On free profinite products of infinitely many absolute Galois groups

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Abstract

We address the question when a free profinite product of infinitely many absolute Galois groups of fields is also an absolute Galois group of some field.

Introduction

Koenigsmann ([Koe]; see also [HJK]) showed that the free profinite product of finitely many absolute Galois groups is also the absolute Galois group of some field. Recently the question to what extent this can be extended to infinitely many groups has arisen ([Bar], [Jar]). Jarden ([Jar]) gives examples of free profinite products of copies of the same absolute Galois group G_0 converging to 1 (aka a free profinite product in the sense of Binz-Neukirch-Wenzel [HJ, Section 4.4]) that are not absolute Galois groups, namely, when G_0 is of order 2 or the absolute Galois group of a p-adic field. Bar-On ([Bar]) gives an exact condition for a free pro-p product of countably many Demushkin groups of rank $\aleph_0$ be an absolute Galois group.

In this note we present three results:

Theorem 2 shows that the free profinite product of copies of a single absolute Galois group over any constant sheaf is an absolute Galois group.

Theorem 9 shows that the free profinite product of absolute Galois groups $\mathrm{Gal}(E_i)$ converging to 1 is an absolute Galois group, provided every E_i contains an algebraically closed field.

Finally, Corollary 4 answers positively the following question about a family $\{G_i\}_{i \in I}$ of profinite groups: In order that the free profinite product in the sense of Binz-Neukirch-Wenzel of this family be realizable as an absolute Galois group, it is necessary that this family be realizable as a family converging to 1 of subgroups of an absolute Galois group. Is this condition also sufficient?

Actually, a more general statement is true: The above holds for every free profinite product. The exact statement is Theorem 3.

1 Free profinite products

In this section we present the rudiments of the theory of free profinite products in a rather intuitive manner, intended for an uninitiated reader. Nevertheless, to follow the proofs, rigorous — and, unfortunately, very technical — definitions are necessary, and we duly refer the reader to them in the existing literature.

To define free profinite products of infinitely many profinite groups, one usually considers a family $\{G_t\}_{t\in T}$ of profinite groups indexed by a profinite space T.

If these groups are subgroups of a profinite group G, then the definition of G being their **(inner) free profinite product** is straightforward: Every continuous map $\bigcup_{t\in T} G_t \to A$ into a profinite group A uniquely extends to a homomorphism $G \to A$ ([R, beginning of Section 5.3] or [HJ, Definition 4.1.1]), but we require that the family be be **étale continuous** ([HJ, Definition 2.1.1]), that is, **continuously indexed by** T ([R, beginning of Section 5.2]).

But we also want to define a free profinite product of $\{G_t\}_{t\in T}$ when these groups are not given as subgroups of one group. To this end we need the concept of a **profinite sheaf** of profinite groups. Intuitively, this is a disjoint union $X = \dot\bigcup_{t\in T} X_t$ of a family $\{X_t\}_{t\in T}$ of profinite groups indexed by a profinite space T so that the group operations on the X_t's extend to continuous operations on X. The precise (and formal) definition is [HJ, Definition 2.2.1] or [R, beginning of Section 5.1] and it writes the sheaf as a triple (X, τ, T), where $\tau\colon X \to T$ is the map which maps X_t onto $\{t\}$, for every $t \in T$.

A particular example of a profinite sheaf is the **constant sheaf** $(T \times G, \mathrm{pr}_T, T)$, where T is a profinite space, G is a profinite group, and $\mathrm{pr}_T\colon T \times G \to T$ is the projection onto T. (Thus $X_t = G$ for every $t \in T$.)

Given any profinite sheaf $\mathbf{X} = (X, \tau, T)$ we can assign to it its **(outer)**

free profinite product $(\mathbf{X},\omega,\hat{G})$ ([HJ, Definition 4.2.1] or [R, Section 5.1]). Here $\hat{G}$ is a profinite group and $\omega\colon X\to\hat{G}$ is a continuous map such that $\{\omega|_{X_t}\colon X_t\to\hat{G}\}_{t\in T}$ are homomorphisms, with the following property: For every continuous map $\alpha\colon X\to A$ into a profinite group A such that $\{\alpha|_{X_t}\colon X_t\to A\}_{t\in T}$ are homomorphisms, there exists a unique homomorphism $\varphi\colon\hat{G}\to A$ such that $\varphi\circ\omega=\alpha$.

