Homogeneous dynamics introduction talk

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1 Lie & lcsc groups; topology and Haar measure

- Informally, a **smooth d-dimensional manifold (G, τ)** is a topological space that locally looks like $\mathbb{R}^d$ in such a way such that if we a map $f : G \to \mathbb{R}^k$ then we can do calculus on the map f via this resemblance, "as if f was a map from $\mathbb{R}^d$ to $\mathbb{R}^k$".
- Formally, (G, τ) is a smooth d-dim manifold if there exist an atlas $(U_\alpha, \varphi_\alpha)_\alpha$ where U_α is a family of open sets with $G = \cup_\alpha U_\alpha$, where φ_α is a map $\varphi_\alpha : U_\alpha \to \mathbb{R}^d$ which is a homeomorphism on its image for all α, and such that for each pair α, β the composition $\varphi_\beta \circ (\varphi_\alpha)^{-1}$ is a smooth map from a subset of $\mathbb{R}^d$ to $\mathbb{R}^d$.
- This also tells us that if we have two manifolds G_1 and G_2 an a map $F : G_1 \to G_2$ we have a way of saying that F is smooth map between the manifolds using both atlases of G_1 and G_2.
- a **Lie group** is a smooth d-dimensional manifold, with the extra property that the group operations of multiplication and inversion are smooth maps. I.e. both of the maps below are smooth

$$G \to G \qquad g \mapsto g^{-1} \tag{1}$$

$$G \times G \to G \qquad (g, h) \mapsto g \cdot h \tag{2}$$

  It is also a **topological-group**, where for topological group we only ask for these maps to be continuous.
- If G is a Lie group then G is a also a lcsc topological group – Hausdorff, locally-compact (every point $g \in G$ has a compact neighbourhood) and second countable (topology admits a countable basis). So it is also separable.

- Most important example from now on: $G = SL_d(\mathbb{R})$ with matrix multiplication, where $SL_d(\mathbb{R})$ is the space of $d \times d$ square matrices with determinant 1.
  - We give $M_d(\mathbb{R})$ (not a group for multiplication) a topology via the natural identification with $\mathbb{R}^{d^2}$.
  - So $SL_d(\mathbb{R})$ has the the subspace topology induced from $M_d(\mathbb{R})$. We have that:
    * $SL_d(\mathbb{R})$ is a closed group – since det is continuous and $SL_d(\mathbb{R}) = \det^{-1}\{1\}$.
    * not bounded, so not compact. Take for example $\{diag(1/t, t, 1, ..., 1) \mid t \in \mathbb{R}\}$.
    * path-connected – exercise. Use the fact that any $g \in SL_d(\mathbb{R})$ can be written as $g = RP$ where R is a rotation matrix and P is a symmetric & positive semi-definite.
  - $SL_d(\mathbb{R})$ is a $(d^2 - 1)$ smooth manifold – exercise. Can use implicit function theorem applied on $M_d(\mathbb{R})$ with det function.
  - It is a $(d^2 - 1)$ Lie group with respect to matrix multiplication.

- Motivation: find a measure on G which is nice – behaves nicely with respect to the topology and to the group operation. We call this kind of measure a **right/left Haar measure on G**, which is a **right/left G-invariant Radon measure**.
- We do not claim that the measure would be both left and right invariant; a-priory these are two different measures. but as we shall see, for some groups they coincide.

1. Given a topological space (G, τ) we define the **Borel σ-algebra** to be the smallest σ-algebra which containes all the open sets of the topology.
   - Particularly, this means that open and closed sets are always measurable.
   - In our case, where G is second-countable, it is the σ-algebra which is generated by a countable basis of the topology.
2. μ is a **Radon measure** if it is Borel (a measure on the Borel σ-algebra), if it is **locally-finite**, and if it is a **regular**. this means that
   - if $K \subseteq G$ is compact then $\mu(K) < \infty$.
   - For any $B \subseteq G$ Borel we have that $\mu(B) = \inf\{\mu(U) | B \subseteq U \ open\} = \sup\{\mu(K) | K \subseteq B \ compact\}$.
3. right/left G-inv means that for all B Borel and every $g \in G$ we have that $\mu(B) = \mu(gB)$ (left) $\qquad \mu(B) = \mu(Bg)$ (right)

Theorem 1. *Let G be an Lie group (or lcsc). Then*

(a) there exists a righ/left G-inv metric on G which induces the topology.

