Homogeneous dynamics introduction talk

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1 Lie & lcsc groups; topology and Haar measure

• Informally, a smooth d-dimensional manifold \((G, \tau)\) is a topological space that locally looks like \(\mathbb{R}^d\) in such a way such that if we a map \(f : G \to \mathbb{R}^k\) then we can do calculus on the map \(f\) via this resemblance, "as if \(f\) was a map from \(\mathbb{R}^d\) to \(\mathbb{R}^k\)."

• Formally, \((G, \tau)\) is a smooth \(d\)-dim manifold if there exist an atlas \((U_\alpha, \varphi_\alpha)_\alpha\) where \(U_\alpha\) is a family of open sets with \(G = \bigcup_\alpha U_\alpha\), where \(\varphi_\alpha\) is a map \(\varphi_\alpha : U_\alpha \to \mathbb{R}^d\) which is a homeomorphism on its image for all \(\alpha\), and such that for each pair \(\alpha, \beta\) the composition \(\varphi_\beta \circ (\varphi_\alpha)^{-1}\) is a smooth map from a subset of \(\mathbb{R}^d\) to \(\mathbb{R}^d\).

• This also tells us that if we have two manifolds \(G_1\) and \(G_2\) an a map \(F : G_1 \to G_2\) we have a way of saying that \(F\) is smooth map between the manifolds using both atlases of \(G_1\) and \(G_2\).

• A Lie group is a smooth d-dimensional manifold, with the extra property that the group operations of multiplication and inversion are smooth maps. i.e. both of the maps below are smooth

\[
\begin{align*}
G \times G & \to G \quad g \mapsto g^{-1} \\
G & \to G \quad (g, h) \mapsto g \cdot h
\end{align*}
\]

(1) (2)

It is also a topological-group, where for topological group we only ask for these maps to be continuous.

• If \(G\) is a Lie group then \(G\) is a also a lcsc topological group – Hausdorff, locally-compact (every point \(g \in G\) has a compact neighbourhood) and second countable (topology admits a countable basis). So it is also separable.

• Most important example from now on: \(G = SL_d(\mathbb{R})\) with matrix multiplication, where \(SL_d(\mathbb{R})\) is the space of \(d \times d\) square matrices with determinant 1.
  – We give \(M_d(\mathbb{R})\) (not a group for multiplication) a topology via the natural identification with \(\mathbb{R}^{d^2}\).
  – So \(SL_d(\mathbb{R})\) has the the subspace topology induced from \(M_d(\mathbb{R})\). We have that:
    * \(SL_d(\mathbb{R})\) is a closed group – since det is continuous and \(SL_d(\mathbb{R}) = \text{det}^{-1}\{1\}\).
    * not bounded, so not compact. Take for example \(\{\text{diag}(1/t, t, 1, ..., 1) \mid t \in \mathbb{R}\}\).
    * path-connected – exercise. Use the fact that any \(g \in SL_d(\mathbb{R})\) can be written as \(g = RP\) where \(R\) is a rotation matrix and \(P\) is a symmetric & positive semi-definite.
  – \(SL_d(\mathbb{R})\) is a \((d^2 - 1)\) smooth manifold – exercise. Can use implicit function theorem applied on \(M_d(\mathbb{R})\) with det function.
  – It is a \((d^2 - 1)\) Lie group with respect to matrix multiplication.
• Motivation: find a measure on \( G \) which is nice – behaves nicely with respect to the topology and to the group operation. We call this kind of measure a right/left Haar measure on \( G \), which is a right/left \( G \)-invariant Radon measure.

• We do not claim that the measure would be both left and right invariant; a-priori these are two different measures. but as we shall see, for some groups they coincide.

1. Given a topological space \((G, \tau)\) we define the Borel \( \sigma \)-algebra to be the smallest \( \sigma \)-algebra which contains all the open sets of the topology.
   - Particularly, this means that open and closed sets are always measurable.
   - In our case, where \( G \) is second-countable, it is the \( \sigma \)-algebra which is generated by a countable basis of the topology.

2. \( \mu \) is a Radon measure if it is Borel (a measure on the Borel \( \sigma \)-algebra), if it is locally-finite, and if it is regular. this means that
   - if \( K \subseteq G \) is compact then \( \mu(K) < \infty \).
   - For any \( B \subseteq G \) Borel we have that \( \mu(B) = \inf\{\mu(U)|B \subseteq U \text{ open}\} = \sup\{\mu(K)|K \subseteq B \text{ compact}\} \).