Historically, the first definition of a free profinite product of infinitely many factors was of a different kind: Given a family $\{G_i\}_{i\in I}$ of profinite groups (the index set I carries no topology) their free product **in the sense of Binz-Neukirch-Wenzel** ([HJ, Section 4.4]) is a profinite group $\hat{G}$ together with a family of homomorphisms $\{\omega_i\colon G_i\to\hat{G}\}_{i\in I}$ **converging to** 1, that is, such that for every open subgroup $\hat{G}_0$ of $\hat{G}$ the set $\{i\in I|\,\omega_i(G_i)\not\leq\hat{G}_0\}$ is finite, with the following property: For every profinite group A and every family of homomorphisms $\{\omega_i\colon G_i\to A\}_{i\in I}$ converging to 1, there is a unique homomorphism $\alpha\colon\hat{G}\to A$ such that $\alpha\circ\omega_i=\alpha_i$ for every $i\in I$. However, it turns out that this free profinite product is the outer free profinite product over the sheaf (X,τ,T), where $X=\dot{\bigcup}_{i\in I}G_i\,\dot{\cup}\,\{1_\infty\}$ and $T=I\,\dot{\cup}\,\{\infty\}$ are the one-point compactifications of $\dot{\bigcup}_{i\in I}G_i$ and I, respectively, and τ maps 1_∞ onto ∞ and every G_i onto $\{i\}$ ([HJ, Corollary 4.4.3(d)]).

We finish this section with some basic facts about profinite groups and profinite spaces can be found outside the standard literature on the subject.

Recall that a **generalized Cantor space** is the topological space $\{0,1\}^A$, for some set A, with the product (Tychonov) topology. It is a profinite space. If $|A|=\aleph_0$, this the homeomorphic to the Cantor ternary set.

Lemma 1. *(a) Every profinite space is a closed subspace of a generalized Cantor space.*

(b) Every infinite profinite group G is homeomorphic to a generalized Cantor space. More precisely, G is homeomorphic to $\{0,1\}^A$, where $|A|$ is the cardinality of the family of open subgroups of G.

(c) Two profinite groups are homeomorphic if and only if they have the same cardinality.

Proof. (a) See [Kop, p. 101]. Explicitly ([Kop, Exercise 5, p. 106]): Let A be the family of clopen subsets of a profinite space T. For every $U\in A$ let

$\chi_U : T \to \{0,1\}$ be its characteristic function, i.e., the function given by

$$\chi_U(t) = \begin{cases} 0 & \text{if } t \notin U \\ 1 & \text{if } t \in U. \end{cases}$$

Then the map $f : T \to \{0,1\}^A$, given by $f(t) = (\chi_U(t))_{U\in A}$ is a homeomorphism of T onto a closed subspace of $\{0,1\}^A$.

(b) The first assertion is [HR, Theorem 9.15].

If G is homeomorphic to $\{0,1\}^A$, then A must be infinite and $|\operatorname{Cl}(G)| = |\operatorname{Cl}(\{0,1\}^A)|$, where $|\operatorname{Cl}(T)|$ denotes the cardinality of the family of the clopen subsets of a profinite space T. As every open subgroup of G has finitely many cosets and every clopen subset of G is the union of finitely many cosets of open subgroups, $|\operatorname{Cl}(G)|$ is the cardinality of the family of open subgroups of G.

On the other hand, every clopen subset of $\{0,1\}^A$ is of the form $\mathrm{pr}_B^{-1}(Z)$, where $B \subseteq A$ is finite, $Z \subseteq \{0,1\}^B$, and $\mathrm{pr}_B : \{0,1\}^A \to \{0,1\}^B$ is the projection on the coordinates in B. Therefore $|\operatorname{Cl}(\{0,1\}^A)|$ is the cardinality of the family of finite subsets of A, which is $|A|$.

(c) This follows from (b). □

2 Constant sheaf

We first realize the free profinite product over a constant sheaf and then deduce a more general result:

Theorem 2. *Let G be the absolute Galois group of a field of characteristic $p \geq 0$ and let T be a profinite space. Then the free profinite product $\hat{G}$ over the constant sheaf $(T \times G, \mathrm{pr}_T, T)$ is the absolute Galois group of a field of characteristic p.*

Proof. We first assume that T is (the underlying topological space of) the absolute Galois group F of a field of characteristic p.

Then F acts on $\hat{G}$ ([HJ, Lemma 4.7.2]) and $\hat{G} \rtimes F$ is isomorphic to the free profinite product $G \coprod F$ ([HJ, Lemma 4.7.4]).

By [Koe], $\hat{G} \rtimes F = G \coprod F$ is the absolute Galois group of a field K of characteristic p. Therefore its closed subgroup $\hat{G}$ is the absolute Galois group of an algebraic separable extension of K.