(b) there exists a right/left Haar measure on G.

(c) Up to scalar multiplication, it is unique.

(d) $\mu(G) < \infty \iff G$ is compact.

(e) If $U \subseteq G$ is open then $\mu(U) > 0$.

- sketch of proof of (a),(b) for Lie groups:
  * if G is a lie group then G admits a right/left G-invariant Rimannian metric which induces the topology of G. (Roughly speaking, an inner product on tangent space)
  * Rimannian metric $\Longrightarrow$ right/left G-inv metric
  * Rimannian metric $\Longrightarrow$ volume form on G.
  * Use Riesz representation theorem to define a Haar measure via this volume form.
- In total,
  * left Rimannian metric $\Longrightarrow$ left Haar measure.
  * right Rimannian metric $\Longrightarrow$ right Haar measure.

- G is **unimodular** if the right and left Haar measures of G coincide.
- Haar measure on $G = SL_d(\mathbb{R})$ – **Cone construction**. Given $B \subseteq SL_d(\mathbb{R})$ Borel define
$$\boldsymbol{Cone(B)} := \{tb \,|\, b \in B \ 0 \leq t \leq 1\} \subset M_d(\mathbb{R}) \approx R^{d^2}.$$
Now define a measure on $SL_d(\mathbb{R})$ by
$$\boldsymbol{m_{SL_d(\mathbb{R})}}(B) = m_G(B) := vol_{M_d(\mathbb{R})}(Cone(B)).$$
  - the map $h \mapsto (\det(h), \det(h)^{-1/d} \cdot h)$ is continuous $\Longrightarrow Cone(B)$ is the inverse image $[0,1] \times B$, so volume is well defined.

- Can check locally-finite and regular due to properties of Vol in $\mathbb{R}^d$.

Theorem 2. *m_G is the left and right Haar measure of G. Particularly, $SL_d(\mathbb{R})$ is unimodular.*

need to show – for all B Bore all $g \in G$ we have $m_G(B) = m_G(gB) = m_G(Bg)$.

Lets show right-inv.

$\iff vol_{M_d(\mathbb{R})}(Cone(B)) = vol_{M_d(\mathbb{R})}(Cone(Bg))$

$\iff vol_{M_d(\mathbb{R})}(Cone(B)) = vol_{M_d(\mathbb{R})}(Cone(B) \cdot g)$

$\implies$ enough to show – for all measurable $A \subseteq M_d(\mathbb{R})$ we have that

$$vol_{M_d(\mathbb{R})}(A) = vol_{M_d(\mathbb{R})}(Ag).$$

For $a \in A$ write $a = \begin{pmatrix} \vec{a}_1 \\ ... \\ \vec{a}_d \end{pmatrix}$. If we identify $M_d(\mathbb{R})$ with $\mathbb{R}^{d^2}$ via the row vector $\overline{a} =: (a_{1,1}, a_{1,2}, ..., a_{1,d}, a_{2,1}, ..., a_{d,d})$ and write $a \approx \overline{a}$, we have that $a\, g \approx \overline{a}\, diag\,(g, g, ..., g)$. In particular

$vol_{M_d(\mathbb{R})}(Ag) = vol_{\mathbb{R}^{d^2}}\left(\overline{A} \cdot diag\,(g, g, ..., g)\right) = det(g)^d\, vol_{\mathbb{R}^{d^2}}(\overline{A}) = vol_{M_d(\mathbb{R})}(A)$, where the last equality holds as $g \in SL_d(\mathbb{R})$.

