RELATIVELY PROJECTIVE PRO-p GROUPS

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Dedicated to Moshe Jarden on the occasion of his 80th birthday

Abstract. It is known that the Kurosh Subgroup Theorem does not hold for pro-p groups of large cardinality. However, a subgroup G of a free pro-p product is projective relative to the Kurosh family of subgroups. In this paper we prove the converse of this fact.

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1. Introduction

It is well-known that the Schreier theorem does not hold for free profinite groups, i.e., a subgroup of a free profinite group need not be free. Subgroups of free profinite groups are groups of cohomological dimension 1; they are called projective, as they satisfy the same universal property as projective modules.

Similarly, the Kurosh Subgroup Theorem does not hold for free products of profinite groups. This led the first author to introduce in [2] the notion of a profinite group projective relative to a family $\mathcal{G}$ of its subgroups, closed under conjugation (see Definition 3.2; these groups are also discussed in [4], cf. Remark 3.3). Then [2, Corollary 5.4] has shown that a subgroup G of a free profinite product $H = \coprod_{x\in X} H_x$ is projective relative to the family of subgroups $\mathcal{G} = \{H_x^h \cap G \mid h \in H, x \in X\}$. An analogous result holds for subgroups of free pro-p products.

However, the converse, that is, whether a profinite group G, projective relative to a continuous family $\mathcal{G}$ of its subgroups, is a subgroup of a free product in the above manner, has been treated only partially.

Assuming that $\mathcal{G}$ is a continuous family closed under conjugation, we have $\mathcal{G} = \{G_t \mid t \in T\}$, where T is a profinite space on which G acts continuously so that the map $t \mapsto G_t$ is G-equivariant and injective on $\{t \in T \mid G_t \neq 1\}$ ([2, Lemma 3.5]). Then the answer to the above question is positive, if T has a closed subset T_0 of representatives of the G orbits ([2, Theorem 9.5]). In particular, this is the case if G is second

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countable ([2, Theorem 8.5]). Moreover, if G is a second countable pro-p group, then G is even a free pro-p product $G = (\coprod_{t\in T_0} G_t) \amalg F$, where F is a free pro-p group ([2, Corollary 9.6]).

On the other hand, the second author has produced an example of a pro-p group G, projective relative to a family $\mathcal{G} = \{G_t \mid t \in T\}$, where the action of G on T has no closed subset of representatives of the G-orbits. Nevertheless, G is a subgroup of a free pro-p product in the above manner ([7, Proposition 4.6]).

The objective of this paper is to give a positive answer to the above question for pro-p groups.

First we note (Lemma 5.12) that if G is a subgroup of a free profinite (or pro-p) product $H = \coprod_{x\in X} H_x$, then, denoting $\mathcal{G} = \{H_x^h \cap G \mid h \in H, x \in X\}$,

(a) there is a profinite space T on which G acts continuously so that the family of the stabilizers $\{G_t \mid t \in T\}$ of this action satisfies
  (a1) the map $t \mapsto G_t$ is injective on $T' = \{t \in T \mid G_t \neq 1\}$;
  (a2) $\mathcal{G} \smallsetminus \{1\} = \{G_t \mid t \in T'\}$;

and, as remarked,

(b) G is projective relative to $\mathcal{G}$.

Conversely:

Theorem 1.1. *Let G be a pro-p group and $\mathcal{G} \subseteq \mathrm{Subgr}(G)$. Assume (a), (b). Then there exists an embedding $\zeta\colon G \to \bar{G} = L \amalg F$ into a free pro-p product of a copy L of G and a free pro-p group F such that $\{\zeta(G_t) \mid t \in T'\} = \{L^\sigma \cap \zeta(G) \mid \sigma \in \bar{G}\} \smallsetminus \{1\}$.*

Theorem 1.1 shows that relatively projective pro-p groups can be characterized as subgroups of free pro-p products. However, the factors of the free product in Theorem 1.1 are as complicated as the original group. Our methods however show that one can find a simpler free product, provided the map $t \mapsto G_t$ does not vary too much, in a sense:

Theorem 1.2. *Under the assumptions of Theorem 1.1 let $\rho\colon G \to L$ be a homomorphism into a pro-p group such that $\rho_{|G_t}$ is a monomorphism for each $t \in T$. Then there exists an embedding $\zeta\colon G \to \bar{G} = L \amalg F$ into a free pro-p product of L and a free pro-p group F such that $\{\zeta(G_t) \mid t \in T'\} = \{L^\sigma \cap \zeta(G) \mid \sigma \in \bar{G}\} \smallsetminus \{1\}$.*

Using Theorem 1.2 one can try to index the family of subgroups $\mathcal{G} = \{G_t \mid t \in T\}$ of a relatively projective pro-p group G such that G embeds into the free pro-p product of the new family according to the following

Theorem 1.3. *Let G be a pro-p group projective relative to a continuous family $\mathcal{G} = \{G_t \mid t \in T\}$ of its subgroups closed under conjugation. Let $\rho\colon G \to L = \coprod_{x\in X} L_x$ be a homomorphism into a free pro-p product such that $\rho_{|G_t}$ is a monomorphism into some conjugate of L_x. Then there exist a free pro-p group F and a monomorphism $\zeta\colon G \to L \amalg F$ such that $\zeta(G_t)$ is conjugate to a subgroup of some L_x, for every $t \in T$.*

Using the notion of pile (Definition 4.1) we have another characterization of relatively projective groups:

Theorem 1.4. *Let a pro-p group G act continuously on a profinite space T and let $\mathcal{G} = \{G_t \mid t \in T\}$. Then the following statements are equivalent:*

(i) G is projective relative to $\mathcal{G}$ and (a1), (a2) above hold;
(ii) (G, T) is a projective pile;
(iii) there exists a profinite space $\hat{T}$ that also satisfies (a1), (a2) and $(G, \hat{T}) = \varprojlim_i (G_i, T_i)$, where $G_i = (\coprod_{x\in X_i} (G_i)_x) \amalg F_i$, is a second countable free pro-p product with X_i a closed set of representatives of the G_i-orbits in T_i and F_i a free pro-p group; in particular, if $i \le j$, then the map π_{ij} of the inverse system maps every $(G_j)_x$ into a conjugate of $(G_i)_{\pi_{ij}(x)}$ in G_i;
(iv) G acts on a pro-p tree with $\mathcal{G} \smallsetminus \{1\}$ being the family of nontrivial stabilizers of vertices and with trivial edge stabilizers.

To illustrate Theorem 1.2 consider the following

Example 1.5. Let G be a pro-p group and let $\mathcal{G} \subseteq \mathrm{Subgr}(G)$. Assume (a), (b) above and assume that the point stabilizers G_t are abelian. By Theorem 1.4, G is an inverse limit of free pro-p products G_i of abelian groups; here we use the fact that a free pro-p group F_i is a free pro-p product of copies of the abelian group $\mathbb{Z}_p$. Each G_i is an inverse limit of free pro-p products G_{ij} of finitely many abelian factors. As $[G_{ij}, G_{ij}]$ intersects factors trivially (say, because the projection on a factor is injective on that factor and maps the commutator into 1) one deduces that $[G, G] \cap G_t = 1$ for each $t \in T$. Let $L = G/[G, G]$ be the abelianization of G. Then the quotient map $\rho\colon G \to L$ is injective on G_t, for every $t \in T$.

Thus, by Theorem 1.2, there exists an embedding $\zeta\colon G \to L \amalg F$, where L is abelian and F is pro-p, such that $\zeta(G_t)$ is conjugate to a subgroup of L for every $t \in T$.

Let $\mathcal{C}$ be a family of finite groups closed under quotients, subgroups, and extensions. In most of our application $\mathcal{C}$ will be the family of p-groups, for a fixed prime p, but sometimes a more general treatment seems to be more appropriate.

Given a subset X of a profinite group H, we denote by $\langle X\rangle^H$ the smallest closed normal subgroup of H containing X.

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2. Profinite spaces and action

Remark 2.1. Let T be a profinite space. A **partition** of T is a finite family $X = \{T_i\}_{i=1}^n$ of nonempty clopen subsets of T such that $T = \dot\bigcup_{i=1}^n T_i$ (a disjoint union). It induces a continuous map $\varphi\colon T \to X$, by mapping every $t \in T_i$ onto $T_i \in X$. Conversely, a continuous map $\varphi\colon T \to X$ into a finite discrete space X defines a partition, namely, $\{\varphi^{-1}(\{x\}) \mid x \in \varphi(T) \subseteq X\}$.

A partition Y is **finer** than (or a **refinement** of) the partition X of a space T, if every $T'' \in Y$ is a subset of some T' of X.

Remark 2.2. Let G be a profinite group acting continuously on a profinite space T. If Z is a closed subset of T, then the **stabilizer**

$$G_Z = \{g \in G \mid Z^g = Z\}$$

of Z in G is a closed subgroup of G; if Z is clopen, then G_Z is open. For $t \in T$ we write G_t instead of $G_{\{t\}}$. Thus, $G_t = \{g \in G \mid t^g = t\}$.

A partition $X = \{T_i\}_{i=1}^n$ of T is a G**-partition** if for every i and every $g \in G$ either $T_i^g = T_i$ or $T_i \cap T_i^g = \emptyset$. It defines an obvious G-action on X. A partition X of T is a G-partition if and only if the induced map $\varphi\colon T \to X$ is G-equivariant.

An element of a G-partition is called a G**-block**.

Lemma 2.3. *Let G be a profinite group acting continuously on a profinite space T. Let $t \in T$ and let $\mathcal{B}(t) = \{U \mid U$ is a G-block, $t \in U\}$.*

(a) Every partition of T can be refined by a G-partition.
(b) T is the inverse limit of its G-partitions.
(c) $\mathcal{B}(t)$ is a basis of neighborhoods of t.
(d) $\{t\} = \bigcap_{U\in\mathcal{B}(t)} U$.
(e) If $U \subseteq T$ is a G-block and $Z \subseteq T$ is a non-empty closed subset, then $G_Z \le G_U$; in particular, $G_t \le G_U$ for every $t \in U$;
(f) $G_t = \bigcap_{U\in\mathcal{B}(t)} G_U$.

Proof. (a) [4, Lemma 7.1.1], Claim B.

(b) [4, Lemma 7.1.1].

(c) Follows from (a).

(d) Follows from (c).

(e) Let $g \in G_Z$. Then $\emptyset \neq Z = Z^g \subseteq U \cap U^g$, so $U \cap U^g \neq \emptyset$, and hence $U = U^g$; thus $g \in G_U$.

(f) By (e), $G_t \leq \bigcap_{U \in \mathcal{B}(t)} G_U$. Conversely, if $g \in \bigcap_{U \in \mathcal{B}(t)} G_U$, then $t^g \in \bigcap_{U \in \mathcal{B}(t)} U^g = \bigcap_{U \in \mathcal{B}(t)} U = \{t\}$, by (d), so $g \in G_t$. □

The next lemma is an analogue of [4, Lemma 2.1.3].

Lemma 2.4. *Let a* ***finite*** *group G act continuously on a profinite space T and let X be a partition of T. Then there is a G-partition Y of T, finer than X, such that for every $V \in Y$ there is $t \in V$ with $G_V = G_t$.*

Proof. By induction on the size of $\mathcal{G} = \{G_t \mid t \in T\}$.