In the general case, by Lemma 1(a), T is a closed subspace of $\{0,1\}^A$ for some set A. Let F be the absolute Galois group of a field of characteristic $p \geq 0$ such that $|F| \geq 2^{|A|}, \aleph_0$. By Lemma 1(b), F is homeomorphic to $\{0,1\}^B$ for some set B. Then $\{0,1\}^A$ is homeomorphic to a closed subspace of $\{0,1\}^B$. Indeed, as $2^{|B|} = |F| \geq 2^{|A|}$, we have $|B| \geq |A|$, and hence there is a surjection $\pi\colon B \to A$. Then the map $\{0,1\}^A \to \{0,1\}^B$ given by $f \mapsto f \circ \pi$ is a continuous injection. Its image is compact, and hence closed in $\{0,1\}^B$.

It follows that T is homeomorphic to a closed subspace of F.

By [HJ, Lemma 4.8.2], $\hat{G}$, the free profinite product over the constant sheaf $(T \times G, \mathrm{pr}_T, T)$, is isomorphic to a closed subgroup of the free profinite product $\hat{G}'$ over the constant sheaf $(F \times G, \mathrm{pr}_F, F)$. By the preceding special case, $\hat{G}'$ is the absolute Galois group of some field K' of characteristic p. Thus $\hat{G}$ is the absolute Galois group of some algebraic separable extension of K'. □

Let G be a profinite group and $\mathcal{G} = \{G_t\}_{t\in T}$ an étale continuous family ([HJ, Definition 2.1.1]) of subgroups of G, over a profinite space T. Let $\mathbf{X} = (X, \tau, T)$ be the **associated sheaf** ([HJ, Proposition 2.3.6]), that is, $X = \{(t,g) \in T \times G |\, g \in G_t\}$ and τ is $(g,t) \mapsto t$.

If G is the inner free profinite product of $\mathcal{G}$, then G is the outer free profinite product over $\mathbf{X}$ ([HJ, Proposition 4.3.1]). But even if this is not the case, we can still form the outer free profinite product $(\mathbf{X}, \mathbf{X} \to \hat{G}, \hat{G})$ over $\mathbf{X}$ ([HJ, Proposition 4.2.2]). We call it **the free profinite product of** $\mathcal{G}$.

For instance, if there is a subgroup H of G such that $G_t = H$ for every $t \in T$, then $\mathbf{X}$ is the constant sheaf $(T \times H, \mathrm{pr}_T, T)$. In this case $\mathbf{X}$, and hence also $\hat{G}$, actually does not depend on G, only on H and T.

Another example in which the associated sheaf does not depend on the group G, (only on the fact that $\mathcal{G}$ is an étale continuous family of subgroups of some group) is the following: T is the one-point compactification $I \mathbin{\dot\cup} \{\infty\}$ of a discrete set I, and $G_\infty = 1$. It follows from definitions that $\{G_t |\, t \in T\}$ is étale continuous if and only if the family of inclusions $\{G_i \to G\}_{i\in I}$ is converging to 1. Moreover, $X = \dot\bigcup_{i\in I} G_i \mathbin{\dot\cup} \{1_\infty\}$, and the topology on X is the one of the one-point compactification of $\dot\bigcup_{i\in I} G_i$. So the free profinite product of $\mathcal{G}$ is the free profinite product of $\{G_t |\, t \in I\}$ in the sense of Binz-Neukirch-Wenzel.

Theorem 3. *Let G be a profinite group, T a profinite space, and $\mathcal{G} = \{G_t\}_{t\in T}$ an étale continuous family of subgroups of G. If G is the absolute Galois group of some field, then so is the free profinite product $\hat{G}$ of $\mathcal{G}$.*

Proof. The associated sheaf $\mathbf{X} = (X, \tau, T)$ of $\mathcal{G}$ is a closed subsheaf of the constant sheaf $\mathbf{Y} = (T \times G, \mathrm{pr}_T, T)$, because $X = \{(t, g) \in T \times G | \, g \in G_t\} \subseteq T \times G$ and $\tau = (\mathrm{pr}_T)|_X$. By definition, $\hat{G}$ is the free profinite product over $\mathbf{X}$. Let H be the free profinite product over $\mathbf{Y}$. By Theorem 2, H is the absolute Galois group of some field E. By [HJ, Lemma 4.8.2], $\hat{G}$ is a closed subgroup of H. Therefore $\hat{G}$ is the absolute Galois group of some algebraic extension of E. □

3 Families converging to 1

We begin with an application of Theorem 3. It appears as Theorem 3(2) of [Bar], proven there by completely different methods.