3. right/left \( G \)-inv means that for all \( B \) Borel and every \( g \in G \) we have that
   \[
   \mu(B) = \mu(gB) \quad \text{(left)} \quad \mu(B) = \mu(Bg) \quad \text{(right)}
   \]

**Theorem 1.** Let \( G \) be an Lie group (or lcsc). Then

(a) there exists a right/left \( G \)-inv metric on \( G \) which induces the topology.
(b) there exists a right/left Haar measure on \( G \).
(c) Up to scalar multiplication, it is unique.
(d) \( \mu(G) < \infty \iff G \) is compact.
(e) If \( U \subseteq G \) is open then \( \mu(U) > 0 \).

- sketch of proof of (a),(b) for Lie groups:
  * if \( G \) is a Lie group then \( G \) admits a right/left \( G \)-invariant Riemannian metric which induces the topology of \( G \). (Roughly speaking, an inner product on tangent space)
  * Riemannian metric \( \Rightarrow \) right/left \( G \)-inv metric
  * Riemannian metric \( \Rightarrow \) volume form on \( G \).
  * Use Riesz representation theorem to define a Haar measure via this volume form.
- In total,
  * left Riemannian metric \( \Rightarrow \) left Haar measure.
  * right Riemannian metric \( \Rightarrow \) right Haar measure.

- \( G \) is unimodular if the right and left Haar measures of \( G \) coincide.

- Haar measure on \( G = SL_d(\mathbb{R}) \) – Cone construction. Given \( B \subseteq SL_d(\mathbb{R}) \) Borel define
  \[
  \text{Cone}(B) := \{tb|b \in B 0 \leq t \leq 1\} \subset M_d(\mathbb{R}) \approx \mathbb{R}^{d^2}.
  \]

Now define a measure on \( SL_d(\mathbb{R}) \) by
\[
\text{m}_{SL_d(\mathbb{R})}(B) = m_G(B) := vol_{M_d(\mathbb{R})}(\text{Cone}(B)).
\]
- the map \( h \mapsto (\det(h),\det(h)^{-1/d}h) \) is continuous \( \Rightarrow \text{Cone}(B) \) is the inverse image \([0,1] \times B\), so volume is well defined.
we have two one-to-one correspondences

\[ \text{Motivation: for an interesting subgroup } \Gamma \]

\[ \bullet \quad \text{Define (a tricky name)} \]

\[ \bullet \quad \text{claim: If } \]

\[ \quad \text{On both } \]

\[ \text{Gauss and Lagrange studied lattices already in the 18th century, while the study of } \]

\[ \text{quotients of Lie groups gained popularity later on.} \]

\[ \quad \text{On both } X_d \text{ and } \overline{X}_d \text{ we have a metric called the Chabauty-Fell metric. Although it is } \]

\[ \text{explicit – its difficult to use it, and it does not help us to construct nice measures.} \]
\[ -d_{CF}(L_1, L_2) := \inf \left\{ \varepsilon \mid \forall x \in L_1 \text{ with } x \in B(0, 1/\varepsilon) \exists y \in L_2 \text{ with } \|x - y\| < \varepsilon \right\} \]

- Claim: The topology induced from this metric coincide with the quotient topology of \( SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \), so in fact the correspondence from above is a homeomorphism.

- We keep this example in mind when we now try to give structure to arbitrary \( G/\Gamma \).

3 The quotient \( X := G/\Gamma \)

- From now on, \( G \) is lscs endowed with right invariant metric \( d_G \) and with right Haar measure \( \mu_G \). The reason why we take right is because we are going to work with right quotient \( G/\Gamma \) (with left cosets of the form \( g\Gamma \)).

- Assume also from now on that \( \Gamma \subseteq G \) is discrete, and define \( X := G/\Gamma \). For \( g \in G \) and \( g_0\Gamma = x_0 \in X \) we denote \( gx_0 := (gg_0)\Gamma \) the left action of \( G \) on \( X \).

- we say \( \Gamma \subseteq G \) is discrete if for all \( \gamma \in \Gamma \) there exists open \( U \subseteq G \) such that \( \Gamma \cap U = \{\gamma\} \). Since \( G \) is lscs, it is the same as asking that there exists an open \( U \) such that \( \Gamma \cap U = \{e\} \) (open mapping theorem for locally compact groups implies that for each \( g \in G \), multiplication by \( g \) is an open map).

- Let \( \pi : G \rightarrow X \) be the natural projection to the quotient defined by \( \pi(g) = g\Gamma \). Recall that if \( G \) is a topological space and \( \Gamma \) divides \( G \) into equivalence classes, we have the quotient topology on \( X \) where \( V \in X \) is open iff \( \pi^{-1}(V) \) is open in \( G \).