We do this analogously for left-invariancy. □

2 Motivation for G/Γ

- Motivation: for an interesting subgroup $\Gamma \leqslant G$ (not necessarily normal), give structure to the space G/Γ (not necessarily a group) via the structure of G.
- $\boldsymbol{L \in \mathbb{R}^d}$ **is a lattice** if there exists linearly-independent $\vec{v}_1, \vec{v}_2 \dots \vec{v}_d$ such that $L = span_{\mathbb{Z}}(\vec{v}_1, \vec{v}_2 \dots \vec{v}_d) \iff$ exists $h \in GL_d(R)$ such that $L = h\mathbb{Z}^d$.
  In this case we also say that h is a basis of L. Example: $\mathbb{Z}^d$ with $h = I_d$.
- Define (a tricky name) $GL_d(\mathbb{Z}) := \{z \in M_{d\times d}(\mathbb{Z}) \text{ such that } \det(z) = \pm 1 \}$.
  Notice this is a group.
- claim: If $L = h_1\mathbb{Z}^d = h_2\mathbb{Z}^d$ then there exists $z \in GL_d(\mathbb{Z})$ such that $h_1 = h_2 z$. I.e. a lattice $L = h\mathbb{Z}^d$ is invariant under the conjugate group $h \cdot GL_d(\mathbb{Z}) \cdot h^{-1}$.
- So we define the *covol*$(L) := |\det h|$, where h is any basis of L. This is well defined by the previous claim.
- Claim: we have two one-to-one correspondences
  - $\overline{X_d} := \{Lattices\ in\ \mathbb{R}^d\} \longleftrightarrow GL_d(\mathbb{R})/GL_d(\mathbb{Z})$ $\qquad h\mathbb{Z}^d \longleftrightarrow h\,GL_d(\mathbb{Z})$.
  - $X_d := \{Lattices\ in\ \mathbb{R}^d\ covol = 1\} \longleftrightarrow SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ $\qquad g\mathbb{Z}^d \longleftrightarrow g\,SL_d(\mathbb{Z})$.
- Gauss and Lagrange studied lattices already in the 18th century, while the study of quotients of Lie groups gained popularity later on.
- On both X_d and $\overline{X_d}$ we have a metric called the Chabauty-Fell metric. Although it is explicit – its difficult to use it, and it does not help us to construct nice measures.

$$- \; d_{CF}(L_1, L_2) := \inf \begin{cases} \varepsilon \mid \forall x \in L_1 \textit{ with } x \in B(\vec{0}, 1/\varepsilon) \; \exists y \in L_2 \textit{ with } \|x-y\| < \varepsilon \\ \quad \forall y \in L_2 \textit{ with } y \in B(\vec{0}, 1/\varepsilon) \; \exists x \in L_1 \textit{ with } \|x-y\| < \varepsilon \\ 1 \end{cases}$$

- Claim: The topology induced from this metric coincide with the quotient topology of $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, so in fact the correspondence from above is a homeomorphism.
- We keep this example in mind when we now try to give structure to arbitrary G/Γ.

3 The quotient $X := G/\Gamma$

- From now on, G is lscs endowed with right invariant metric d_G and with right Haar measure μ_G. The reason why we take right is because we are going to work with right quotient G/Γ (with left cosets of the form $g\Gamma$).
- Assume also from now on that $\Gamma \leqslant G$ is discrete, and define $X := G/\Gamma$.
  For $g \in G$ and $g_0\Gamma = x_0 \in X$ we denote $gx_0 := (gg_0)\Gamma$ the left action of G on X.
- we say $\Gamma \leqslant G$ is **discrete** if for all $\gamma \in \Gamma$ there exists open $U \subseteq G$ such that $\Gamma \cap U = \{\gamma\}$. Since G is lcsc, it is the same as asking that there exists an open U such that $\Gamma \cap U = \{e\}$ (open mapping theorem for locally compact groups implies that for each $g \in G$, multiplication by g is an open map).
- Let $\pi : G \to X$ be the natural projection to the quotient defined by $\pi(g) = g\Gamma$.
  Recall that if G is a topological space and Γ divides G into equivalence classes, we have the quotient topology on X where $V \in X$ is open iff $\pi^{-1}(V)$ is open in G.
- Define a metric $d_X(\cdot, \cdot)$ on X by

  $$d_X(g_1\Gamma, g_2\Gamma) := \inf_{\gamma_1,\gamma_2 \in \Gamma} d_G(g_1\gamma_1, g_2\gamma_2) = \inf_{\gamma \in \Gamma} d_G(g_1, g_2\gamma) = \min_{\gamma \in \Gamma} d_G(g_1, g_2\gamma)$$

  where the first equality follows from the fact that d_G is right-inv, and the second follows from the fact that Γ is discrete.