Corollary 4. *Let $\mathcal{G} = \{G_i\}_{i \in I}$ be a family of profinite groups. The free profinite product in the sense of Binz-Neukirch-Wenzel of $\mathcal{G}$ is realizable as an absolute Galois group if and only if $\mathcal{G}$ is realizable as a family of subgroups, converging to* 1*, of an absolute Galois group.*

Proof. Suppose that $\{G_i\}_{i \in I}$ is a family of subgroups of a profinite group G. Let $T = I \cup \{\infty\}$ be the one-point compactification of I and put $G_\infty = 1$. It follows from definitions that $\{G_i\}_{i \in I}$ is converging to 1 if and only if $\{G_t | \, t \in T\}$ is étale continuous. Moreover, by [HJ, Corollary 4.4.3], G is the free profinite product of $\{G_i\}_{i \in I}$ in the sense of Binz-Neukirch-Wenzel if and only if G is the free profinite product of $\{G_t | \, t \in T\}$. So the result follows by Theorem 3. □

We now want to show that the free profinite product of a family of certain absolute Galois groups, converging to 1, is an absolute Galois group. By Corollary 4 it would suffice to realize this family as converging to 1, in an absolute Galois group. However, the latter task seems to be as difficult as the former one. So we adopt another approach.

Lemma 5 ([HJK], Proposition 2.3(b))**.** *Let F be a valued field with residue field $\bar{F}$. Let K be an extension of $\bar{F}$ and κ a cardinality. Then there is an extension L of F with* $\mathrm{Gal}(L) \cong \mathrm{Gal}(K)$ *and* $\mathrm{tr.deg}(L/F) \geq \kappa$*.*

Proof. This is [HJK, Proposition 2.3(b)], except for the last assertion. To obtain it too, replace F by $F(T)$, where T is an algebraically independent set over F of cardinality $\geq \kappa$ and extend the valuation of F to a valuation of $F(T)$ with the same residue field $\bar{F}$. □

We will need a variant of [HJK, Proposition 2.5]:

Lemma 6. *Let F, K be fields, each containing an algebraically closed subfield, such that either* $\operatorname{char} K = \operatorname{char} F$ *or* $\operatorname{char} K = 0$*. Let c be a cardinality. Then there is a field extension L/K such that* $\operatorname{Gal}(L) \cong \operatorname{Gal}(F)$ *and* $\operatorname{tr.deg} L/K \geq c$*.*

Proof. Let F_1, K_1 be the algebraic closures of the prime subfields F_0, K_0 of F, K, respectively. By assumption, $F_1 \subseteq F$ and $K_1 \subseteq K$.

The unique place $K_0 \to F_0 \cup \{\infty\}$ extends to a place $\varphi_1 \colon K_1 \to F_1 \cup \{\infty\}$. As K_1 is algebraically closed, so is its residue field, hence the residue field of φ_1 is F_1. We can extend φ_1 to a place $\varphi_1 \colon K \to F_1 \cup \{\infty\}$ with the same residue field F_1.

Choose a transcendence base S for F/F_1 and choose an algebraically independent set T over K such that $\operatorname{card}(T) \geq c, \operatorname{card}(S)$. Then there is a surjective map $\varphi_2 \colon T \to S$. Extend φ_1 and φ_2 to a place $\varphi \colon K(T) \to F_1(S) \cup \{\infty\}$ with residue field $F_1(S)$. This place gives a valuation of $K(T)$ with residue field $F_1(S)$. By [HJK, Proposition 2.3(b)] there is an algebraic extension L of $K(T)$ such that $\operatorname{Gal}(L) \cong \operatorname{Gal}(F)$. $\square$

Lemma 7. *Let E, E' be fields. Assume that E' contains an algebraically closed subfield and either* $\operatorname{char} E = \operatorname{char} E'$ *or* $\operatorname{char} E = 0$*. Then there is an extension K/E and separable algebraic extensions $K_1/K, K_2/K$ such that*

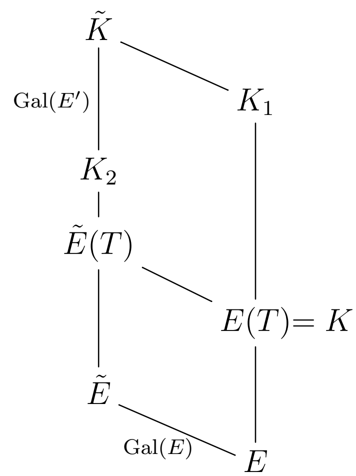
(a) $\operatorname{Res} \colon \operatorname{Gal}(K_1) \to \operatorname{Gal}(E)$ *is an isomorphism,*

(b) $\operatorname{Res} \colon \operatorname{Gal}(K_2) \to \operatorname{Gal}(E)$ *is the trivial map, and*

(c) $\operatorname{Gal}(K_2) \cong \operatorname{Gal}(E')$*.*

Proof. Let $\tilde{E}$ be the algebraic closure of E. Replace E by its inseparable closure in $\tilde{E}$ to assume that $\tilde{E}/E$ is separable. By Lemma 6 there is an extension $K_2/\tilde{E}$ such that (c) holds. As $\tilde{E} \subseteq K_2$, (b) holds.