- Define a metric \( d_X(\cdot, \cdot) \) on \( X \) by

\[ d_X(g_1\Gamma, g_2\Gamma) := \inf_{\gamma_1, \gamma_2 \in \Gamma} d_G(g_1\gamma_1, g_2\gamma_2) = \inf_{\gamma \in \Gamma} \min_{\gamma \in \Gamma} d_G(g_1, g_2\gamma) \]

where the first equality follows from the fact that \( d_G \) is right-inv, and the second follows from the fact that \( \Gamma \) is discrete.

**Theorem 3.** We have the followings:

\( a \) The topology induced by the metric \( d_X \) coincide with the quotient topology. Furthermore, \( G \) and \( X \) are "locally isometric": for any \( x_0 \subseteq X \) there exists \( r = r(x_0) > 0 \) such that the following map is an isometry

\[ B_G(e, r) \rightarrow B_X(x_0, r) \quad g \mapsto g : x_0 \]

\( b \) \( g_n\Gamma \rightarrow g_0\Gamma \iff \exists \gamma_n \text{ such that } g_n\gamma_n \rightarrow g_0. \)

- In this case, we say say that \( X \) is a locally \( G \)-homogeneous space – a quotient space of \( G \) which locally looks the same everywhere.

- There is a bit of ambiguity in the literature regarding the definition of a "homogeneous space". The definition of a locally homogeneous space is fairly new, but it is more suitable due to Theorem 3.

- Motivation: construct a Borel measure on \( X \) which is invariant under the left \( G \)-action.

- We say \( \Omega \subseteq G \) is a fundamental domain of \( X = G/\Gamma \) (usually just say "fundamental domain") if \( \Omega \) is Borel, and if for all \( g \in G \) there exists a unique \( \gamma \in \Gamma \) such that \( g\gamma \in \Omega \).

\[ \iff \Omega \text{ is a choice of representative from each coset of } X, \text{ which is also measurable.} \]
Theorem 4. If $\Gamma \leq G$ is discrete there exists a fundamental domain $\Omega$ for $X = G/\Gamma$.

- proof
  
  * For $g \in G$ define $r_g > 0$ by $r_g := \sup \{ r > 0 | \pi_{|B_{G}(g, r)} \text{ is injective} \}$. This is well defined by discreteness of $\Gamma$. Indeed, if not then there exists $h_n \neq k_n$ with $h_n\Gamma = k_n\Gamma$ with $h_n \rightarrow g$ and $h_n \rightarrow g$
  
  $\implies \exists \gamma_n \in \Gamma$ with $k_n h_n^{-1} = \gamma_n$
  
  $\implies \gamma_n \rightarrow e$, cannot hold for since $\Gamma$ discrete.

  * Notice that if for some $g, k \in G$ we have that $d_G(g, k) < \frac{r_k}{r_k}$, then $r_g \geq \frac{r_k}{r_k}$.
  
  So we have $k \in B_G(g, \frac{r_k}{r_k})$.

  * Now take a countable dense sequence $(g_n)_{n=1}^\infty$ in $G$ (lcsc $\rightarrow$ separable), and denote for convenience $r_n := r_{g_n}$. and $B_n = B_G(g_n, \frac{r_n}{r_n})$.

  * So $G = \bigcup_{n=1}^{\infty} B_n$, and $\pi_{|B_n}$ is injective on for each $n$.

  Indeed, for $k \in G$ and by density, there exists $n$ such that $d_G(g_n, k) < r_k$
  
  $\implies k \in B_G(g_n, \frac{r_n}{r_n}) = B_n$.

  * Now define $\Omega := \bigcup_{n=1}^{\infty} B_n \setminus (\bigcup_{j<n} B_j \Gamma)$. check that $\pi_{|\Omega}$ is a bijection and that $\Omega$ is Borel.

Theorem 5. Let $\Omega_1, \Omega_2 \subset G$ be two fundamental domains of $X$. Then for every Borel $A \subset X$ we have $\mu_G(\Omega_1 \cap \pi^{-1} A) = \mu_G(\Omega_2 \cap \pi^{-1} A)$. In particular, $\mu_G(\Omega_1) = \mu_G(\Omega_2)$.

- Note that this is true for any right invariant measure on $G$, not only right Haar measure.