For every maximal $\Gamma \in \mathcal{G}$ let $C(\Gamma) = \{t \in T \mid G_t = \Gamma\}$. This is a closed subset of T, because it is the complement of the open subset $\bigcup_{\Gamma' \in \mathcal{G},\, \Gamma' \neq \Gamma} \{t \in T \mid G_t \leq \Gamma'\}$ of T. If $g \in G$, then Γ^g is also maximal in $\mathcal{G}$, and $C(\Gamma)^g = C(\Gamma^g) = \{t \in T \mid G_t = \Gamma^g\}$.

The union $C = \bigcup_{g \in G} C(\Gamma)^g$ is also closed; it is G-invariant. Clearly, $C = \dot\bigcup_{g \in R} C(\Gamma)^g$, where $S = N_G(\Gamma)$ is the stabilizer of $C(\Gamma)$ in G and R is a set of left coset representatives of G modulo S. Thus this is a G-partition of C. Deduce that if $C' \subseteq C(\Gamma)$, then $G_{C'} = S_{C'}$.

By Lemma 2.3(a) we may assume that X is a G-partition.

By Lemma 2.3(d),(f), every $t \in C(\Gamma)$ is contained in an S-block C_t of $C(\Gamma)$ with $S_{C_t} = S_t = G_t = \Gamma$. As $C(\Gamma)$ is compact, finitely many of these C_t cover $C(\Gamma)$. Their intersections constitute a partition X' of $C(\Gamma)$. By Lemma 2.3(a), $C(\Gamma)$ has an S-partition $C(\Gamma) = \dot\bigcup_{i=1}^m C_i$, finer than X' and finer than the partition induced by X on $C(\Gamma)$.

Choose $t' \in C_i$ and $t \in C(\Gamma)$ such that $C_i \subseteq C_t$. By Lemma 2.3(e), $\Gamma = S_{t'} \leq S_{C_i}$ and $S_{C_i} \leq S_{C_t} = \Gamma$. Thus $S_{C_i} = \Gamma = G_t$ for every $t \in C_i$.

Now $C = \dot\bigcup_{g \in R} \dot\bigcup_i C_i^g$ is a G-partition of C and $G_{C_i^g} = S^g_{C_i^g} = G_t$ for every $t \in C_i^g$. We write this partition as $\{C_i\}_{i=1}^n$.

There are disjoint open subsets $T_1, \dots, T_n$ of T such that $C_i \subseteq T_i$ for every i. Let $1 \leq i \leq n$. Every $t \in C_i$ has a clopen neighborhood in T_i such that $G_{t'} \leq \Gamma$ for every t' in that neighborhood. As C_i is compact, we may replace T_i by a union of finitely many such neighborhoods, so that T_i is clopen in T and $G_{t'} \leq \Gamma$, for every $t' \in T_i$.

If $1 \leq i, k \leq n$, and $C_i^g = C_k$, for some $g \in G$, then, without loss of generality, $T_i^g = T_k$, otherwise replace T_i by its subset $V_i = \bigcap_{(j,h)} T_j^{h^{-1}}$, where (j,h) runs through all pairs of $1 \leq j \leq n$ and $h \in G$ such that

$C_i^h = C_j$. Indeed, if $C_i^g = C_k$, then

$$V_i^g = \Big(\bigcap_{\substack{(j,h) \\ C_i^h = C_j}} T_j^{h^{-1}} \Big)^g = \bigcap_{\substack{(j,h) \\ C_k^{g^{-1}h} = C_j}} T_j^{(g^{-1}h)^{-1}} = \bigcap_{\substack{(j,\sigma) \\ C_k^{\sigma} = C_k}} T_j^{\sigma^{-1}} = V_k.$$

Thus $U = \dot\bigcup_{i=1}^n T_i$ is is a G-invariant clopen subset of T, and $\{T_i\}_{i=1}^n$ is its G-partition, such that $G_{T_i} = G_{C_i}$ for every i. Choose $t_i \in C_i \subseteq T_i$, then $G_{T_i} = G_{t_i}$ for every i.

Also $T' = T \smallsetminus U$ is a clopen G-invariant subset. As the set $\{G_t \mid t \in T'\}$ does not contain Γ, its size is strictly smaller than $|\mathcal{G}|$. Therefore, by induction on $|\mathcal{G}|$, there is a partition of T', which, together with the partition $\{T_i\}_{i=1}^n$ of U, gives the required G-partition of T. $\square$

Let $F(T)$, resp. $F(T,*)$, denote the free pro-$\mathcal{C}$ group on a profinite space T, resp. pointed profinite space $(T,*)$ ([9, Section 3.3]).

Lemma 2.5. *Let G be a pro-$\mathcal{C}$ group acting on a profinite space T and let $\tilde{G} = F(T) \rtimes G$ be the induced semidirect product. Then $C_{F(T)}(G) = \langle T^G \rangle = F(T^G)$, where T^G is the subspace of points of T fixed by G.*

Proof. As $C_{F(T)}(G) = \bigcap_{\sigma \in G} C_{F(T)}(\langle \sigma \rangle)$ and by [8, Proposition 5.5.3], $\bigcap_{\sigma \in G} F(T^{\langle \sigma \rangle}) = F(\bigcap_{\sigma \in G} T^{\langle \sigma \rangle})$, it suffices to prove the assertion for G procyclic. Since the centralizer is contained in the normalizer, it suffices to prove in this case a stronger assertion, namely, that $N_{F(T)}(G) = F(T^G)$; equivalently, $N_{\tilde{G}}(G) = F(T^G) \times G$.

First assume that T is finite.

Suppose G acts on T transitively. If G acts trivially on T, then $N_{F(T)}(G) = F(T)$. Thus it suffices to show that if G acts nontrivially on T, then $N_{F(T)}(G) = 1$; equivalently, $N_{\tilde{G}}(G) = G$. Replacing G by its image in $\mathrm{Sym}(T)$ we may assume that G is finite and acts faithfully on T. In particular, since G is cyclic, it acts freely on T, and so, by [4, Lemma 4.7.4], $\tilde{G} = \langle t \rangle \amalg G$. Hence, by a theorem of Herfort and Ribes ([5, Theorem A]), $N_{\tilde{G}}(G) = G$.

Now write T as the disjoint union $T = \dot\bigcup_{i=1}^n T_i$ of its G-orbits. Then $F(T) = \coprod_{i=1}^n F(T_i)$, hence $\tilde{G} = \coprod_{G,i} \tilde{G}_i$ is a free product amalgamating G, where $\tilde{G}_i = F(T_i) \rtimes G$ for each i. By [1, Proposition 2.8] $N_{\tilde{G}}(G) = \coprod_{G,i} N_{\tilde{G}_i}(G)$ and so $N_{\tilde{G}}(G)/G = \coprod_i (N_{\tilde{G}_i}(G)/G)$. Since $N_{\tilde{G}}(G) = N_{F(T)}(G) \times G$ and $N_{\tilde{G}_i}(G) = N_{F(T_i)}(G) \times G$, one has $N_{F(T)}(G) \cong N_{\tilde{G}}(G)/G \cong \coprod_i (N_{\tilde{G}_i}(G)/G) = \coprod_i N_{F(T_i)}(G)$. By the preceding paragraph $N_{F(T_i)}(G) = F(T_i^G)$ for each i and so the lemma is proved if T is finite.

In the general case T is an inverse limit of finite G-spaces, $T = \varprojlim T_k$. Then $T^G = \varprojlim T_k^G$, and $F(T) = \varprojlim F(T_k)$ and $\tilde{G} = \varprojlim (F(T_k) \rtimes G)$. Hence $N_{F(T)}(G) = \varprojlim N_{F(T_k)}(G) = \varprojlim F(T_k^G) = F(T^G)$. □

Lemma 2.6. *Let $F = F(T,*)$ be a free pro-$\mathcal{C}$ group on a pointed profinite space $(T,*)$. Then there is a profinite space Y such that F is $F(Y)$, the free pro-$\mathcal{C}$ group on Y.*

Proof. If T is finite, then $Y = T \smallsetminus \{*\}$ satisfies $F = F(Y)$. So assume that T is infinite. By [9, 3.5.12], $F(T,*)$ and $F(T)$ are free pro-$\mathcal{C}$ groups of the same rank, and hence $F(T,*) \cong F(T)$. □

3. Relatively projective groups

Let G be a profinite group and $(\mathcal{G},T) = \{G_t \mid t \in T\}$ be a family of subgroups indexed by a profinite space T. For a group A, we shall find it convenient to indicate by $A \in \mathcal{G}$ the existence of $t \in T$ with $A = G_t$. Following [8, Section 5.2] we say that $(\mathcal{G},T)$ is **continuous** if for any open subgroup U of G the subset $\{t \in T \mid G_t \leq U\}$ is open.

For instance, $(\mathcal{G},T)$ is continuous if it is **locally constant**, i.e., if T is the disjoint union of finitely many clopen subsets T_i and for each i there is a subgroup A_i of G such that $G_t = A_i$ for every $t \in T_i$.

Lemma 3.1 ([8, Lemma 5.2.1])**.** *Let G be a profinite group and let $\{G_t \mid t \in T\}$ be a collection of subgroups indexed by a profinite space T. Then the following conditions are equivalent:*

- *(a) $\{G_t \mid t \in T\}$ is continuous;*
- *(b) The set $\hat{\mathcal{G}} = \{(g,t) \in G \times T \mid t \in T, g \in G_t\}$ is closed in $G \times T$;*
- *(c) The map $\varphi\colon T \to \mathrm{Subgr}(G)$, given by $\varphi(t) = G_t$, is continuous, where $\mathrm{Subgr}(G)$ is endowed with the étale topology;*
- *(d) $\bigcup_{t\in T} G_t$ is closed in G.*

The set $\hat{\mathcal{G}}$ in (b), together with the projection $\pi\colon \hat{\mathcal{G}} \to T$ on the second coordinate, is a **sheaf** of profinite groups. Given a profinite group H, a **sheaf morphism** $\alpha\colon \hat{\mathcal{G}} \to H$ is a continuous map such that the restriction of α to $\hat{\mathcal{G}}(t) = \pi^{-1}(t)$ is a group homomorphism, for every $t \in T$. For instance, the map $\hat{\mathcal{G}} \to G$, given by $(g,t) \mapsto g$, is a sheaf morphism. These notions are instrumental in the construction of free profinite products ([8, Section 5.1]).

By abuse of notation we write simply $\mathcal{G}$ instead of $(\mathcal{G},T)$ and also instead of $\hat{\mathcal{G}}$.

For a family $\mathcal{G}$ of subgroups of a profinite group G we denote by $\mathrm{Env}(\mathcal{G})$ its **envelope**, the family of all closed subgroups of the groups in $\mathcal{G}$.

Definition 3.2. Consider the category of **profinite pairs** $(G, \mathcal{G})$, where G is a profinite group and $\mathcal{G}$ is a continuous family of closed subgroups of G, closed under the conjugation in G. We denote by $\mathcal{G}^G$ the set of all G-conjugates of the groups in $\mathcal{G}$, that is, $\mathcal{G}^G = \{g^{-1}\Gamma g \mid \Gamma \in \mathcal{G},\, g \in G\}$. A **morphism** $\varphi\colon (G, \mathcal{G}) \to (A, \mathcal{A})$ in this category is a homomorphism $\varphi\colon G \to A$ of profinite groups such that $\varphi(\mathrm{Env}(\mathcal{G})) \subseteq \mathrm{Env}(\mathcal{A})$, that is, for every $\Gamma \in \mathcal{G}$ there is $\Delta \in \mathcal{A}$ such that $\varphi(\Gamma) \le \Delta$; it is an **epimorphism**, if $\varphi(\mathrm{Env}(\mathcal{G})) = \mathrm{Env}(\mathcal{A})$.