Let T be a separating transcendence base for $K_2/\tilde{E}$ and put $K = E(T)$. Then $K_2/\tilde{E}(T)$ and $\tilde{E}(T)/K$ are separable algebraic, hence so is K_2/K. By [HJK, Proposition 2.4], $\operatorname{Res} \colon \operatorname{Gal}(K) \to \operatorname{Gal}(E)$ has a section, hence there is an separable algebraic extension K_1/K such that (a) holds.



$\square$

Remark 8. (a) Let I be a set. It can be well-ordered, i.e., there is an ordinal ν and a bijection $f\colon \nu \to I$. For every $\lambda \le \nu$ we define $I_\lambda = \{f(\kappa) \mid \kappa < \lambda\}$. (Thus $I_0 = \emptyset$, $I_1 = \{f(0)\}$, $I_2 = \{f(0), f(1)\}$, ..., $I_\omega = \{f(0), f(1), \ldots\}$, $I_{\omega+1} = \{f(0), f(1), \ldots, f(\omega)\}$, ..., $I_\nu = I$.) Thus for every $\lambda \le \nu$ we have: If λ is not a limit ordinal, then $I_\lambda = I_{\lambda-1} \mathbin{\dot\cup} \{f(\lambda - 1)\}$; if λ is a limit ordinal, then $I_\lambda = \bigcup_{\kappa<\lambda} I_\kappa = \varinjlim_{\kappa<\lambda} I_\kappa$.

(b) Let I, f, and ν be as in (a). Let $\{X_i\}_{i\in I}$ be a family of profinite spaces. For every $\lambda \le \nu$ let

$$\bar{X}_\lambda = (\dot{\bigcup_{i\in I_\lambda}} X_i) \mathbin{\dot\cup} \{\infty\}$$

be the one-point compactification of $\dot\bigcup_{i\in I_\lambda} X_i$ ([HJ, Lemma 4.4.2]). In particular, $\bar{X}_0 = \{\infty\}$ and $\bar{X}_\nu$ is the one-point compactification of $\dot\bigcup_{i\in I} X_i$.

For $\kappa \le \lambda \le \nu$ define $\pi_{\lambda,\kappa}\colon \bar{X}_\lambda \to \bar{X}_\kappa$ by

$$\pi_{\lambda,\kappa}(x) = \begin{cases} x & \text{if } x \in X_i, \text{where } i \in I_\kappa \\ \infty & \text{if } x \in X_i, \text{where } i \in I_\lambda \smallsetminus I_\kappa \\ \infty & \text{if } x = \infty \end{cases}$$

Then $\pi_{\lambda,\kappa}$ is surjective. It is continuous.

Indeed, let U be a basic open subset of $\bar{X}_\kappa$, that is ([HJ, Lemma 4.4.2]) either a clopen subset of X_i for some $i \in I_\kappa$ or $U = \{\infty\} \cup \dot{\bigcup}_{i \in I'_\kappa} X_i$ for some cofinite subset I'_κ of I_κ. In the former case $\pi^{-1}_{\lambda,\kappa}(U) = U$ is clopen in X_i. In the latter case $I'_\lambda = I'_\kappa \cup (I_\lambda \smallsetminus I_\kappa)$ is a cofinite subset of I_λ and $\pi^{-1}_{\lambda,\kappa}(U) = \{\infty\} \cup \dot{\bigcup}_{i \in I'_\lambda} X_i$. So in both cases $\pi^{-1}_{\lambda,\kappa}(U)$ is a basic open subset of $\bar{X}_\lambda$.

