Presrving regularity of κ in HOD.

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Let

$$\vec{U} = \langle U(\alpha,\beta) \mid \alpha \in \mathrm{dom}(\vec{U}), \beta < o^{\vec{U}}(\alpha)\rangle$$

be a coherent sequence of normal ultrafilters such that

1. $\kappa = \max(\mathrm{dom}(\vec{U}))$,
2. each $U(\alpha,\beta)$ is a normal ultrafilter on $\mathcal{P}_\kappa(\kappa^+)$,
3. $o^{\vec{U}}(\alpha) < \alpha^+$, for every $\alpha < \kappa$,
4. $o^{\vec{U}}(\kappa) = \kappa^+$.

Let $\langle R_{\vec{U}}, \leq, \leq^* \rangle$ be the supercompact Radin forcing with $\vec{U}$.
For every $\beta < \kappa^+$, let $h_\beta : \kappa \to \kappa$ be the $\beta-$th canonical function. Set

$$A_\beta = \{P \in \mathcal{P}_\kappa(\kappa^+) \mid \beta \in P \wedge o^{\vec{U}}(P\cap\kappa) = \mathrm{otp}(P\cap\beta) = h_\beta(P\cap\kappa)\},$$

for every $\beta < \kappa^+$.

Lemma 0.1 *The sequence $\langle A_\beta \mid \beta < \kappa^+\rangle$ consists of disjoint subsets of $\mathcal{P}_\kappa(\kappa^+)$ and $A_\beta \in U(\kappa,\beta)$, for every $\beta < \kappa^+$.*

Proof. Clearly, $A_\beta \in U(\kappa,\beta)$, for every $\beta < \kappa^+$. Show disjointness. Let $\beta < \gamma < \kappa^+$.
If there is $P \in A_\beta \cap A_\gamma$, then $\beta,\gamma \in P$. Hence, $\mathrm{otp}(P\cap\beta) < \mathrm{otp}(P\cap\gamma)$ But $\mathrm{otp}(P\cap\gamma) = o^{\vec{U}}(P\cap\kappa) = \mathrm{otp}(P\cap\beta)$, which is impossible.
□

Without loss of generality, let us assume that $\langle A_\beta \mid \beta < \kappa^+\rangle$ are in the weakest condition of $R_{\vec{U}}$.

Lemma 0.2 *Both κ and κ^+ change their cofinality to ω in $V^{R_{\vec{U}}}$.*

Proof. Define a sequence

$$\langle P_n \mid n < \omega \rangle$$

as follows. Let P_0 be the first member of the Radin sequence. Pick P_{n+1} to be the first element of the Radin sequence in $A_{\sup(P_n)}$.
Then

$$\bigcup_{n<\omega} P_n = \kappa^+.$$

Just otherwise for some $\beta < \kappa^+$, $P \in A_\beta$ we have $\bigcup_{n<\omega} P_n = P$. But $\beta \in P$, hence $\beta \in P_n$, for some $n < \omega$. On the other hand, $\bigcup_{n<\omega} P_n = P$ implies that $P \in A_\gamma$,
for some $\gamma \geq \bigcup_{n<\omega} \sup(P_n)$. Contradiction.
□

Let $\vec{U}^*$ be the projection of $\vec{U}$ to measures on ordinals, i.e.

$$U^*(\alpha, \beta) = \{\{P \cap \alpha \mid P \in X\} \mid X \in U(\alpha, \beta)\}.$$

Then the Radin forcing $R_{\vec{U}^*}$ is the projection of $R_{\vec{U}}$. The following holds:

Lemma 0.3 *Both κ and κ^+ remain regular in $V^{R_{\vec{U}^*}}$. κ is not Mahlo in $V^{R_{\vec{U}^*}}$.*

Proof. The length of the projected sequence over κ is κ^+. Hence κ remains regular but not Mahlo.
□.

Let C denotes a Radin sequence generic for $R_{\vec{U}}$ and C^* denotes a Radin sequence generic for $R_{\vec{U}^*}$ which is the projection of C.

Note that the forcings $R_{\vec{U}}$ and $R^*_{\vec{U}}$ are non-homogeneous. However, let us show the following:

Lemma 0.4 *The forcing $R_{\vec{U}}/C^*$ is homogeneous.*

Proof.

Consider a bit simplified situation. Thus let

$$p = \langle\langle P, X\rangle, \langle \mathcal{P}_\kappa(\kappa^+), Y\rangle\rangle, p' = \langle\langle P', X'\rangle, \langle \mathcal{P}_\kappa(\kappa^+), Y'\rangle\rangle$$

be two conditions in $R_{\vec{U}}$ with $P \cap \kappa = P' \cap \kappa$. Denote $P \cap \kappa$ by η.
It looks at first problematic if P and P' come from different measures. Thus say, $P \in A_\beta$

and $P' \in A_\gamma$, for some $\beta < \gamma < \kappa^+$. However, $P \cap \kappa = \eta$ already determines the coherent sequence used over η:

$$\langle U(\eta, \beta) \mid \beta < o^{\vec{U}}(\eta) \rangle.$$

Hence, it is the same in both p, p'.
Now it is easy to define an automorphism of $R_{\vec{U}}$ that moves an extension of p to those of p'.
Just pick some $Q \in Y \cap Y' \cap A_0$ which contains both P and P'.
Consider

$$q = \langle \langle P, X \cap X' \rangle, Q, \langle \mathcal{P}_\kappa(\kappa^+), Z \rangle \rangle \text{ and } q' = \langle \langle P', X \cap X' \rangle, Q, \langle \mathcal{P}_\kappa(\kappa^+), Z \rangle \rangle,$$

where

$$Z = \{T \in Y \cap Y' \mid T \supseteq Q\}.$$

Clearly, q extends p and q' extends p'.
Take finally the automorphism of $R_{\vec{U}}$ that moves q to q' and q' to q.
□

Finally we code C^* into HOD, but not C. Then by the lemmas above the following will hold:

Theorem 0.5 *κ is regular strong limit non-Mahlo cardinal in $(HOD)^{V[C]}$ and it is singular of cofinality ω in $V[C]$.*