An **embedding problem** for $(G, \mathcal{G})$ (cf. [4, Definition 5.1.1] or [2, Definition 4.1]) is a pair of morphisms

$$\begin{array}{ccc} & & (G,\mathcal{G}) \\ & & \big\downarrow{\scriptstyle\varphi} \\ (B,\mathcal{B}) & \xrightarrow{\ \alpha\ } & (A,\mathcal{A}) \end{array} \tag{3.1}$$

such that α is an epimorphism and for every $\Gamma \in \mathcal{G}$ there exists $\Delta \in \mathcal{B}$ and a homomorphism $\gamma_\Gamma\colon \Gamma \to \Delta$ such that $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$.

We say that (3.1) is **finite**, if B is finite. We say that (3.1) is **rigid**, if α is **rigid**, i.e., $\alpha|_\Delta$ is injective for every $\Delta \in \mathcal{B}$.

A **solution** of (3.1) is a morphism $\gamma\colon (G, \mathcal{G}) \to (B, \mathcal{B})$ such that $\alpha \circ \gamma = \varphi$.

We say that G is **projective relative to** $\mathcal{G}$ or $\mathcal{G}$**-projective**, if every finite embedding problem (3.1) for $(G, \mathcal{G})$ has a solution. Equivalently ([4, Corollary 5.1.5]), every finite rigid embedding problem (3.1) for $(G, \mathcal{G})$ has a solution. Moreover, replacing $\mathcal{A}$ by the subset $\mathcal{A}_0$ of maximal groups in $\mathcal{A}$, and $\mathcal{B}$ by $\{\Delta \in \mathcal{B} \mid \alpha(\Delta) \in \mathcal{A}_0\}$, we may assume that α maps every $\Delta \in \mathcal{B}$ isomorphically onto some group in $\mathcal{A}$.

Remark 3.3. We remark that [4] uses the term *strongly* $\mathcal{G}$*-projective* instead of $\mathcal{G}$-projective and both [4] and [2] do not assume that $\mathcal{G}$ is a continuous family, only that it is étale compact. We do not know whether this is the same ([4, Problem 2.1.11 and Proposition 2.1.8]).

However, since we do assume here that $\mathcal{G}$ is indexed by a profinite space, it would be desirable to take this space into account in the above definition. This is the purpose of the following two sections.

4. Piles

Definition 4.1. A **pile** $\mathbf{G} = (G, T)$ consists of a profinite group G, a profinite space T, and a continuous action of G on T (from the right). Denoting by G_t the G-stabilizer of t, for every $t \in T$, we note that $\mathcal{G} = \{G_t \mid t \in T\}$ is a continuous family of closed subgroups of G ([8, Lemma 5.2.2]) closed under the conjugation in G, such that $G_{t^g} = G_t^g$ for all $t \in T$ and $g \in G$.

A pile $\mathbf{G} = (G, T)$ is **finite** if both G and T are finite.

A **morphism** of group piles $\alpha\colon \mathbf{B} = (B, Y) \to \mathbf{A} = (A, X)$ consists of a group homomorphism $\alpha\colon B \to A$ and a continuous map $\alpha\colon Y \to X$ such that $\alpha(y^b) = \alpha(y)^{\alpha(b)}$ for all $y \in Y$ and $b \in B$. This implies $\alpha(B_y) \le A_{\alpha(y)}$ for every $y \in Y$; in particular, denoting $\mathcal{A} = \{A_x \mid x \in X\}$ and $\mathcal{B} = \{B_y \mid y \in Y\}$, we have $\alpha(\mathrm{Env}(\mathcal{B})) \subseteq \mathrm{Env}(\mathcal{A})$.

The **kernel** $\mathrm{Ker}\,\alpha$ **of** α is the kernel of the group homomorphism $\alpha\colon B \to A$.

The above morphism α is an **epimorphism** if $\alpha(B) = A$, $\alpha(Y) = X$, and for every $x \in X$ there is $y \in Y$ such that $\alpha(y) = x$ and $\alpha(B_y) = A_x$. (Then $\alpha(\mathrm{Env}(\mathcal{B})) = \mathrm{Env}(\mathcal{A})$.) It is **rigid**, if α maps B_y isomorphically onto $A_{\alpha(y)}$, for all $y \in Y$, and the induced map of the orbit spaces $Y/B \to X/A$ is a homeomorphism.

Remark 4.2. Let $\alpha\colon \mathbf{B} = (B, Y) \to \mathbf{A} = (A, X)$ be a morphism and let K be its kernel. The quotient map $\pi\colon \mathbf{B} \to \mathbf{B}/K := (B/K, Y/K)$ is an epimorphism of piles and there is a unique morphism $\bar{\alpha}\colon \mathbf{B}/K \to \mathbf{A}$ such that $\alpha = \bar{\alpha} \circ \pi$.

Moreover, α is a rigid epimorphism if and only if $\bar{\alpha}$ is an isomorphism and $K \cap B_y = 1$ for every $y \in Y$.

Construction 4.3. Let G be a profinite group and $\{G_t \mid t \in T_0\}$ a continuous family of subgroups of G. We construct the **standard G-extension** T of T_0 such that (G, T) is a pile, T_0 is a set of representatives of the G-orbits in T, and G_t is the G-stabilizer of t, for every $t \in T_0$.

Let $T = \{(t, G_t g) \mid t \in T_0, G_t g \in G/G_t\}$ and let G act on T by $(t, G_t g)^\sigma = (t, G_t g\sigma)$. Then the G-stabilizer of $(t, G_t g) \in T$ is $\{\sigma \in G \mid G_t g\sigma = G_t g)\} = G_t^g$.

Identifying $t \in T_0$ with $(t, G_s 1) \in T$ we may view T_0 as a subset of T such that T_0 is a set of representatives of the G-orbits in T. Then G_t is the G-stabilizer of t, for every $t \in T_0$.

If G and T_0 are finite, and we regard T as a discrete space, then the above map and the action are continuous, and hence (G, T) is a finite pile.

In the general case we view T as the quotient space of the profinite space $T_0 \times G$ via the map $\pi\colon T_0 \times G \to T$ given by $(t, g) \mapsto (t, G_t g)$. By [8, Proposition 5.2.3] this is a profinite space. The G-action on T is induced via π from the continuous G-action on $T_0 \times G$ by multiplying the second coordinate from the right, and therefore is continuous.

Hence (G,T) is a pile. The embedding $T_0 \to T$ is also continuous, and hence, as every continuous map of profinite spaces is closed, T_0 is a closed subset of T.

Moreover, the above construction is functorial in the following sense. Let H be another profinite group, with a continuous family of subgroups $\mathcal{H} = \{H_s \mid s \in S_0\}$, and let $\varphi\colon G \to H$ be a homomorphism and $\varphi'\colon T_0 \to S_0$ a continuous map, such that $\varphi(G_t) \le H_{\varphi'(t)}$ for every $t \in T_0$. Let S be the standard H-extension of S_0. Then φ, φ' induce a continuous map $T \to S$, namely, $(t, G_t g) \mapsto (\varphi'(t), H_{\varphi'(t)}\varphi(g))$, which, together with φ, form a morphism of piles $(G,T) \to (H,S)$.

Later we shall need the following, easily verified, lemma:

Lemma 4.4. *Let $\mathbf{G} = (G,T)$ be a pile and $\mathbf{A} = (A,X)$, $\mathbf{B} = (B,Y)$ finite piles. Let $\varphi\colon \mathbf{G} \to \mathbf{A}$ be a morphism and $\psi\colon \mathbf{G} \to \mathbf{B}$ an epimorphism. Assume that* $\mathrm{Ker}(\psi) \le \mathrm{Ker}(\varphi)$ *and the partition $\{\psi^{-1}(y) \mid y \in Y\}$ is finer than $\{\varphi^{-1}(x) \mid x \in X\}$. Then there is a morphism $\alpha\colon \mathbf{B} \to \mathbf{A}$ such that $\alpha \circ \psi = \varphi$.*

Lemma 4.5. *Let $\mathbf{G} = (G,T)$ be a pile and let N_0 be an open normal subgroup of G.*

- *(a) Let X be a partition of T. Then there is a finite pile $\mathbf{B} = (B,Y)$ and an epimorphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that* $\mathrm{Ker}(\psi) \le N_0$ *and the partition $\{\psi^{-1}(y) \mid y \in Y\}$ is finer than X.*
- *(b) Let $\varphi\colon \mathbf{G} \to \mathbf{A}$ be a morphism into a finite pile. Then there is a finite pile $\mathbf{B}$, an epimorphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that* $\mathrm{Ker}(\psi) \le N_0$*, and a morphism $\alpha\colon \mathbf{B} \to \mathbf{A}$ such that $\alpha \circ \psi = \varphi$.*

Proof. (a) If N is an open normal subgroup of G, then $(G/N, T/N)$ is a pile and the pair of quotient maps $(G \to G/N, T \to T/N)$ is a rigid epimorphism $\mathbf{G} \to (G/N, T/N)$.

As T is the inverse limit of T/N, where N runs through the open normal subgroups of G, there is an open $N \triangleleft G$ such that $N \le N_0$ and the map $T \to X$ factors through T/N ([4, Lemma 1.1.16(b)]). Thus, replacing $\mathbf{G}$ by $(G/N, T/N)$, we may assume that G is finite and $N_0 = 1$.

Put $B = G$ and let $\psi\colon G \to B$ be the identity. By Lemma 2.4 there is a G-partition $Y = \{T_1, \dots, T_n\}$ of T finer than X, and for every i there is $t_i \in T_i$ such that $G_{T_i} = G_{t_i}$.

Then $\mathbf{B} = (G,Y)$ is a finite pile and the identity of G together with the map $T \to Y$ define an epimorphism $\psi\colon \mathbf{G} \to \mathbf{B}$ with the required properties.

(b) Write $\mathbf{A} = (A,X)$. By (a) there is a finite pile $\mathbf{B} = (B,Y)$ and an epimorphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that $\mathrm{Ker}(\psi) \le N_0$ and the partition

$\{\psi^{-1}(y) \mid y \in Y\}$ is finer than the partition $\{\varphi^{-1}(x) \mid x \in X\}$. By Lemma 4.4 there is a morphism $\alpha\colon B \to A$ such that $\alpha \circ \psi = \varphi$. □

Lemma 4.6. *Let* $\mathbf{G} = (G, T)$ *and* $\mathbf{A} = (A, X)$ *be piles,* $\mathbf{A}$ *finite. Let* $\varphi\colon \mathbf{G} \to \mathbf{A}$ *be a morphism. Assume that* $G_{t_1} \cap G_{t_2} = 1$ *for all distinct* $t_1, t_2 \in T$. *Then there is a finite pile* $\hat{\mathbf{A}} = (\hat{A}, \hat{X})$ *and a factorization of* φ *into an epimorphism* $\hat{\varphi}\colon \mathbf{G} \to \hat{\mathbf{A}}$ *and a morphism* $\varphi_0\colon \hat{\mathbf{A}} \to \mathbf{A}$ *such that if* $\hat{x}_1, \hat{x}_2 \in \hat{X}$ *satisfy* $\varphi_0(\hat{x}_1) \neq \varphi_0(\hat{x}_2)$, *then* $\hat{A}_{\hat{x}_1} \cap \hat{A}_{\hat{x}_2} \leq \operatorname{Ker} \varphi_0$.