It is easy to see that if $\kappa \le \lambda \le \mu \le \nu$, then $\pi_{\lambda,\kappa} \circ \pi_{\mu,\lambda} = \pi_{\mu,\kappa}$. Thus $(\bar{X}_\lambda, \pi_{\lambda,\kappa} | \, \kappa \le \lambda \le \nu)$, is an inverse system. Therefore, if $\mu \le \nu$ is a limit ordinal, then $(\bar{X}_\lambda, \pi_{\lambda,\kappa} | \, \kappa \le \lambda < \mu)$, is also an inverse system and the maps $(\pi_{\mu,\lambda} | \, \lambda < \mu)$ define a surjective continuous map $\pi \colon \bar{X}_\mu \to \varprojlim_{\lambda<\mu} \bar{X}_\lambda$. It is injective as well. Indeed, if $x, x' \in \bar{X}_\mu$, there is $\lambda < \mu$ such that $x, x' \in \bar{X}_\lambda$. So if $x \ne x'$, then $\pi_{\mu,\lambda}(x) \ne \pi_{\mu,\lambda}(x')$, whence $\pi(x) \ne \pi(x')$. Thus $\bar{X}_\mu \cong \varprojlim_{\lambda<\mu} \bar{X}_\lambda$. If $\mu \le \nu$ is not a limit ordinal, then $\bar{X}_\mu$ is the disjoint union of $\bar{X}_{\mu-1}$ and $X_{f(\mu)}$.

(c) Let I, f, and ν be as in (a). Let $\{G_i\}_{i \in I}$ be a family of profinite groups and let $\lambda \le \nu$. Consider the sheaf of profinite groups $\mathbf{X}_\lambda = (\bar{X}_\lambda, \tau_\lambda, T_\lambda)$, where $\bar{X}_\lambda = (\dot{\bigcup}_{i \in I_\lambda} G_i) \cup \{\infty\}$ and $T_\lambda = I_\lambda \cup \{\infty\}$ are the one-point compactifications, and $\tau_\lambda(G_i) = \{i\}$, for every $i \in I_\lambda$ and $\tau_\lambda(\infty) = \infty$.

Let $\lambda \le \nu$. The free profinite product H_λ of $\{G_i\}_{i \in I_\lambda}$ in the sense of Binz-Neukirch-Wenzel is the outer free profinite product over $\mathbf{X}_\lambda$, by [HJ, Corollary 4.4.3]. Thus $H_0 = 1$ and $H_\nu = \coprod_{i \in I} G_i$. Let $\mu \le \nu$. By (b), if μ is a limit ordinal, then $\mathbf{X}_\mu = \varprojlim_{\lambda<\mu} \mathbf{X}_\lambda$, hence, by [HJ, Lemma 4.2.5], $H_\mu = \varprojlim_{\lambda<\mu} H_\lambda$. If μ is not a limit ordinal, then $\mathbf{X}_\mu$ is the disjoint union of $\mathbf{X}_{\mu-1}$ and $(G_i, \tau_i, \{i\})$, where $i = f(\mu - 1)$ and τ_i maps every $g \in G_i$ onto i; hence $H_\mu = H_{\mu-1} \coprod G_i$.

Theorem 9. *Let $\{E\} \cup \{E_i\}_{i \in I}$ be a family of fields. Assume that either all of them have the same characteristic or* $\mathrm{char}\, E = 0$*, and suppose that every E_i contains an algebraically closed subfield. Let $H = \coprod_{i \in I} \mathrm{Gal}(E_i)$ be the free profinite product of the* $\mathrm{Gal}(E_i)$ *in the sense of Binz-Neukirch-Wenzel. Then there is an extension F/E with* $\mathrm{Gal}(F) \cong H$.

Proof. By Remark 8 there is an ordinal ν, a bijection $f \colon \nu \to I$, and an inverse system of epimorphisms of profinite groups $(H_\lambda, \pi_{\lambda,\kappa} | \, \kappa \le \lambda \le \nu)$, such that

(a) $H_0 = 1$ and $H_\nu = H$;

(b) if λ is a limit ordinal, then $H_\lambda = \varprojlim_{\kappa<\lambda} H_\kappa$; and

(c) if $\lambda > 0$ is not a limit ordinal, then $H_\lambda = H_{\lambda-1} \coprod G_{\lambda-1}$, where $G_{\lambda-1} = \mathrm{Gal}(E_{f(\lambda-1)})$.

We construct, by transfinite induction, a family of field extensions $\{L_\lambda\}_{\lambda \leq \nu}$ of E such that

(d) for every $\lambda \leq \nu$ there is an isomorphism $\theta_\lambda\colon \mathrm{Gal}(L_\lambda) \to H_\lambda$;

(e) if $\kappa \leq \lambda \leq \nu$, then $L_\kappa \subseteq L_\lambda$ and the following diagram commutes

$$\begin{array}{ccc} \mathrm{Gal}(L_\lambda) & \xrightarrow{\theta_\lambda} & H_\lambda \\ \downarrow{\scriptstyle \mathrm{Res}_{\lambda,\kappa}} & & \downarrow{\scriptstyle \pi_{\lambda,\kappa}} \\ \mathrm{Gal}(L_\kappa) & \xrightarrow{\theta_\kappa} & H_\kappa \end{array} \tag{1}$$

Here $\mathrm{Res}_{\lambda,\kappa}$ is the restriction map of the Galois groups.