- proof

  * Notice that by definition of fundamental domains we have $\Omega_1 \cap \pi^{-1} (A) = \bigcup_{\gamma \in \Gamma} [\Omega_1 \cap \pi^{-1} (A)] \cap [\Omega_2 \cap \pi^{-1} (A)] \cdot \gamma$

  * By discreteness of $\Gamma$ and the fact that is lcsc, $\Gamma$ is countable. so we have $\mu_G(\Omega_1 \cap \pi^{-1} (A)) = \sum_{\gamma \in \Gamma} \mu_G([\Omega_1 \cap \pi^{-1} (A)] \cap [\Omega_2 \cap \pi^{-1} (A)] \cdot \gamma) = \sum_{\gamma \in \Gamma} \mu_G([\Omega_1 \cap \pi^{-1} (A)] \cdot \gamma^{-1} \cap [\Omega_2 \cap \pi^{-1} (A)]) = \mu_G(\Omega_2 \cap \pi^{-1} (A))$.

- Using theorem 2 and 3, we can finally construct a nice measure on the quotient.

Theorem 6. Assume $\mu_G(\Omega) < \infty$ for some (all) fundamental domain $\Omega \subset G$ of $X$. Then $G$ is unimodular.

Theorem 7. Assume that $G$ is unimodular, and let $\mu_G$ be the Haar measure on $G$.

(a) For Borel $A \subset X$ define

$$
\mu_X(A) := \mu_G(\pi^{-1}(A) \cap \Omega) \quad \text{if } \mu_G(\Omega) = \infty
$$

$$
\mu_X(A) := \mu_G(\Omega)^{-1} \mu_G(\pi^{-1}(A) \cap \Omega) \quad \text{if } \mu_G(\Omega) < \infty
$$

Then $\mu_X$ is a radon (probability) measure on $X$ which is invariant for the left $G$-action on $X$.

(b) Up to scalar multiplication, $\mu_X$ is the unique $G$-inv radon measure on $X$.

- proof

  * First notice that if $\Omega$ is a fundamental domain for $X$ then so does $g\Omega$, for all $g \in G$.

  * Now let $A \subset X$ Borel. Then by theorem 5 and left $G$-inv of $\mu_G$ we have

$$
\mu_X(gA) = \mu_G(\pi^{-1}(gA) \cap \Omega) = \mu_G(g \pi^{-1}(A) \cap \Omega)
$$

$$
= \mu_G(\pi^{-1}(A) \cap g^{-1} \Omega) = \mu_G(\pi^{-1}(A) \cap \Omega) = \mu_X(A).
$$
• For $\Gamma \leq G$ discrete we say $\Gamma$ is a lattice in $G$ if $X$ admits a left $G$-inv probability measure. For example, if $G$ is unimodular and there exists $\Omega$ a fundamental domain for $X$ with $\mu_G(\Omega) < \infty$.

• $\Gamma$ is called a uniform (or cocompact) lattice if in addition $X$ is compact. Otherwise, $\Gamma$ is called non-uniform.

• A sequence $(x_n)_{n=1}^{\infty}$ in $X$ is called divergent (notation $x_n \to \infty$) if the sequence eventually leaves any compact set; if for all $K \subseteq X$ compact we have $x_n \notin K$ for all sufficiently large $n$.

• Back to $G = SL_d(\mathbb{R})$. Define $\Gamma = SL_d(\mathbb{Z})$ (which is discrete), and define from on $X_d := G/\Gamma$. Recall that we have seen the correspondence $X_d := \{\text{Lattices in } \mathbb{R}^d \text{ covol = 1}\} \leftrightarrow SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \quad \text{g} \mathbb{Z}^d \leftrightarrow g SL_d(\mathbb{Z})$.

• We have seen that the measure $m_G$ (cone construction) is both left and right Haar measure on $G$, and that any lcsc group admits a fundamental domain. So by theorem 7 we have a radon left $G$-inv measure on $X_d$, which we denote by $m_{X_d}$. At this point we do not know weather it is finite or not.

• Motivation: although it is difficult to explicitly find a fundamental domain $\Omega \subset SL_d(\mathbb{R})$ of $X_d$, we want to show that $m_{X_d}$ is a probability measure.

• Notice that the names are a bit confusing with this correspondence – $g \mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$ of covolume 1 for all $g \in G$, and we also want to show that $\Gamma$ is a lattice in $G$; the correspondence is between the space of all lattices in $\mathbb{R}^d$ of covolume 1, and the quotient space $G/\Gamma$ when $\Gamma$ is a lattice in $G$.

Theorem 8. We have the following:

(a) **Siegel summation formula (SSF):** Let $f \in L^1(\mathbb{R}^d, \text{Vol})$. Then

$$\int_{\mathbb{R}^d} f(\vec{v}) \text{Vol}(\vec{v}) = \int_{X_d} \sum_{\vec{v} \notin L} f(\vec{v}) \ dm_{X_d}(L)$$

(b) $SL_d(\mathbb{Z})$ is a lattice in $SL_d(\mathbb{R})$.