Proof. Let $R = \{(t_1, t_2) \mid \varphi(t_1) \neq \varphi(t_2)\}$. This is a clopen subset of $T \times T$, since X is finite. If $(t_1, t_2) \in R$, then $t_1 \neq t_2$, and hence $G_{t_1} \cap G_{t_2} = 1$. Therefore there is an open $N \triangleleft G$ such that

$$G_{t_1} N \cap G_{t_2} N \leq \operatorname{Ker} \varphi. \tag{4.1}$$

If $(t_1', t_2') \in T \times T$ is sufficiently close to (t_1, t_2), then $G_{t_i'} N \leq G_{t_i} N$, for $i = 1, 2$, hence $G_{t_1'} N \cap G_{t_2'} N \leq \operatorname{Ker} \varphi$ as well.

As R is compact, there is an open $N \triangleleft G$ such that (4.1) holds simultaneously for all $(t_1, t_2) \in R$.

By Lemma 4.5(b) there is a finite pile $\hat{\mathbf{A}} = (\hat{A}, \hat{X})$, an epimorphism $\hat{\varphi}\colon \mathbf{G} \to \hat{\mathbf{A}}$, and a morphism $\varphi_0\colon \hat{\mathbf{A}} \to \mathbf{A}$ such that $\varphi_0 \circ \hat{\varphi} = \varphi$ and $\operatorname{Ker} \hat{\varphi} \leq N$.

Let $\hat{x}_1, \hat{x}_2 \in \hat{X}$ such that $\varphi_0(\hat{x}_1) \neq \varphi_0(\hat{x}_2)$. Then there are $t_1, t_2 \in T$ such that $\hat{\varphi}(t_i) = \hat{x}_i$ and $\hat{\varphi}(G_{t_i}) = \hat{A}_{\hat{x}_i}$, for $i = 1, 2$. Then $(t_1, t_2) \in R$, hence (4.1) holds. Since $\operatorname{Ker} \hat{\varphi} \leq N$, we have $\operatorname{Ker} \hat{\varphi} \leq G_{t_i} N, G_{t_i} N$, for $i = 1, 2$, and hence

$$\hat{A}_{\hat{x}_1} \cap \hat{A}_{\hat{x}_2} = \hat{\varphi}(G_{t_1}) \cap \hat{\varphi}(G_{t_2}) \leq \hat{\varphi}(G_{t_1} N) \cap \hat{\varphi}(G_{t_2} N) = \hat{\varphi}(G_{t_1} N \cap G_{t_2} N) \leq \hat{\varphi}(\operatorname{Ker} \varphi) = \operatorname{Ker} \varphi_0.$$

□

Definition 4.7. A commutative diagram of piles

$$\begin{array}{ccc} \hat{\mathbf{B}} & \xrightarrow{\hat{\alpha}} & \hat{\mathbf{A}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varphi_0} \\ \mathbf{B} & \xrightarrow{\alpha} & \mathbf{A} \end{array} \qquad \begin{array}{c} \hat{\mathbf{B}} = (\hat{B}, \hat{Y}),\ \hat{\mathbf{A}} = (\hat{A}, \hat{X}), \\ \\ \mathbf{B} = (B, Y),\ \mathbf{A} = (A, X) \end{array} \tag{4.2}$$

is called a **cartesian square** if, up to an isomorphism, $\hat{\mathbf{B}} = \mathbf{B} \times_{\mathbf{A}} \hat{\mathbf{A}}$, that is, $\hat{B} = B \times_A \hat{A}$, $\hat{Y} = Y \times_X \hat{X}$, and $p, \hat{\alpha}$ are the coordinate projections.

Lemma 4.8. *Let* (4.2) *be a cartesian diagram. If* α *is a rigid epimorphism, then so is* $\hat{\alpha}$.

Proof. Let $(y,\hat{x}) \in Y \times_X \hat{X}$. Then

$$\hat{B}_{(y,\hat{x})} = \{(b,\hat{a}) \in \hat{B} \mid (y^b, \hat{x}^{\hat{a}}) = (y,\hat{x})\} = \\ \{(b,\hat{a}) \in \hat{B} \mid b \in B_y,\ \hat{a} \in \hat{A}_{\hat{x}}\} = B_y \times_{A_x} \hat{A}_{\hat{x}},$$

where $x = \alpha(y) = \varphi_0(\hat{x})$. As α maps B_y isomorphically onto A_x, $\hat{\alpha}$ maps $B_y \times_{A_x} \hat{A}_{\hat{x}}$ isomorphically onto $\hat{A}_{\hat{x}}$.

As $\alpha\colon Y \to X$ is surjective, so is $\hat{\alpha}\colon \hat{Y} \to \hat{X}$, and therefore also the induced map $\hat{Y}/\hat{B} \to \hat{X}/\hat{A}$. We have to show that it is injective, i.e., that $(y,\hat{x}), (y',\hat{x}') \in Y \times_X \hat{X}$, such that $\hat{x}' = \hat{x}^{\hat{a}}$ for some $\hat{a} \in \hat{A}$, are in the same $\hat{B}$-orbit.

Replacing $(y',\hat{x}')$ by $(y',\hat{x}')^{\hat{b}^{-1}}$, where $\hat{\alpha}(\hat{b}) = \hat{a}$, we may assume that $\hat{x}' = \hat{x}$. Put $x = \varphi_0(\hat{x})$, then $\alpha(y) = \alpha(y') = x$. As $Y/B \to X/A$ is a bijection, there is $b \in B$ such that $y' = y^b$. Apply α to get that $x = x^{\alpha(b)}$. Therefore $\alpha(b) \in A_x = \alpha(B_y)$, whence $b = \beta\kappa$, where $\beta \in B_y$ and $\kappa \in \mathrm{Ker}(\alpha)$.

It follows that $(y',\hat{x}') = (y^{\beta\kappa},\hat{x}) = (y^{\kappa},\hat{x}) = (y,\hat{x})^{(\kappa,1)}$. □

5. Projective piles

Definition 5.1. An **embedding problem** for a pile $\mathbf{G}$ is a pair

$$(\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A}) \tag{5.1}$$

of morphisms of group piles such that α is a rigid epimorphism. It is **finite**, if $\mathbf{B}$ is finite.

A **solution** of (5.1) is a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \gamma = \varphi$.

A pile $\mathbf{G}$ is **projective**, if every finite embedding problem for $\mathbf{G}$ has a solution.

Example 5.2. Let $\mathcal{C}$ be a family of finite groups closed under quotients, subgroups, and extensions. Let $\{G_t \mid t \in T_0\}$ be a finite family of $\mathcal{C}$-groups and F a finitely generated free pro-$\mathcal{C}$ group. Form the free pro-$\mathcal{C}$ product $G = F \amalg (\coprod_{t \in T_0} G_t)$ and let T be the standard G-extension of T_0 (Construction 4.3). It is easy to see that $\mathbf{G} = (G,T)$ is a projective pile.

We call a pile of this type a **basic pro-$\mathcal{C}$ pile**.

Lemma 5.3. *Let $\hat{\alpha}\colon \hat{\mathbf{B}} \to \hat{\mathbf{A}}$ be a rigid epimorphism. Assume that* $\mathrm{Ker}\,\hat{\alpha}$ *is a finite group. Then there is a cartesian square* (4.2) *in which* $\alpha\colon \mathbf{B} \to \mathbf{A}$ *is a rigid epimorphism of finite piles.*

Proof. Write $\hat{\mathbf{B}} = (\hat{B},\hat{Y})$ and let $K = \mathrm{Ker}\,\hat{\alpha}$. By Remark 4.2 we have $(\bigcup_{\hat{y} \in \hat{Y}} \hat{B}_{\hat{y}}) \cap (K \smallsetminus \{1\}) = \emptyset$ and we may assume that $\hat{\mathbf{A}} = \hat{\mathbf{B}}/K$ and $\hat{\alpha}$ is the quotient map. By [8, Lemma 5.2.1], $\bigcup_{\hat{y} \in \hat{Y}} \hat{B}_{\hat{y}}$ is a closed subset of $\hat{B}$. As K is finite and $(\bigcup_{\hat{y}\in\hat{Y}} \hat{B}_{\hat{y}}) \cap (K \smallsetminus \{1\}) = \emptyset$, there is an open $N \triangleleft \hat{B}$ such that $K \cap N = 1$ and $(\bigcup_{\hat{y}\in\hat{Y}} \hat{B}_{\hat{y}}) \cap (K \smallsetminus \{1\})N = \emptyset$, whence $\hat{B}_{\hat{y}}N \cap KN = N$, for every $\hat{y} \in \hat{Y}$. For every such N the diagram

$$\begin{array}{ccc} \hat{\mathbf{B}} & \xrightarrow{\hat{\alpha}} & \hat{\mathbf{B}}/K = \hat{\mathbf{A}} \\ \downarrow & & \downarrow \\ \hat{\mathbf{B}}/N & \longrightarrow & \hat{\mathbf{B}}/\hat{\alpha}(N) \end{array}$$

is cartesian and by Remark 4.2 the bottom map is a rigid epimorphism. Since a composition of cartesian diagrams is again a cartesian diagram, we may replace $\hat{\mathbf{B}}$ by $\hat{\mathbf{B}}/N$ to assume that $\hat{B}$ is a finite group.

As $\hat{\alpha}$ is rigid, $K_{\hat{y}} = 1$ for every $\hat{y} \in \hat{Y}$. Thus, by Lemma 2.4, applied to the pile $(K, \hat{Y})$, there is a partition $Y = \{\hat{Y}_i\}_{i=1}^n$ of $\hat{Y}$ such that $K_{\hat{Y}_i} = 1$ for every i. Replacing Y by a refinement we may assume that Y is a $\hat{B}$-partition. Thus $\hat{Y} \to Y$, together with the identity of $\hat{B}$, induces a morphism of piles $p\colon \hat{\mathbf{B}} \to \mathbf{B} := (\hat{B}, Y)$. Then

$$\begin{array}{ccc} \hat{\mathbf{B}} & \xrightarrow{\hat{\alpha}} & \hat{\mathbf{B}}/K \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbf{B} & \longrightarrow & \mathbf{B}/K \end{array}$$

is a cartesian square, because $K_{\hat{Y}_i} = 1$ implies that $\hat{Y}_i \cap \hat{Y}_i^{\kappa} = \emptyset$ for every $1 \neq \kappa \in K$, hence $\hat{Y}_i$ contains at most one element of every K-orbit in $\hat{Y}$. Moreover, $K \cap \hat{B}_{\hat{Y}_i} = K_{\hat{Y}_i} = 1$, hence the bottom map is a rigid epimorphism. □

Proposition 5.4. *Let* $\mathbf{G}$ *be a projective pile. Then every embedding problem (not necessarily finite) for* $\mathbf{G}$ *has a solution.*

Proof. Let $(\hat{\varphi}\colon \mathbf{G} \to \hat{\mathbf{A}},\ \hat{\alpha}\colon \hat{\mathbf{B}} \to \hat{\mathbf{A}})$ be an embedding problem for $\mathbf{G} = (G, T)$. We may assume that $\hat{\alpha}$ is the quotient map $\hat{\mathbf{B}} \to \hat{\mathbf{B}}/K$ for some $K \triangleleft G$.