Then $L = L_\nu$ is an extension of E with $\mathrm{Gal}(L) \cong H$.

The construction is by transfinite induction. As $H_0 = 1$, we may set $L_0 = \tilde{E}$. If μ is a limit ordinal, we let $L_\mu = \varinjlim_{\kappa<\mu} L_\kappa$. Then $\mathrm{Gal}(L_\mu) = \varprojlim_{\kappa<\mu} \mathrm{Gal}(L_\kappa)$ and $H_\mu = \varprojlim_{\kappa<\mu} H_\kappa$. The commutativity of diagrams (1) for $\kappa \leq \lambda < \mu$ gives an isomorphism $\theta_\mu\colon \mathrm{Gal}(L_\mu) \to H_\mu$ such that the left diagram

$$\begin{array}{ccc} \mathrm{Gal}(L_\mu) & \xrightarrow{\theta_\mu} & H_\mu \\ \downarrow{\scriptstyle \mathrm{Res}_{\mu,\kappa}} & & \downarrow{\scriptstyle \pi_{\mu,\kappa}} \\ \mathrm{Gal}(L_\kappa) & \xrightarrow{\theta_\kappa} & H_\kappa \end{array} \qquad \begin{array}{ccc} \mathrm{Gal}(L_\mu) & \xrightarrow{\theta_\mu} & H_\mu \\ \downarrow{\scriptstyle \mathrm{Res}_{\mu,\mu-1}} & & \downarrow{\scriptstyle \pi_{\mu,\mu-1}} \\ \mathrm{Gal}(L_{\mu-1}) & \xrightarrow{\theta_{\mu-1}} & H_{\mu-1} \end{array} \tag{2}$$

commutes for all $\kappa \leq \mu$.

Suppose that μ is not a limit ordinal. By Lemma 7 there is an extension $K/L_{\mu-1}$ and algebraic extensions $K_1/K, K_2/K$ such that

(i) $\mathrm{Res}\colon \mathrm{Gal}(K_1) \to \mathrm{Gal}(L_{\mu-1})$ is an isomorphism,

(ii) $\mathrm{Res}\colon \mathrm{Gal}(K_2) \to \mathrm{Gal}(L_{\mu-1})$ is the trivial map, and

(iii) $\mathrm{Gal}(K_2) \cong G_{\mu-1} = \mathrm{Gal}(E_{f(\mu-1)})$.

Replace K by $K_1 \cap K_2$ to assume that $K = K_1 \cap K_2$. Then replace K_1, K_2 by the separable closures of K in them to assume that $K_1/K, K_2/K$ are separable. Then, by [HJK, Theorem 3.3], there are fields L_μ, F_1, F_2 such that

(iv) $F_1/L_\mu, F_2/L_\mu$ are algebraic and $F_1 \cap F_2 = L_\mu$;

(v) $K_i \subseteq F_i$ and $\mathrm{Res}\colon \mathrm{Gal}(F_i) \to \mathrm{Gal}(K_i)$ is an isomorphism, for $i = 1, 2$;

(vi) $\mathrm{Gal}(L_\mu) = \mathrm{Gal}(F_1) \amalg \mathrm{Gal}(F_2)$; moreover,

(vii) $L_{\mu-1} \subseteq L_\mu$, because $L_{\mu-1} \subseteq K = K_1 \cap K_2 \subseteq F_1 \cap F_2 = L_\mu$,

We construct an isomorphism $\theta_\mu\colon \mathrm{Gal}(L_\mu) \to H_\mu$ such that the left diagram in (2) commutes, for every $\kappa < \mu$. As (1) commutes for $\lambda = \mu - 1$, it suffices to construct θ_μ such that the right diagram in (2) commutes.

Consider the subgroups $\mathrm{Gal}(F_1), \mathrm{Gal}(F_2)$ of $\mathrm{Gal}(L_\mu)$.

By (v) and (i), $\mathrm{Res}_{\mu,\mu-1}$ maps $\mathrm{Gal}(F_1)$ isomorphically onto $\mathrm{Gal}(L_{\mu-1})$. So $\theta_{\mu-1} \circ \mathrm{Res}_{\mu,\mu-1}|_{\mathrm{Gal}(F_1)}$ is an isomorphism $\alpha_1\colon \mathrm{Gal}(F_1) \to H_{\mu-1}$. Recall that $H_\mu = H_{\mu-1} \amalg G_{\mu-1}$ and $\pi_{\mu,\mu-1}$ maps the subgroup $H_{\mu-1}$ identically onto itself. Thus $\pi_{\mu,\mu-1} \circ \alpha_1 = \theta_{\mu-1} \circ \mathrm{Res}_{\mu,\mu-1}|_{\mathrm{Gal}(F_1)}$.