  Theorem 3. *We have the followings:*

  (a) The topology induced by the metric d_X coincide with the quotient topology. Furthermore, G and X are "locally isometric": for any $x_0 \subset X$ there exists $r = r(x_0) > 0$ such that the following map is an isometry

  $$B_G(e, r) \to B_X(x_0, r) \qquad g \mapsto g \cdot x_0$$

  (b) $g_n\Gamma \to g_0\Gamma \iff \exists \gamma_n$ such that $g_n\gamma_n \to g_0$.

- In this case, we say say that X **is a locally G-homogeneous space** – a quotient space of G which locally looks the same everywhere.
- There is a bit of ambiguity in the literature regarding the definition of a "homogeneous space". The definition of a locally homogeneous space is fairly new, but it is more suitable due to Theorem 3.

- Motivation: construct a Borel measure on X which is invariant under the left G-action.
- We say $\boldsymbol{\Omega \subset G}$ **is a fundamental domain of** $\boldsymbol{X = G/\Gamma}$ (usually just say "fundamental domain") if Ω is Borel, and if for all $g \in G$ there exists a unique $\gamma \in \Gamma$ such that $g\gamma \in \Omega$.
  $\iff$ Ω is a choice of representative from each coset of X, which is also measurable.

Theorem 4. *If $\Gamma \leqslant$ is discrete there exists a fundamental domain Ω for $X = G/\Gamma$.*

- proof
  - For $g \in G$ define $r_g > 0$ by $r_g := \sup\{r > 0 |\, \pi_{|B_G(g,r_g)}$ is injective$\}$.
    This is well defined by discreetness of Γ. Indeed, if not then there exists $h_n \neq k_n$ with $h_n\Gamma = k_n\Gamma$ with $h_n \longrightarrow g$ and $h_n \longrightarrow g$
    $\Longrightarrow \exists \gamma_n \in \Gamma$ with $k_n h_n^{-1} = \gamma_n$
    $\Longrightarrow \gamma_n \to e$, cannot hold for since Γ discrete.
  - Notice that if for some $g, k \in G$ we have that $d_G(g,k) < \frac{r_k}{4}$, then $r_g \geq \frac{r_k}{2}$. So we have $k \in B_G(g, \frac{r_g}{2})$.
  - Now take a countable dense sequence $(g_n)_{n=1}^{\infty}$ in G (lcsc $\to$ separable), and denote for convenience $r_n := r_{g_n}$ and $B_n = B_G(g_n, \frac{r_n}{2})$.
  - So $G = \cup_{n=1}^{\infty} B_n$, and $\pi_{|B_n}$ is injective on for each n.
    Indeed, for $k \in G$ and by density, there exists n such that $d_G(g_n, k) < \frac{r_k}{4}$,
    $\Longrightarrow k \in B_G(g_n, \frac{r_n}{2}) = B_n$.
  - Now define $\Omega := \cup_{n=1}^{\infty} B_n \setminus (\cup_{j<n} B_j \Gamma)$. check that $\pi_{|\Omega}$ is a bijection and that Ω is Borel.

Theorem 5. *Let $\Omega_1, \Omega_2 \subset G$ be two fundamental domains of X. Then for every Borel $A \subset X$ we have $\mu_G(\Omega_1 \cap \pi^{-1}A) = \mu_G(\Omega_2 \cap \pi^{-1}A)$. In particular, $\mu_G(\Omega_1) = \mu_G(\Omega_2)$.*

- Note that this is true for any right invariant measure on G, not only right Haar measure.
- proof
  - Notice that by definition of fundamental domains we have
    $\Omega_1 \cap \pi^{-1}(A) = \sqcup_{\gamma \in \Gamma}\ [\Omega_1 \cap \pi^{-1}(A)] \cap [\Omega_2 \cap \pi^{-1}(A)] \cdot \gamma$
  - By discreetness of Γ and the fact that is lcsc, Γ is countable. so we have
    $\mu_G(\Omega_1 \cap \pi^{-1}(A)) = \Sigma_{\gamma \in \Gamma}\ \mu_G(\,[\Omega_1 \cap \pi^{-1}(A)] \cap [\Omega_2 \cap \pi^{-1}(A)] \cdot \gamma\,) =$
    $\Sigma_{\gamma \in \Gamma}\ \mu_G(\,[\Omega_1 \cap \pi^{-1}(A)] \cdot \gamma^{-1} \cap [\Omega_2 \cap \pi^{-1}(A)]\,) = \mu_G(\Omega_2 \cap \pi^{-1}(A))$.