- Proof of (b) assuming (a)
  
  * Let $f = 1_E$ where $E = [-2,2]^d$. $\text{Vol}(E) = 2^d$, so by Minkowski’s 1st theorem, any lattice $L \in SL_d(\mathbb{R})$ contains at least one non-zero point in $E$, so $\sum_{\vec{v} \notin L} 1_E(\vec{v}) \geq 1$ for all $L \in X_d$. Let $1_{X_d}$ be the indicator function of the all space $X_d$.
  
  * So we have

$$m_{X_d}(X_d) = \int_{X_d} 1_{X_d}(L) \ dm_{X_d}(L) \leq \int_{X_d} \sum_{\vec{v} \notin L} 1_E(\vec{v}) \ dm_{X_d}(L) = \text{SSF} \int_{\mathbb{R}^d} 1_E \text{Vol} \leq \text{Vol}(E) < \infty.$$

Theorem 9. **Mahler compactness criterion** – the following are equivalent

(a) A sequence of lattices $(g_n \mathbb{Z}^d)_{n=1}^{\infty} = (L_n)_{n=1}^{\infty}$ is divergent (eventually leaves any compact set)

(b) There exists a sequence $0 \neq \vec{v}_n \in L_n$ such that $\|\vec{v}_n\| \to 0$
4 actions of subgroups \( H \leq G \) on \( X \)

- \( G \) is a Lie group. \( H \leq G \) is closed, then \( H \) is also a Lie group.
- For \( H \leq G \) and for \( g_0 \Gamma = x \in X \), the stabilizer of \( x \) w.r.t the action of \( H \) on \( X \) is
  \[ \text{stab}_H(x) := \{ h \in H \mid hx = x \} = H \cap g_0 \Gamma g_0^{-1}. \]

So in particular, \( \text{stab}_H(x) \) is discrete.

- By the orbit-stabilizer Theorem, there is a 1-1 correspondence via the orbit map
  \[ Hx \rightarrow H/\text{stab}_H(x) \quad hx \mapsto h\text{stab}_H(x). \]

- This is only a 1-1 map in terms of sets, but this is not necessarily a homeomorphism.

- So now, after we have developed a good theory on quotients of discrete subgroups of Lie groups, we can try to use this theory to understand better the orbits \( Hx \).

**Theorem 10.** \( Hx \) is closed in \( X \) \iff the orbit map is a homeomorphism.

**Theorem 11.** Assume \( H \) is unimodular. If \( \Gamma \) is a lattice in \( G \) and \( \text{stab}_H(x) \) is a lattice in \( H \), then \( Hx \) is closed.

- Motivation: although \( Hx \) is not necessarily homeomorphic to \( H/\text{stab}_H(x) \), we want to use the quotient to put a measure on \( Hx \).

- Recall that for \( A \subseteq X \) we have defined
  \[ \mu_X(A) := \mu_G(\Omega \cap \pi^{-1}(A)) = \mu_G(\{ \omega \in \Omega \mid \omega \cdot e_X \in A \}). \]

- Let \( \mu_H \) be the right Haar measure on \( H \), \( \Omega_H \) a fundamental domain for \( H/\text{stab}_H(x) \).

- If \( H \) is unimodular, then we can define a "new" measure on \( X \) by
  \[ \text{vol}_{Hx}(A) := \mu_H(\{ \omega \in \Omega_H \mid \omega x \in A \}). \]

- Claim: \( \text{vol}_{Hx} \) is Borel and left \( H \)-inv.

- But it is not necessarily Radon because it is not necessarily locally finite.

- Notice that since \( H \) is unimodular, there is a unique left \( H \)-inv Radon on \( H/\text{stab}_H(x) \).

- So if \( Hx \) is closed and Theorem 10 holds, \( \text{vol}_{Hx} \) is locally finite, hence the unique Radon left \( H \)-inv measure on \( X \).

- Let \( U = \{ u_t \mid t \in \mathbb{R} \} \leq G \) be a one-parameter subgroup of \( G \).

- Let \( \nu_X \) be a probability measure on \( X \) which is left \( U \)-inv.
  We say \( \nu_X \) is homogenous (or algebraic) if there exists a closed & connected & unimodular subgroup \( H \) with \( U \leq H \leq G \) and there exists \( x \in X \) with \( Hx \) a closed orbit, such that \( \nu_X = \text{vol}_{Hx} \), where in this case \( \text{vol}_{Hx} \) is the unique left \( H \)-inv Radon measure on \( X \).

- In the following talks will study some Ergodic theory, and discuss Ratner theorem’s which deals with the classification of homogeneous measures.