If K is finite, then by Lemma 5.3 we have a commutative diagram

$$\begin{array}{ccc} & & \mathbf{G} \\ & & {\scriptstyle \hat{\varphi}}\downarrow \\ \hat{\mathbf{B}} & \xrightarrow{\hat{\alpha}} & \hat{\mathbf{A}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ \mathbf{B} & \xrightarrow{\alpha} & \mathbf{A} \end{array} \qquad (\varphi\colon \mathbf{G} \to \mathbf{A}) \tag{5.2}$$

with a cartesian square, in which α is a rigid epimorphism of finite piles. By assumption there is a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha\circ\gamma = \varphi = \varphi_0 \circ \hat{\varphi}$. By the universal property of fiber products there is a unique morphism $\hat{\gamma}\colon \mathbf{G} \to \hat{\mathbf{B}}$ such that $\hat{\alpha}\circ\hat{\gamma} = \hat{\varphi}$ (and $p\circ\hat{\gamma} = \gamma$). Thus $\hat{\gamma}$ is a solution of the given embedding problem.

The general case is verbally identical with Part II in the proof of [3, Lemma 7.3]. □

Lemma 5.5. *Let $\varphi\colon \mathbf{G} \to \mathbf{A}$ and $\alpha\colon \mathbf{B} \to \mathbf{A}$ be morphisms of piles, $\mathbf{B} = (B, Y)$ and $\mathbf{A} = (A, X)$ finite, and $\mathbf{G} = (G, T)$. Let $\psi\colon G \to B$ be a group homomorphism such that $\alpha \circ \psi = \varphi$. Then ψ can be completed to a morphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \psi = \varphi$ if and only if for every $t \in T$ there is $y \in Y$ such that*

$$\alpha(y) = \varphi(t) \text{ and } \psi(G_t) \le B_y. \tag{5.3}$$

Proof. If ψ extends to $\mathbf{G} \to \mathbf{A}$, then (5.3) holds with $y = \psi(t)$.

Conversely, assume that the condition holds. Let $t \in T$ and fix $y \in Y$ that satisfies (5.3). As $\varphi\colon T \to X$ is continuous and X is finite, there is a clopen neighborhood U_t of t in T such that $\varphi(U_t) = \{\alpha(y)\}$. As $\psi(G_t) \le B_y$, by Lemma 2.3(c),(f) we may assume that $\psi(G_{U_t}) \le B_y$.

As T is compact, $\{U_t \mid t \in T\}$ has a finite subcovering. Therefore there is a partition $T = \dot{\bigcup}_{i=1}^n T_i$ and there are $y_1, \dots, y_n \in Y$ such that $\varphi(T_i) = \{\alpha(y_i)\}$ and $\psi(G_{T_i}) \le B_{y_i}$, for every i.

If we now define $\psi\colon T \to Y$ by mapping T_i onto y_i, then ψ is continuous and $\alpha \circ \psi = \varphi$ on T.

By Lemma 2.3(a) we may assume that $\{T_i\}_{i=1}^n$ is a G-partition. Without loss of generality, if $T_j = T_i^g$, with $g \in G$, then $y_j = y_i^{\psi(g)}$. Indeed, we fix a representative T_i of a G-orbit in $\{T_i\}_{i=1}^n$ and for $T_j = T_i^g$ redefine y_j to be $y_j = y_i^{\psi(g)}$. This definition is good: If $T_i^g = T_i^h$, then $hg^{-1} \in G_{T_i}$, hence $\psi(h)\psi(g)^{-1} = \psi(hg^{-1}) \in B_{y_i} = B_{y_i}$, whence $y_i^{\psi(g)} = y_i^{\psi(h)}$.

It then follows that $\psi(t^g) = \psi(t)^{\psi(g)}$ for every $g \in G$. □

Corollary 5.6. *Let $\mathbf{G} = (G, T)$ be a pile, $\mathbf{B} = (B, Y)$ a finite pile, and $\psi\colon G \to B$ a group homomorphism. Assume that for every $t \in T$ there is $y \in Y$ such that $\psi(G_t) \le B_y$. Then ψ can be completed to a morphism $\psi\colon \mathbf{G} \to \mathbf{B}$.*

Moreover, let X be a partition of T, and put $\mathcal{B} = \{B_y \mid y \in Y\}$. Then there is $N \in \mathbb{N}$ such that if

$$|\{y \in Y \mid B_y = B'\}| \ge N \text{ for all } B' \in \mathcal{B}, \tag{5.4}$$

then ψ can be completed so that the partition $\{\psi^{-1}(\{y\}) \mid y \in Y\}$ is finer than X.

Proof. Let $\mathbf{A} = (1, \{*\})$ be the trivial pile and let $\varphi\colon \mathbf{G} \to \mathbf{A}$ and $\alpha\colon \mathbf{B} \to \mathbf{A}$ be the unique morphisms of piles. Then the group homomorphisms satisfy $\alpha \circ \psi = \varphi$, and $\alpha(y) = * = \varphi(t)$ for all $t \in T$ and $y \in Y$. By Lemma 5.5, ψ can be completed to a morphism of piles.

Moreover, given a partition X of T, by Lemma 2.3 we may replace $\{T_i\}_{i=1}^n$, in the proof of Lemma 5.5 for this particular case, by a partition finer than X. Then, if (5.4) holds for a sufficiently large N, we can choose $y_1, \dots, y_n$ above from distinct B-orbits in Y. Then $y_1, \dots, y_n$ remain distinct even after we replace them by their conjugates, and hence $\{\psi^{-1}(\{y\}) \mid y \in Y\} = \{\psi^{-1}(\{y_i\})\}_{i=1}^n = \{T_i\}_{i=1}^n$. □

Proposition 5.7. *A pile $\mathbf{G} = (G, T)$ is projective if and only if*

(a) G is $\{G_t \mid t \in T\}$-projective; and
(b) if $t, t' \in T$ are distinct, then $G_t \cap G_{t'} = 1$.

Proof. Assume that $\mathbf{G}$ is projective. Thus every finite embedding problem (5.1) for piles has a solution. Put $\mathcal{G} = \{G_t \mid t \in T\}$.

(a) We have to show that every finite rigid embedding problem

$$\big(\varphi\colon (G, \mathcal{G}) \to (A, \mathcal{A}),\ \alpha\colon (B, \mathcal{B}) \to (A, \mathcal{A})\big) \tag{3.1}$$

for the pair $(G, \mathcal{G})$ has a solution.

Write $\mathcal{B}$ as $\{B_y \mid y \in Y_0\}$ and for every $y \in Y_0$ let $A_y = \alpha(B_y)$. Then $\mathcal{A} = \{A_y \mid y \in Y_0\}$, and $\mathcal{A} = \mathcal{A}^A$, $\mathcal{B} = \mathcal{B}^B$. Let Y be the standard B-extension of Y_0 with respect to $\mathcal{B}$, and let X be the standard A-extension of Y_0 with respect to $\mathcal{A}$ (Construction 4.3). Then $\mathbf{B} = (B, Y)$ and $\mathbf{A} = (A, X)$ are piles and the identity map $Y_0 \to Y_0$ extends α to a rigid epimorphism $\alpha\colon \mathbf{B} \to \mathbf{A}$.

As φ is a morphism of pairs, for every $t \in T$ there is $x \in X$ such that $\varphi(G_t) \le A_x$. Hence by Corollary 5.6, φ can be completed to a morphism $\varphi\colon \mathbf{G} \to \mathbf{A}$.

As $\mathbf{G}$ is projective, there is a morphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \psi = \varphi$. In particular, the group homomorphism $\psi\colon G \to B$ satisfies $\alpha \circ \psi = \varphi$ and $\psi(G_t) \le B_{\psi(t)}$, for every $t \in T$, that is, $\psi(\mathcal{G}) \subseteq \mathrm{Env}(\mathcal{B})$. Thus ψ solves (3.1).

(b) Let $t, t' \in T$ be distinct. It suffices to show that $G_t \cap G_{t'}$ is contained in every open normal subgroup N of G.

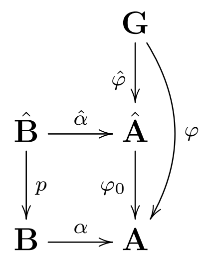
By Lemma 4.5(a) there is an epimorphism $\varphi\colon \mathbf{G} \to \mathbf{A}$ onto a finite pile $\mathbf{A} = (A, X)$ such that $\mathrm{Ker}(\varphi) \le N$ and $\varphi(t), \varphi(t') \in X$ are distinct.

Let Y_0 be a set of representatives of the A-orbits of X. Let $B = A \amalg (\amalg_{y \in Y_0} A_y)$ and let $\alpha\colon B \to A$ be the epimorphism that maps A, A_y identically to the corresponding subgroups of A. Put $\mathcal{B} = \{A_y^b \mid y \in Y_0,\ b \in B\}$. Let Y be the standard B-extension of Y_0 with respect to $\mathcal{B}$, so that $\mathbf{B} = (B, Y)$ is a pile. Using the functoriality in Construction 4.3, α together with the identity of Y_0 extend to a morphism $\alpha\colon \mathbf{B} \to \mathbf{A}$. It is easy to see that α is a rigid epimorphism.

By Proposition 5.4 there is a morphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha\circ\psi = \varphi$. Then $\psi(t) \neq \psi(t')$, because $\alpha\circ\psi(t) = \varphi(t) \neq \varphi(t') = \alpha\circ\psi(t')$. By [4, Lemma 3.1.10], $B_{\psi(t)} \cap B_{\psi(t')} = 1$. Thus $\varphi(G_t \cap G_{t'}) = \alpha(\psi(G_t \cap G_{t'})) \leq \alpha(\psi(G_t) \cap \psi(G_{t'})) \leq \alpha(B_{\psi(t)} \cap B_{\psi(t')}) = 1$, whence $G_t \cap G_{t'} \leq \mathrm{Ker}(\varphi) \leq N$.

Conversely, assume that (a) and (b) hold. Let (5.1) be a finite embedding problem (φ, α) for $\mathbf{G}$, with $\mathbf{A} = (A, X)$ and $\mathbf{B} = (B, Y)$.

Let $\hat{\mathbf{A}} = (\hat{A}, \hat{X})$ and $\hat{\varphi}, \varphi_0$ be as in Lemma 4.6. Then there is a commutative diagram with a cartesian square



(5.2)

By Lemma 4.8, $(\hat{\varphi}, \hat{\alpha})$ is also a finite embedding problem for $\mathbf{G}$.

As G is $\mathcal{G}$-projective, there is a group homomorphism $\hat{\psi}\colon G \to \hat{B}$ such that $\hat{\alpha} \circ \hat{\psi} = \hat{\varphi}$ and for every $t \in T$ there is $\hat{y} \in \hat{Y}$ such that

$$\hat{\psi}(G_t) \leq \hat{B}_{\hat{y}}. \tag{5.5}$$

Put $\psi = p \circ \hat{\psi}$ and $y = p(\hat{y}) \in Y$. Then $\alpha \circ \psi = \varphi$ and $\psi(G_t) \leq B_y$.