- Using theorem 2 and 3, we can finally construct a nice measure on the quotient.

Theorem 6. *Assume $\mu_G(\Omega) < \infty$ for some (all) fundamental domain $\Omega \subset G$ of X. Then G is unimodular.*

Theorem 7. *Assume that G is unimodular, and let μ_G be the Haar measure on G.*

(a) For Borel $A \subset X$ define

$$\mu_X(A) := \mu_G(\pi^{-1}(A) \cap \Omega) \qquad if\ \mu_G(\Omega) = \infty$$

$$\mu_X(A) := \mu_G(\Omega)^{-1}\mu_G(\pi^{-1}(A) \cap \Omega) \qquad if\ \mu_G(\Omega) < \infty$$

Then μ_X is a radon (probability) measure on X which is invariant for the left G-action on X.

(b) Up to scalar multiplication, μ_X is the unique G-inv radon measure on X.

- proof
  - First notice that if Ω is a fundamentla domain for X then so does $g\Omega$, for all $g \in G$.
  - Now let $A \subset X$ Borel. Then by theorem 5 and left G-inv of μ_G we have

$$\mu_X(gA) = \mu_G(\pi^{-1}(gA) \cap \Omega) = \mu_G(g\,\pi^{-1}(A) \cap \Omega)$$

$$= \mu_G(\pi^{-1}(A) \cap g^{-1}\Omega) = \mu_G(\pi^{-1}(A) \cap \Omega)) = \mu_X(A).$$

- For $\Gamma \leqslant G$ discrete we say **Γ is a lattice in G** if X admits a left G-inv probality measure. For example, if G is unimodular and there exists Ω a fundamental domain for X with $\mu_G(\Omega) < \infty$.
- Γ is called **a uniform (or cocompact) lattice** if in addition X is compact. Otherwise, Γ is called non-uniform.
- a sequence $(x_n)_{n=1}^{\infty}$ in X is called **divergent** (notation $x_n \to \infty$) if the sequence eventually leaves any compat set; if for all $K \subseteq X$ compact we have $x_n \notin K$ for all sufficiently large n.

- Back to $G = SL_d(\mathbb{R})$. Define $\Gamma = SL_d(\mathbb{Z})$ (which is discrete), and define from on $X_d := G/\Gamma$. Recall that we have seen the correspondence

  $$X_d := \{Lattices\ in\ \mathbb{R}^d\ covol = 1\} \longleftrightarrow SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \qquad g\mathbb{Z}^d \longleftrightarrow g\, SL_d(\mathbb{Z}),$$

- We have seen that the measure m_G (cone construction) is both left and right Haar measure on G, and that any lcsc group admits a fundamental domain. So by theorem 7 we have a radon left G-inv measure on X_d, which we denote by m_{X_d}. At this point we do not know weather it is finite or not.
- <u>motivation:</u> although it is difficult to explicitly find a fundamental domain $\Omega \subset SL_d(\mathbb{R})$ of X_d, we want to show that m_{X_d} is a probability measure.
- Notice that the names are a bit confusing with this correspondence – $g\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$ of covolume 1 for all $g \in G$, and we also want to show that Γ is a lattice in G; the correspondence is between the space of all lattices in $\mathbb{R}^d$ of covolume 1, and the quotient space G/Γ when Γ is a lattice in G.

  Theorem 8. *We have the followings:*

  (a) ***Siegel summation formula (SSF):*** *Let $f \in L^1(\mathbb{R}^d, Vol)$. Then*

  $$\int_{\mathbb{R}^d} f(\vec{v})\ dVol(\vec{v}) = \int_{X_d} \Sigma_{0\neq\vec{v}\in L}\ f(\vec{v})\ dm_{X_d}(L)$$

  (b) $SL_d(\mathbb{Z})$ is a lattice in $SL_d(\mathbb{R})$.