By (v) and (iii) there is an isomorphism $\alpha_2\colon \mathrm{Gal}(F_2) \to G_{\mu-1}$. By (v) and (ii), $\mathrm{Res}_{\mu,\mu-1}$ maps $\mathrm{Gal}(F_2)$ onto 1. As $\pi_{\mu,\mu-1}$ maps $G_{\mu-1}$ onto 1, we have $\pi_{\mu,\mu-1} \circ \alpha_2 = \theta_{\mu-1} \circ \mathrm{Res}_{\mu,\mu-1}|_{\mathrm{Gal}(F_2)}$.

By (vi), α_1, α_2 extend to a unique isomorphism $\theta_\mu\colon \mathrm{Gal}(L_\mu) \to H_\mu$ such that the right diagram in (2) commutes. □

Corollary 10. *Let $\{E_i\}_{i \in I}$ be a family of fields such that all but finitely many E_i contain algebraically closed subfields. Let $H = \coprod_{i \in I} \mathrm{Gal}(E_i)$ be the free profinite product of the* $\mathrm{Gal}(E_i)$ *in the sense of Binz-Neukirch-Wenzel. Then there is a field F of characteristic* 0 *with* $\mathrm{Gal}(F) \cong H$. *If all E_i have the same characteristic p, then there is a field F of characteristic p with* $\mathrm{Gal}(F) \cong H$.

Proof. Let $I' \subseteq I$ be the subset of all $i \in I$ such that E_i contains an algebraically closed subfield. By Theorem 9 there is a field F' of characteristic 0 (or of characteristic p, if all E_i are of characteristic p), such that $\mathrm{Gal}(F') = \coprod_{i \in I'} G_i$. Then $H = \mathrm{Gal}(F') \amalg (\coprod_{i \in I \smallsetminus I'} G_i)$. By [HJK, Theorem 3.4] there is a field F of the desired characteristic with $\mathrm{Gal}(F) \cong H$. □

Although the condition of Corollary 10 need not be necessary, it is significant in the following sense (we again use the notation $\coprod_{i \in I} G_i$ for the free profinite product in the sense of Binz-Neukirch-Wenzel):

Proposition 11. *Let $\{G_i\}_{i\in I}$ be a family of absolute Galois groups. Then $\coprod_{i\in I} G_i$ is an absolute Galois group if and only if there is a countable subset J of I such that $\coprod_{i\in J} G_i$ is an absolute Galois group and for every $i \in I \smallsetminus J$, G_i is the absolute Galois group of a field that contains an algebraically closed subfield.*

Proof. Let $H = \coprod_{i\in I} G_i$. It follows from the universal property of a free profinite product that if J is a subset of I, then H is the free profinite product of $\coprod_{i\in J} G_i$ and $\coprod_{i\in I\smallsetminus J} G_i$. In particular, these two groups are subgroups of H.

Assume that there is a field F such that $H = \mathrm{Gal}(F)$. Let F_0 be the prime subfield of F. The restriction $\pi\colon \mathrm{Gal}(F) \to \mathrm{Gal}(F_0)$ is an epimorphism. As the family $\{G_i\}_{i\in I}$ is converging to 1, also its image $\{\pi(G_i)\}_{i\in I}$ is a family of subgroups of $\mathrm{Gal}(F_0)$ is converging to 1. Thus for every open normal subgroup U of $\mathrm{Gal}(F_0)$ there is a finite subset I_U of I such that $\pi(G_i) \le U$ for all $i \in I \smallsetminus I_U$.

As F_0 is a countable field, the family of open normal subgroups U of $\mathrm{Gal}(F_0)$ is countable, and hence $J := \bigcup_U I_U$ is a countable set. As $\coprod_{i\in J} G_i$ is a subgroup of $H = \mathrm{Gal}(F)$, it is the absolute Galois group of an algebraic separable extension of F. And if $i \in I \smallsetminus J$, then $\pi(G_i) \le \bigcap_U U = 1$; hence the fixed field E_i of G_i (in the algebraic closure of F) contains the algebraic closure of F_0 and $G_i = \mathrm{Gal}(E_i)$.

Conversely, assume that there is J as in this proposition. By assumption, $\coprod_{i\in J} G_i$ is an absolute Galois group. By Theorem 9, so is $\coprod_{i\in I\smallsetminus J} G_i$. Hence by [HJK, Theorem 3.4] so is their free profinite product H. $\square$

Thus realizability of a free profinite product as an absolute Galois group is reduced to free profinite products over a countable set.

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