If $\varphi(G_t) = 1$, use $\alpha(Y) = X$ to choose $y' \in Y$ such that $\alpha(y) = \varphi(y')$. We have $\alpha(\psi(G_t)) = \varphi(G_t) = 1$, and α is injective on B_y, hence also on $\psi(G_t)$, so $\psi(G_t) = 1 \leq B_{y'}$. Thus condition (5.3) of Lemma 5.5 holds with y' instead of y.

If $\varphi(G_t) \neq 1$, then $\hat{\varphi}(G_t) \not\leq \mathrm{Ker}(\varphi_0)$. But, by (5.5),

$$\hat{\varphi}(G_t) = \hat{\alpha} \circ \hat{\psi}(G_t) \leq \hat{\alpha}(\hat{B}_{\hat{y}}) = A_{\hat{\alpha}(\hat{y})}$$

and $\hat{\varphi}(G_t) \leq \hat{A}_{\hat{\varphi}(t)}$, hence $\hat{A}_{\hat{\alpha}(\hat{y})} \cap \hat{A}_{\hat{\varphi}(t)} \not\leq \mathrm{Ker}(\varphi_0)$. By Lemma 4.6, $\varphi_0(\hat{\alpha}(\hat{y})) = \varphi_0(\hat{\varphi}(t))$, that is, $\alpha(y) = \varphi(t)$. Thus condition (5.3) of Lemma 5.5 holds.

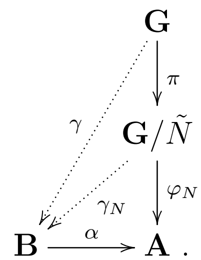
By Lemma 5.5, ψ can be completed to a solution of (5.1). $\square$

Remark 5.8. If G is a $\mathcal{G}$-projective group, then $\Gamma \cap \Gamma' = 1$ for all distinct $\Gamma, \Gamma' \in \mathcal{G}$ ([4, Proposition 5.5.3(b)]). Hence we can replace (b) in Proposition 5.7 by

(b') the map $t \mapsto G_t$ is injective on $T' = \{t \in T \mid G_t \neq 1\}$.

Lemma 5.9. *Let $\mathbf{G} = (G, T)$ be a projective pile. Let N be a normal subgroup of G and $\tilde{N} = \langle N \cap G_t \mid t \in T\rangle$. Then the quotient pile $\mathbf{G}/\tilde{N} := (G/\tilde{N}, T/\tilde{N})$ is projective.*

Proof. Let $\pi \colon \mathbf{G} \to \mathbf{G}/N$ be the quotient map. We have to find a solution γ_N of a finite embedding problem (φ_N, α) as depicted in the lower part of the following diagram



(5.6)

Then $(\varphi_N \circ \pi, \alpha)$ is a finite embedding problem for $\mathbf{G}$. Since $\mathbf{G}$ is projective, this embedding problem has a solution γ.

Note that $N \cap G_t \leq \mathrm{Ker}(\gamma_{|G_t}) \leq \mathrm{Ker}(\gamma)$ (as $(\varphi_N \circ \pi)(N \cap G_t) = 1$ and $\alpha_{|\gamma(N \cap G_t)}$ is injective), for every $t \in T$. Hence $\gamma(\tilde{N}) = 1$, whence γ factors via π, and so we obtain a solution of (φ_N, α). □

Since projective pro-p groups are free pro-p ([9, Theorem 7.7.4]), we get:

Corollary 5.10. *In the above setting $N/\tilde{N}$ is a projective profinite group. If G is pro-p, then $N/\tilde{N}$ is free pro-p.*

Corollary 5.11. *In the above setting $G/\langle G_t \mid t \in T\rangle$ is a projective profinite group. If G is pro-p, then it is free pro-p.*

Proof. Let $N = \langle G_t \mid t \in T\rangle$. Then $\tilde{N} = \langle N \cap G_t \mid t \in T\rangle = N$. By Lemma 5.9, $\mathbf{G}/N$ is projective. By Proposition 5.7, G is $\{1\}$-projective, i.e., projective. □

Lemma 5.12. *Let H be a free pro-$\mathcal{C}$ product of a continuous family $\mathcal{H}$ of its subgroups and let G be a closed subgroup of H. Put $\mathcal{G} = \{G \cap \Delta^h \mid \Delta \in \mathcal{H},\ h \in H\}$. Then there is a profinite G-space T such that*

(a) $\mathcal{G} = \{G_t := G_t \mid t \in T\}$;
(b) if $t, t' \in T$ are distinct, then $G_t \cap G_{t'} = 1$;
(c) the map $t \mapsto G_t$ is injective on $\{t \in T \mid G_t \neq 1\}$;
(d) G is $\mathcal{G}$-projective.

Thus, $\mathbf{G} = (G, T)$ is a projective pile.

Proof. (a) Write $\mathcal{H} = \{H_t \mid t \in T_0\}$ and let T be the standard H-extension of T_0. Then $\mathcal{H}^H = \{H_t \mid t \in T\}$. For $t \in T$ put $G_t = G \cap H_t$. Then $\mathcal{G} = \{G_t \mid t \in T\}$. Furthermore, G, as a subgroup of H, acts on T. By Construction 4.3(a), $G_t = G \cap H_t = G \cap H_t = G_t$.

(b) By [8, Corollary 7.1.5(a)], $H_t \cap H_{t'} = 1$. As $G_t \leq H_t$ and $G_{t'} \leq H_{t'}$, also $G_t \cap G_{t'} = 1$.

(c) Follows from (b).

(d) By [4, Proposition 5.2.2], H is $\mathcal{H}^H$-projective. (This result is stated in [4] only for a free profinite product H, but the proof goes through, mutatis mutandis, also for a free pro-$\mathcal{C}$ product.) Now apply [4, Proposition 5.4.2].

It follows from (b) and (d) by Proposition 5.7 that $\mathbf{G}$ is projective. □

Lemma 5.13. *Let G be a $\mathcal{G}$-projective pro-p group. Let $\rho\colon G \to L$ be an epimorphism, injective on every $\Gamma \in \mathcal{G}$. Extend ρ to an epimorphism $\rho_L\colon G \amalg L \to L$ by the identity of L. Then $\operatorname{Ker}\rho_L$ is a free pro-p group.*

Proof. Let $H = G \amalg L$ and $K = \operatorname{Ker}(\rho_L)$. Put $\mathcal{H} = \mathcal{G} \cup \{L\} \subseteq \operatorname{Subgr}(H)$.

Clearly, L is $\{L\}$-projective. By [4, Lemma 5.2.1], H is $\mathcal{H}$-projective, and hence also $\mathcal{H}^H$-projective ([4, Lemma 5.2.4]). As ρ_L is injective on every $\Gamma \in \mathcal{H}$, and hence on every $\Gamma \in \mathcal{H}^H$, we have $K \cap \Gamma = 1$ for every $\Gamma \in \mathcal{H}^H$. By [4, Proposition 5.4.2], K is $\{1\}$-projective, that is, projective. Therefore K is free pro-p. □

6. HNN-extensions

Let G be a pro-$\mathcal{C}$ group, let $\mathcal{G} = \{G_t \mid t \in T\}$ be a continuous family of its subgroups, and let $\phi\colon \mathcal{G} \to G$ be a sheaf morphism such that $\phi_t := \phi|_{G_t}\colon G_t \to G$ is injective, for every $t \in T$.

A **pro-$\mathcal{C}$ HNN-extension** $\tilde{G} := \operatorname{HNN}(G, T, \mathcal{G}, \phi)$ is a special case of the fundamental pro-$\mathcal{C}$ group of a profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ (see [8, Example 6.2.3(e)]). Namely, $\tilde{G}$ can be thought of as $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$, where

(a) Γ is a bouquet of loops (i.e., a profinite graph having just one vertex v) with T as the space of edges, such that T is closed in $\Gamma = \{v\} \cup T$;
(b) $\mathcal{G}$ turns into a sheaf over Γ by putting $G_v = G$;
(c) the boundary maps $\partial_0, \partial_1\colon \mathcal{G} \to G$ are the inclusion and ϕ, respectively.

Thus an HNN-extension can be explicitly defined as follows. Let $F(T)$ denote the free pro-$\mathcal{C}$ group on T (see [9, Section 3.3]).

Definition 6.1. The **HNN-extension** $\tilde{G} := \mathrm{HNN}(G, T, \mathcal{G}, \phi)$ is the quotient of the free pro-$\mathcal{C}$ product $G \amalg F(T)$ modulo the relations

$$\phi_t(g_t) = g_t^t, \qquad g_t \in G_t,\ t \in T. \tag{6.1}$$

We call G the **base group**, T the set of **stable letters**, and the subgroups G_t and $\phi(G_t)$ **associated**. The inclusion $G \to G \amalg F(T)$ induces a homomorphism $\xi \colon G \to \tilde{G}$, the **base map**.

Obviously, $\tilde{G}$ has the following **universal property**: Given a pro-$\mathcal{C}$ group L, a homomorphism $\beta \colon G \to L$, and a continuous map $\zeta \colon T \to L$, such that $\beta(\phi_t(g_t)) = \beta(g_t)^{\zeta(t)}$, for all $g_t \in G_t$ and all $t \in T$, then there exists a unique homomorphism $\tilde{\beta} \colon \tilde{G} \to L$ such that $\beta = \tilde{\beta} \circ \xi$.

We call the HNN-extension **special**, if ϕ is the inclusion (so that every t centralizes G_t) and $\mathcal{G}$ is locally constant, that is, there is a partition $T = \dot{\bigcup}_{i=1}^n T_i$ of T and there are subgroups $G_1, \ldots, G_n$ of G such that $\mathcal{G} = \dot{\bigcup}_{i=1}^n T_i \times G_i$. In particular, $G_t = G_i$ for every $t \in T_i$. In this case we usually write $\tilde{G} := \mathrm{HNN}(G, T, \mathcal{G})$, omitting ϕ.

Remark 6.2. One could conceive a more general definition of a bouquet Γ and an HNN-extension $\tilde{G}$. Namely, (a) $\Gamma = \{v\} \,\dot\cup\, T$ is a profinite graph having just one vertex v, with T as the space of edges, this time not necessarily closed in Γ; (b) $\mathcal{G}$ is a sheaf over Γ such that $G_v = G$; and (c) above holds. Let $\phi \colon \Gamma \to G$ be a morphism such that $\phi_v = \mathrm{id}_G$ and $\phi_t \colon G_t \to G$ is injective, for every $t \in T$. Then $\tilde{G} := \mathrm{HNN}(G, \Gamma, \mathcal{G}, \phi)$ is the quotient of the free pro-$\mathcal{C}$ product $G \amalg F(\Gamma, v)$ modulo the relations (6.1), where $F(\Gamma, v)$ is the free pro-$\mathcal{C}$ group on the pointed profinite space (Γ, v) (see [9, Section 3.3]). (We could also add the relations $\phi_v(g_v) = g_v^v$ for all $g_v \in G_v = G$, but v, as an element of $F(\Gamma, v)$, is 1, and $\phi_v = \mathrm{id}_G$, so these relations are redundant.)

