space of unimodular grids and μ is the natural probability measure on Y_2, $T(\sigma)\subseteq\mathbb{R}^2$ triangle with vertices $(0,0)$, $(1,0)$, $(1,2\sigma)$ of area σ.

There is a proper map between Y_d to X_d.

Gap distribution for visible lattice points

Let $\mathcal{L}_0$ be a unimodular grid in $\mathbb{R}^d$, $d\ge 2$, let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ with normalized volume. Let $T,\sigma>0$ and for $v\in\mathbb{S}^{d-1}$ define $D_{T,\sigma,v}$ be the disk around v in $\mathbb{S}^{d-1}$ of radius $r=r(T,\sigma)\asymp T^{-\frac{d}{d-1}}$ chosen so that $\mathrm{vol}\left(D_{T,\sigma,v}\right)\cdot\mathrm{vol}\left(B(0,T)\right)=\sigma$. Let

$$N_{T,\sigma}(v)=\#\left\{u\in\mathcal{L}_0:\|u\|\le T,\ \frac{u}{\|u\|}\in D_{T,\sigma,v}\right\}$$

our normalization insures that

$$\int_{\mathbb{S}^{d-1}}N_{T,\sigma}(v)d\mathrm{vol}(v)=\mathrm{vol}\left(D_{T,\sigma,v}\right)\#\left(\mathcal{L}_0\cap B(0,T)\right)\sim\sigma.$$

Call $\mathcal{L}_0=\mathcal{L}+x$ rational grid, if $\exists q\in\mathbb{N}$ such that $qx\in\mathcal{L}$, otherwise, it is irrational.

Theorem 173 (Marklof-Strombergson '07). *$\exists c_1>0$ $\forall\mathcal{L}_0$ irrational grid,*

$$\lim_{T\to\infty}vol\left(\left\{v\in\mathbb{S}^{d-1}:N_{T,\sigma}(v)=r\right\}\right)=\mu\left(\{\mathcal{L}\in Y_d:|\mathcal{L}\cap\mathcal{C}(\sigma)|=r\}\right)$$

where $\mathcal{C}(\sigma)=\left\{x\in\mathbb{R}^d:x_1\in(0,1),\ \|(x_2,\ldots,x_d)\|\le x_1c_1\sigma^{\frac{1}{d-1}}\right\}$.

Note that $\mu\left(\{\mathcal{L}\in Y_d:|\mathcal{L}\cap\mathcal{C}(\sigma)|=r\}\right)$ can be expressed by

$$\int_\sigma^\infty\Phi_r(x)dx$$

for some positive continuous piecewise analytic function.

Free path length statistics for a "periodic Lorentz gas".

Fix a grid $\mathcal{L} \in Y_d$ and $\rho_0 > 0$ such that $0 \notin \mathcal{L}^{(\rho_0)} = \bigcup_{x\in\mathcal{L}} B(x,\rho_0)$. Fix $v \in \mathbb{S}^{d-1}$ and $\rho < \rho_0$ define $\tau = \tau(\mathcal{L}, v, \rho) = \inf\left\{t > 0 : tv \in \mathcal{L}^{(\rho)}\right\}$. Physical intuition: the probability of hitting a scatterer is inverse proportional to the surface area of a ball so thinking of v as chosen randomly, the expectation of τ should behave like $\frac{1}{\rho^{d-1}}$.

Theorem 174 (Marklof-Strombergson '07). *There is a piecewise analytic positive Φ such that for any irrational $\mathcal{L}$ for any $\xi > 0$*

$$\lim_{T\to\infty} vol\left(\left\{v \in \mathbb{S}^{d-1} : \rho^{d-1}\tau(\mathcal{L}, v, \rho) \geq \xi\right\}\right) = \mu\left(\left\{\mathcal{L}_0 \in Y_d : |\mathcal{L}_0 \cap \mathcal{D}(\xi)| = \varnothing\right\}\right),$$

where $\mathcal{D}(\xi) = \left\{x \in \mathbb{R}^d : x_1 \in (0,1),\ \|(x_2,\ldots,x_d)\| \leq \frac{1}{\xi^{d-1}}\right\}$, and it equals $\int_\xi^\infty \Phi(s)ds$. (In particular it does not depend the original grid)

Part 2. Common point in the proof of all three theorems is to set up a sequence of measures ν_T on a homogenuous space (in applications, Y_d). Show that the asymptotics that one wish to analyze follow from a result of the form $\nu_T \to \mu$ weak-* convegence. Similar problem comes up in the proof of Ratner's theorems and in the Dani-Smillie equidistribution theorems. 2⇒ 3 is modelled on the same argument used to prove Dani theorem: If $G/\Gamma = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, then any ergodic measure is either μ or is the length measure on a closed periodic orbit.

Theorem 175 (Dani and Smillie). *Any h_t orbit on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is either periodic or equidistributed.*

How to use Dani's theorem in order to prove Dani&Smille? Define $\nu_T = \frac{1}{T}\int_0^T f(h_t x)\,dt$. Goal: prove that $\nu_T \to \mu$ in weak-*. Step 1: Form the one-point compactification to establish non-escape of mass, i.e., one of the three equivalent statements:

(1) If $\overline{X} = SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \cap \{\infty\}$ and $\nu_{T_n} \to \nu$ then $\nu(\{\infty\}) = 0$.
(2) $\forall\varepsilon > 0\ \exists K \subseteq SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ compact such that $\forall T\ \nu_T(K) \geq 1-\varepsilon$ (This follows from the Dani-Margulis non-divergence).