  – proof of (b) assuming (a)
    * Let $f = \mathbf{1}_E$ where $E = [-2,2]^d$. $Vol(E) > 2^d$, so by Minkowski's 1st theorem, any lattice $L \in SL_d(\mathbb{R})$ contains at least one non-zero point in E, so $\Sigma_{0\neq\vec{v}\in L}\ \mathbf{1}_E(\vec{v}) \geq 1$ for all $L \in X_d$. Let $\mathbf{1}_{X_d}$ be the indicator function of the all space X_d.
    * So we have $m_{X_d}(X_d) = \int_{X_d} \mathbf{1}_{X_d}(L)\ dm_{X_d}(L) \leq \int_{X_d} \Sigma_{0\neq\vec{v}\in L}\ \mathbf{1}_E(\vec{v})\ dm_{X_d}(L) =^{SSF} \int_{\mathbb{R}^d} \mathbf{1}_E\ dVol = Vol(E) < \infty$.

Theorem 9. ***Mahler compactness criterion*** *– the following are equivalent*

(a) a sequence of lattices $(g_n\mathbb{Z}^d)_{n=1}^{\infty} = (L_n)_{n=1}^{\infty}$ is divergent (eventually leaves any compact set)

(b) there exists a sequence $0 \neq \vec{v}_n \in L_n$ such that $\|\vec{v}_n\| \to 0$

4 actions of subgroups $H \leqslant G$ on X

- G is a Lie group. $H \leqslant G$ is closed, then H is also a Lie group.
- For $H \leqslant G$ and for $g_0\Gamma = x \in X$, the stabilizer of x w.r.t to the action of H on X is

  $$stab_H(x) := \{h \in H \mid hx = x\} = H \cap g_0\Gamma g_0^{-1}.$$

  So in particular, $stab_H(x)$ is discrete.
- By the orbit-stabilizer Theorem, there is a 1-1 correspondence via the orbit map

  $$Hx \to H/stab_H(x) \qquad hx \mapsto hstab_H(x).$$

- This is only a 1-1 map in terms of sets, but this is not necesserily a homeomorphism.
- So now, after we a have developed a good theory on quotients of discrete subgroups of Lie groups, we can try to use this theory to understand better the orbits Hx.

Theorem 10. *Hx is closed in X $\Longleftrightarrow$ the orbit map is an homeomorphism.*

Theorem 11. *Assume H is unimodular. If Γ is a lattice in G and $stab_H(x)$ is a lattice in H, then Hx is closed.*

- <u>Motivation:</u> although Hx is not necessarily homeomorphic to $H/stab_H(x)$, we want to use the quotient to put a measure on Hx.
- recall that for $A \subseteq X$ we have defined

  $$\mu_X(A) := \mu_G(\Omega \cap \pi^{-1}(A)) = \mu_G(\{\omega \in \Omega \mid \omega \cdot e_X \in A\}).$$

- Let μ_H be the right Haar measure on H, Ω_H a fundamental domain for $H/stab_H(x)$.
- If H is unimodular, then we can define a "new" measure on X by

  $$vol_{Hx}(A) := \mu_H(\{\omega \in \Omega_H \mid \omega x \in A).$$

- <u>claim:</u> vol_{Hx} is Borel and left H-inv.
- But it is not necessarily Radon because it is not necessarily locally finite.
- Notice that since H is unimodular, there is a unique left H-inv Radon on $H/stab_H(x)$. So if Hx is closed and Theorem 10 holds, vol_{Hx} is locally finite, hence the unique Radon left H-inv measure on X.
- Let $U = \{u_t \mid t \in \mathbb{R}\} \leqslant G$ be a one-parameter subgroup of G.
- Let ν_X be a probality measure on X which is left U-inv.
  We say **$\boldsymbol{\nu_X}$ is homogenuous (or algebraic)** if there exists a closed & connected & unimodular subgroup H with $U \leqslant H \leqslant G$ and there exists $x \in X$ with Hx a closed orbit, such that $\nu_X = vol_{Hx}$, where in this case vol_{Hx} is the unique left H-inv Radon measure on X.
- In the following talks will study some Ergodic theory, and discuss Ratner thoerem's which deals with the classification of homogeneous measures.