In this more general setting the definition of a special HNN-extension is the same as in Definition 6.1, except that the sets T_i in the partition $T = \dot{\bigcup}_{i=1}^n T_i$ are not only clopen in T, but also open in Γ (or, equivalently, the sets $T_i \cup \{v\}$ are closed in Γ).

This definition of a special HNN-extension is the one adopted in [10] (see [10, Definition 2.9, Remark 2.10]), from which we are going to quote some results. However, it turns out that it is equivalent to a special HNN-extension of our Definition 6.1.

Indeed, let $T = \dot{\bigcup}_{i=1}^n T_i$ with sets $T_i \cup \{v\}$ closed in $\Gamma = \{v\} \,\dot\cup\, T$. If ϕ is the inclusion, then (6.1) is equivalent to

$$t^{g_t} = t, \qquad g_t \in G_i,\ t \in T_i,\ 1 \le i \le n,$$

that is,

$$t^{g_t} = t, \qquad g_t \in G_i,\ t \in F(T_i, v).$$

Then $F(T_i \cup \{v\}, v) \cong \coprod_{i=1}^n F(T_i \cup \{v\}, v)$. By Lemma 2.6, there are profinite spaces, $T_1', \ldots, T_n'$ such that $F(T_i \cup \{v\}, v) \cong F(T_i')$ for each i. Put $T' = \dot\bigcup_{i=1}^n T_i'$, then $F(\Gamma, v) \cong F(T')$.

Remark 6.3. Let $\tilde{G} := \mathrm{HNN}(G, T, \mathcal{G})$ be a special HNN-extension. The identity of G extends to an epimorphism $\iota \colon \tilde{G} \to G$ that maps T to the trivial element of G. Then $\tilde{G} = G \ltimes \mathrm{Ker}(\iota)$. In particular, the map $G \to \tilde{G}$ is injective. On the other hand, the identity of $F(T)$ together with the map that maps G to the trivial element extend to an epimorphism $\tilde{G} \to F(T) = \tilde{G}/\langle G\rangle^G$, and so the map $F(T) \to \tilde{G}$ is injective.

Moreover, if $\tilde{G} = G \ltimes \tilde{F}$ is another expression as semidirect product, then we can choose the space of stable letters to be in $\tilde{F}$. Indeed, for every $t \in T$ we have $t = h_t t'$, for unique $h_t \in G$, $t' \in \tilde{F}$, and so $T' = \{t' \mid t \in T\}$ is the needed space of stable letters, homeomorphic to T, with associated subgroups $G_t^{h_t}$. Thus $\tilde{F}$ becomes the normal closure $\tilde{F} = \langle T\rangle^{\tilde{G}}$ of T in $\tilde{G}$.

Theorem 6.4. [10, Corollary 5.2] *Let $G = L \ltimes F$ be a semidirect product of a finite p-group L and a free pro-p group F. Suppose that every torsion element of G is F-conjugate into L. Then G is a special pro-p HNN-extension with base group L.*

Lemma 6.5. *Let $\mathcal{L} = \{L_s \mid s \in S\}$ be a continuous family of subgroups of a finite group L, and let $\tilde{G} = \mathrm{HNN}(L, S, \mathcal{L})$ be the pro-p special HNN-extension. Put $\tilde{F} = \langle S\rangle^{\tilde{G}}$ and let*

$$S' = \{s \in S \mid L_s \neq 1\}, \quad S'' = S \smallsetminus S', \quad R' = \langle C_{\tilde{F}}(L_s)) \mid s \in S'\rangle^{\tilde{G}}.$$

Then

(a) *For $M \leq L$ we have $C_{\tilde{F}}(M) = \langle s^l \mid s \in S,\, l \in L,\, M \leq L_s^l\rangle$.*
(b) *$R' = \langle S'\rangle^{\tilde{G}}$.*
(c) *$\tilde{G}/\langle S'\rangle^{\tilde{G}} = L \amalg F(S'')$.*

Proof. (a) By [10, Proposition 2.11], $\tilde{F}$ is the free pro-p group on its subspace $\tilde{S} = \{s^l \mid s \in S,\, l \in L\}$, and the L-stabilizer of $s^l \in \tilde{S}$ is L_s^l. Therefore by Lemma 2.5, $C_{\tilde{F}}(M) = \langle s^l \in \tilde{S} \mid M \leq L_s^l\rangle$.

(b) Let $M = L_s$. Then $M \neq 1$, hence, by (a), $C_{\tilde{F}}(M)$ is contained in $\langle (s')^{l'} \in \tilde{S} \mid s' \in S,\, l' \in L,\, L_{s'} \neq 1\rangle \leq \langle S'\rangle^{\tilde{G}}$. Thus $R' \leq \langle S'\rangle^{\tilde{G}}$. But $s \in C_{\tilde{F}}(L_s)$ for all $s \in S'$, so $\langle S'\rangle^{\tilde{G}} \leq R'$.

(c) From the presentation (6.1) of a special HNN-extension

$$\tilde{G}/\langle S'\rangle^{\tilde{G}} = L \amalg F(S'') \amalg F(S')/\langle S'\rangle^{\tilde{G}} = L \amalg F(S''). \qquad \square$$

Apart from special HNN-extensions we shall be using HNN-extensions only of the following two kinds:

Definition 6.6. Let $\mathbf{G} = (G, T)$ be a pile of pro-$\mathcal{C}$ groups, T_0 a closed subset of T, and $\rho\colon G \to L$ be a homomorphism of pro-$\mathcal{C}$ groups, injective on G_t, for every $t \in T_0$.

Put $\mathcal{G} = \{G_t \mid t \in T_0\}$ and let $G_L = G \amalg L$ be the free pro-$\mathcal{C}$ product of G and L. Then $G, L \subseteq G_L$. Hence $\mathcal{G}$ is also a continuous family of subgroups of G_L and ρ can be viewed as a sheaf morphism $\mathcal{G} \to G_L$. This point of view allows us to form the **HNN'-extension**

$$\widetilde{\mathrm{HNN}}(G, T_0, \rho, L) := \mathrm{HNN}(G_L, T_0, \mathcal{G}, \rho). \tag{6.2}$$

Thus, explicitly, $\widetilde{\mathrm{HNN}}(G, T_0, \rho, L)$ is the quotient of the free pro-$\mathcal{C}$ product $G \amalg L \amalg F(T_0)$ modulo the relations

$$\rho(g_t) = g_t^t, \qquad g_t \in G_t,\ t \in T_0. \tag{6.3}$$

The inclusion $G \to G \amalg L \amalg F(T_0)$ induces the **HNN-map** $\eta_G\colon G \to \widetilde{\mathrm{HNN}}(G, T_0, \rho, L)$. Recall that the inclusion $G \amalg L \to G \amalg L \amalg F(T_0)$ induces the base map $\xi\colon G \amalg L \to \widetilde{\mathrm{HNN}}(G, T_0, \rho, L)$.

By (6.3), ρ extends to a homomorphism $\tilde{\rho}\colon \widetilde{\mathrm{HNN}}(G, T_0, \rho, L) \to L$, the **HNN-extension of** ρ, that maps L identically onto itself and T to 1.

Remark 6.7. Both T_0 and G are subsets of the above mentioned group $G \amalg L \amalg F(T_0)$. If $T_0 = T$, then G also acts on T. In this case the notation $t^g \in G \amalg L \amalg F(T)$, for $t \in T$ and $g \in G$, is ambiguous: It could be either the outcome of the action of g on tT or the conjugate $g^{-1}tg$ of t in $G \amalg L \amalg F(T)$.) To avoid this ambiguity we write henceforth $t^{\bullet g}$ for the former and reserve the notation t^g for the latter.

If $T_0 = T$ and $g \in G$, then each relation $\rho(g_t) = g_t^t$, for $g_t \in G_t$, produces the conjugate relation $\rho(g_t)^{\rho(g)} = g_t^{t\rho(g)}$, in addition to the relation $\rho(g_t^g) = (g_t^g)^{t^{\bullet g}}$, which follows from (6.3) observing that $g_t^g \in G_{t^{\bullet g}}$. This produces a lot of centralizers $(gt^{\bullet g})(t\rho(g))^{-1}$ of G_t, which makes it difficult to embed G into a free pro-p product in such a way that G_t is embedded in one of the free factors, because in a free pro-p product the centralizer of a factor is contained in this free factor cf. [8, Corollary 7.1.6(b)]). Therefore we shall give a definition of an HNN-extension of a pile, which is an HNN'-extension with set of stable letters T_0, the image of a continuous section $T/G \to T$, if such a section exists.

Definition 6.8. Let $\mathbf{G} = (G, T)$ be a pile of pro-$\mathcal{C}$ groups, and $\rho\colon G \to L$ be a homomorphism of pro-$\mathcal{C}$ groups, injective on G_t, for every $t \in T$. The **pile HNN-extension**

$$\bar{G} := \overline{\mathrm{HNN}}(G, T, \rho, L)$$

is the quotient of the free pro-$\mathcal{C}$ product $G \amalg L \amalg F(T)$ modulo the relations:

$$g^{-1} t \rho(g) = t^{\bullet g} \text{ for all } t \in T \text{ and } g \in G. \tag{6.4}$$

Here $t^{\bullet g} \in T \subseteq F(T) \le G \amalg L \amalg F(T)$ is the outcome of the action of $g \in G$ on $t \in T$, which is a part of the definition of $\mathbf{G}$; we reserve the notation t^g to denote the conjugate $g^{-1}tg$ of t in $G \amalg L \amalg F(T)$. We call $G \amalg L$ the **base group**, T the set of **stable letters**, and the subgroups G_t and $\rho(G_t)$ **associated**. The inclusion $G \to G \amalg L \amalg F(T)$ induces the **HNN-map** $\zeta_G\colon G \to \bar{G}$ and ρ extends to $\bar{\rho}\colon \bar{G} \to L$, the **HNN-extension of** ρ, that maps L identically onto itself and T to 1.

This construction has an obvious functorial property:

Lemma 6.9. *Let $\mathbf{G} = (G, T), \mathbf{G}' = (G', T')$ be piles of pro-$\mathcal{C}$ groups and let $\rho\colon G \to L$ and $\rho'\colon G' \to L'$ be homomorphisms of pro-$\mathcal{C}$ groups, injective on every G_t, resp. every G'_t. Put $\bar{G} := \overline{\mathrm{HNN}}(G, T, \rho, L)$ and $\bar{G}' := \overline{\mathrm{HNN}}(G', T', \rho', L')$ and let $\bar{\rho}, \bar{\rho}'$ be the HNN-extensions of ρ, ρ', respectively. Let $\psi = (\psi_G, \psi_T)\colon \mathbf{G} \to \mathbf{G}'$ be a pile morphism and let*

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & L \\ {\scriptstyle \psi_G}\downarrow & & \downarrow{\scriptstyle \lambda} \\ G' & \xrightarrow[\ \rho'\ ]{} & L' \end{array} \tag{6.5}$$

be a commutative diagram of homomorphisms of profinite groups. Then ψ_G, λ, ψ_T induce a homomorphism $\bar{\psi}\colon \bar{G} \to \bar{G}'$ such that

$$\begin{array}{ccccc} G & \xrightarrow{\ \zeta_G\ } & \bar{G} & \xrightarrow{\ \bar{\rho}\ } & L \\ {\scriptstyle \psi_G}\downarrow & & \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \lambda} \\ G' & \xrightarrow[\ \zeta_{G'}\ ]{} & \bar{G}' & \xrightarrow[\ \bar{\rho}'\ ]{} & L' \end{array} \tag{6.6}$$

commutes.