Step 2: Since $\overline{X}$ is compact, the space of ptobability measures on $\overline{X}$ is also compact (in the weak-*). So any sequence (ν_{T_n}) has a convergent subsequence. Enough to show that any such convergent subsequence tends to μ.

Step 3: Suppose that $\nu_{T_n} \xrightarrow[n\to\infty]{} \nu$ and show $\nu = \mu$. It is easy to show that ν is h_t invariant. Using the measure classification result we know that ν is some convex combination of μ and additional measures on closed orbits. One needs to rule out positive mass of measures which are length measures on closed orbits. For this, prove that $\{h_t x : t \in [0,T]\}$ spends a negligible proportion of time near any specific closed orbit.

Idea of proof: $\rho^{d-1}\tau(\mathcal{L}, v, \rho) \geq \xi \iff \mathcal{L}$ has no points in the ρ neighborhood of the segment $\left\{tv : t \in \left[0, \frac{\xi}{\rho^{d-1}}\right]\right\}$ (up to boundary effects ignoring the caps) $\iff \mathcal{L}$ has no points in $R = R_{\rho,v,\xi} = O\left(\left[0, \frac{\xi}{\rho^{d-1}}\right] \times B^{d-1}(0,\rho)\right)$ where O is an orthoganal transformation that takes e_1 to $v \iff \mathcal{L}' = O^{-1}\mathcal{L}$ has no points in $\left[0, \frac{\xi}{\rho^{d-1}}\right] \times B^{d-1}(0,\rho) \iff \begin{pmatrix} \frac{\rho^{d-1}}{\xi} & & & \\ & \frac{\xi^{\frac{1}{d-1}}}{\rho} & & \\ & & \ddots & \\ & & & \frac{\xi^{\frac{1}{d-1}}}{\rho} \end{pmatrix} \mathcal{L}' \cap \mathcal{D}(\xi) = \varnothing.$

Choosing v at random corresponds to choosing O at random with respect to normalized Haar measure on $K = SO_d(\mathbb{R})$. Let ν_0 be the pushforward of Haar measure on K under the map $k \mapsto k\mathcal{L}$. Need to prove $g_t\nu_0 \to \mu$ and then by applying equidistribution to the indicator of $\{\mathcal{L}_0 : \mathcal{L}_0 \cap \mathcal{D}(\xi) = \varnothing\}$ we get what we wanted.

Idea for 2: Find a matrix $O \in K$ which rotates v to e_1. For $S = \left\{u \in \mathbb{R}^d : \|u\| \leq T,\ \frac{u}{\|u\|} \in D_{T,\sigma,v}\right\}$ find a diagonal matrix g_t which maps $O(S)$ to a fixed sector (independent of T, v). This sector turns out to be $C(\sigma)$. The theorem follows from $g_t\nu_0 \to \mu$ as before.

Idea for 1: We want

$$\frac{1}{N}\#\left\{i : N\left|b_{i+1} - b_i\right| \in (\alpha,\beta)\right\} \xrightarrow[N\to\infty]{} \mu\left(\{\mathcal{L} \in Y_2 : \mathcal{L} \cap T(\alpha) = \varnothing,\ \mathcal{L} \cap T(\beta) \neq \varnothing\}\right).$$

By Elkies and McMullen, it suffices to analyze (for x, t fixed, N parameter) the probability (when choosing $x \in [0,1]$ uniformly) that there is $0 \leq n < N$ such that $\{\sqrt{n}\} \in \left[x, x + \frac{t}{N}\right]$. I.e., $\iff \exists a \in \mathbb{Z}\ 0 \leq n < N$ such that $a + x \leq \sqrt{n} \leq a + x + \frac{t}{N}$. $\iff \exists a \in \mathbb{Z}\ 0 \leq n < N$ such that $(a+x)^2 \leq n \leq (a+x)^2 + \frac{2t}{N}(a+x) + \frac{t^2}{N^2}$. $\iff$ (up to boundary effects) $\exists a \in \mathbb{Z}\ 0 \leq n < N$ such that $(a+x)^2 \leq n \leq (a+x)^2 + \frac{2t}{N}(a+x) \iff$

$$2(a+x)x \leq n + x^2 - a^2 \leq 2(a+x)\left(x + \frac{t}{N}\right).$$

Let $b = n - a^2$ and note that $b \in \mathbb{Z}$ and therefore this is $\iff T \cap \mathbb{Z}^2 \neq \varnothing$ where

$$T = \left\{(a,b) : 2(a+x)x \leq b + x^2 \leq 2(a+x)\left(x + \frac{t}{N}\right)\right\}.$$

T is a triangle of area t. Let g be an affine map which maps T to a standard triangle T' of area t. T' is the triangle with vertices $(0,0), (1,0), (1,2t)$ to get $\iff$

$$T' \cap \Lambda_{x,t,N} \neq \varnothing$$

where $\Lambda_{x,t,N}$ is the image of $\mathbb{Z}^2$ under g. After doing explicit calculations, for fixed t and varying x between 0 and 1 the grids $\Lambda_{x,t,N}$ are described by $\begin{pmatrix} e^{t'} & \\ & e^{-t'} \end{pmatrix}\left[\begin{pmatrix} 1 & 2x \\ & 1 \end{pmatrix}, \begin{pmatrix} x^2 \\ x \end{pmatrix}\right]\mathbb{Z}^2$ where $t' \xrightarrow[N\to\infty]{} \infty$. $\left[\begin{pmatrix} 1 & 2x \\ & 1 \end{pmatrix}, \begin{pmatrix} x^2 \\ x \end{pmatrix}\right]\mathbb{Z}^2$ is a one-parameter unipotent group and it is a closed loop in Y_2. Therefore the diagonal flow equidistributes.