Proof. The maps $\psi_G\colon G \to G'$, $\lambda\colon L \to L'$, and $\psi_T\colon T \to T'$ extend to a homomorphism $\hat{\psi}\colon G \amalg L \amalg F(T) \to G' \amalg L' \amalg F(T')$. Let $t \in T$ and $g \in G$. The commutativity of (6.5) gives

$$\hat{\psi}(g^{-1} t \rho(g)) = \psi_G(g)^{-1} \psi_T(t) \lambda(\rho(g)) = \psi_G(g)^{-1} \psi_T(t) \rho'(\psi_G(g))$$

and, as ψ is a morphism of piles, $\hat{\psi}(t^{\bullet g}) = \psi_T(t)^{\bullet \psi_G(g)}$. Therefore $\hat{\psi}$ preserves relations (6.4) and hence induces a homomorphism $\bar{\psi}\colon \bar{G} \to \bar{G}'$. Diagram (6.6) commutes because $\hat{\psi}$ is ψ_G on G, λ on L, and $\psi_T(T) \subseteq T'$. □

In the situation of Lemma 6.9 we say that the homomorphisms ρ and ρ' are **compatible**, if diagram (6.5) commutes.

Using the left-exactness of the $\varprojlim$-functor we get:

Corollary 6.10. *Let $\{\mathbf{G}_i = (G_i, T_i)\}_{i\in I}$ be an inverse system of piles and $\mathbf{G} = (G, T) = \varprojlim_i \mathbf{G}_i$. Let $\{\rho_i\colon G_i \to L\}_{i\in I}$ be a compatible system of homomorphisms, and let $\rho\colon G \to L = \varprojlim_i \rho_i$. Assume that for every $i \in I$ and every $t_i \in T_i$ the restriction of ρ_i to $(G_i)_{t_i}$ is injective. Then the restriction of ρ to G_t is injective for every $t \in T$ and $\overline{\mathrm{HNN}}(G, T, \rho, L) = \varprojlim_i \overline{\mathrm{HNN}}(G_i, T_i, \rho_i, L)$.*

Remark 6.11. In (6.4), if $g \in G_t$, then $t^{\bullet g} = t$, so (6.4) reads $g^{-1}t\rho(g) = t$, that is, $\rho(g) = g^t$, relation (6.1). Thus for a closed subset T_0 of T the identity maps of G, L and $T_0 \to T$ induce a homomorphism $\theta\colon \widetilde{\mathrm{HNN}}(G, T_0, \rho|_{T_0}, L) \to \overline{\mathrm{HNN}}(G, T, \rho, L)$. If $T = \{t^{\dot{g}} \mid t \in T_0, g \in G\}$, then θ is an epimorphism. By (6.4), $\mathrm{Ker}\,\theta$ is the normal closure of $\{(t^{\bullet g})^{-1}g^{-1}t\rho(g) \mid t \in T_0,\ g \in G\}$.

Lemma 6.12. *Let $\mathbf{G} = (G, T)$ be a pile of pro-$\mathcal{C}$ groups. Suppose that there is a closed set T_0 of representatives of the G-orbits of T. Let $\bar{G} = \overline{\mathrm{HNN}}(G, T, \rho, L)$ and $\tilde{G} = \widetilde{\mathrm{HNN}}(G, T_0, \rho|_{T_0}, L)$. Then the identities of G, L, and T_0 induce an isomorphism $\tilde{G} \to \bar{G}$.*

Proof. By Remark 6.11, these identities induce an epimorphism $\theta\colon \tilde{G} \to \bar{G}$.

Conversely, define a continuous map $T \to \tilde{G}$ by $t^{\bullet g} \mapsto g^{-1}t\rho(g)$, for $t \in T_0$ and $g \in G$. It is well defined: If $t^{\bullet g} = t$, then $g \in G_t$, hence $\rho(g) = g^t$ in $\tilde{G}$, that is, $g^{-1}t\rho(g) = t$; it follows that if $t^{\bullet g} = t^{\bullet h}$, then $g^{-1}t\rho(g) = h^{-1}t\rho(h)$ in $\tilde{G}$. Thus this map, together with the identities of G and L, defines a homomorphism $\lambda\colon \bar{G} \to \tilde{G}$. Notice that λ is the identity on T_0 and we have $\bar{G} = \langle G, L, T_0\rangle$, by (6.4). Therefore λ is the inverse of θ. □

We shall later need the following simple observation:

Lemma 6.13. *Let G be a pro-p group and L a subgroup of G. Suppose $G = \varprojlim_{i\in I} G_i$, where $G_i = F_i \amalg L_i$ is the free pro-p product of a free pro-p group F_i and the image L_i of L in G_i. Then there is a free pro-p subgroup F of G such that G is the free pro-p product $G = F \amalg L$ of F and L.*

Proof. Put $K = L^G$. Then $K \triangleleft G$. Note that $G/K = \varprojlim_{i\in I} G_i/L_i^{G_i} \cong \varprojlim_{i\in I} F_i$ is a free pro-p group. Therefore the quotient map $G \to G/L^G$ splits. We denote by F the image of this splitting. Then G is the semidirect product of F and K. The natural embeddings of F and L into G induce the homomorphisms of cohomology groups

$$H^1(G, \mathbb{F}_p) \to H^1(F, \mathbb{F}_p) \oplus H^1(L, \mathbb{F}_p), \tag{6.7}$$

$$H^2(G, \mathbb{F}_p) \to H^2(L, \mathbb{F}_p) \oplus H^2(L, \mathbb{F}_p). \tag{6.8}$$

We first claim that the map in (6.7) is an isomorphism.

To show that it is injective, let $\rho\colon G \to \mathbb{F}_p$ be a homomorphism such that $\rho|_F = 0$ and $\rho|_L = 0$. Then, for all $x \in L$ and $\sigma \in G$, we have $\rho(x^\sigma) = -\rho(\sigma) + \rho(x) + \rho(\sigma) = \rho(x) = 0$, hence $\rho|_K = 0$. Since also $\rho|F = 0$ and $G = FK$, we have $\rho = 0$.

To show that the map in (6.7) is surjective, let $\varphi\colon F \to \mathbb{F}_p$ and $\psi\colon L \to \mathbb{F}_p$ be two homomorphisms. We have to show that φ, ψ extend to a homomorphism $\rho\colon G \to \mathbb{F}_p$.

Since $\mathbb{F}_p$ is finite, there is $i \in I$ such that ψ factors as $\psi = \bar\psi \circ \pi_{i,L}$, where $\pi_{i,L}\colon L \to L_i$ is the restriction to L of the map $\pi_i\colon G \to G_i$ of the inverse system and $\bar\psi\colon L_i \to \mathbb{F}_p$ is a homomorphism. Since $G_i = F_i \amalg L_i$, we can extend $\bar\psi$ to a homomorphism $\bar\psi'\colon G \to \mathbb{F}_p$. Then $\psi' = \bar\psi' \circ \pi_i\colon G \to \mathbb{F}_p$ is a homomorphism that extends ψ.

For all $x \in L$ and $\sigma \in G$ we have $\psi(x^\sigma) = \psi(x)^{\psi(\sigma)} = \psi(x)$. Therefore, $\psi(x^\sigma) = \psi(x)$, for all $x \in K$ and $\sigma \in G$. Finally, we extend $\psi|_K$ and φ to a homomorphism $\rho\colon G = FK \to \mathbb{F}_p$ by $\rho(yx) = \varphi(y)\psi(x)$, for $y \in F$ and $x \in K$.

Now, the map in (6.8) is an isomorphism as well. Indeed, since $G = \varprojlim_{i\in I} G_i$, $L = \varprojlim_{i\in I} L_i$, $H^2(F, \mathbb{F}_p) = 0$, and $H^2(F_i, \mathbb{F}_p) = 0$ for every $i \in I$, the map in (6.8) is the direct limit of the homomorphisms

$$H^2(G_i, \mathbb{F}_p) \to H^2(L_i, \mathbb{F}_p) \oplus H^2(F_i, \mathbb{F}_p),$$

that are isomorphisms by [9, Theorem 9.3.10]. Hence by [9, Theorem 9.3.10] again, $G = F \amalg L$. □

7. HNN-extensions of relatively projective pro-p groups

In this section we fix a prime p. Unless otherwise stated, all groups and free constructions will be carried out in the category of pro-p groups. Thus, for any profinite space T, let $F(T)$ denote the free pro-p group on T.

For the rest of this section we fix the following setup:

Setup 7.1. We fix a projective pile $\mathbf{G} = (G, T)$ (Definition 5.1) of pro-p groups. Denote the action of G on T by $t^{\bullet g}$, for $t \in T$ and $g \in G$. Let $\rho\colon G \to L$ be a homomorphism of pro-p groups, injective when restricted to the point stabilizers G_t, for every $t \in T$.

According to Definitions 6.6 and 6.8 we form two variants of HNN-extensions

$$\tilde{G} = \widetilde{\mathrm{HNN}}(G, T, \rho, L) \quad \text{and} \quad \bar{G} = \overline{\mathrm{HNN}}(G, T, \rho, L), \tag{7.1}$$

the corresponding HNN-maps $\eta_G\colon G \to \tilde{G}$ and $\zeta_G\colon G \to \bar{G}$, as well as the base map $\xi\colon G \amalg L \to \tilde{G}$, and the HNN-extensions $\tilde{\rho}\colon \tilde{G} \to L$ and $\bar{\rho}\colon \bar{G} \to L$ of ρ. Let $F = \operatorname{Ker}\rho$, $\tilde{F} = \operatorname{Ker}\tilde{\rho}$, and $\bar{F} = \operatorname{Ker}\bar{\rho}$. Then

$$\tilde{G} = \tilde{F} \rtimes L \qquad \text{and} \qquad \bar{G} = \bar{F} \rtimes L. \tag{7.2}$$

By [6, Lemma 10], $\tilde{F}$ is a free pro-p group.

Lemma 7.2. $\bar{G}/L^{\bar{G}} \cong G/\langle G_t \mid t \in T\rangle \amalg F(T/G)$.

Proof. Let $K = \langle G_t \mid t \in T\rangle$. This is a normal subgroup of G. Using the presentation (6.4) we see that $\bar{G}/L^{\bar{G}} = (G \amalg F(T))/N$, where $N = \langle t^{\bullet g} t^{-1} g \mid t \in T,\ g \in G\rangle^{G \amalg F(T)}$.

If $g \in G_t$, then $t = t^{\bullet g}$, so $g \in N$, and hence $K \le N$. If $g, h \in G$ satisfy $g \equiv h \pmod K$, then $g \equiv h \pmod N$, hence $t^{\bullet g} \equiv g^{-1} t \equiv h^{-1} t = t^{\bullet h} \pmod N$. Thus $\bar{G}/L^{\bar{G}} = (G/K \amalg F(T/K))/(N/K)$.

As K contains the stabilizers of all $t \in T$, the action of G/K on T/K is with trivial stabilizers, and hence T/K has a closed set $\bar{T}_0$ of representatives of the G/K-orbits (see [9, Lemma 5.6.5]).

Now note that the homomorphism

$$G/K \amalg F(\bar{T}_0) \to (G/K \amalg F(T/K))/(N/K)$$

has an inverse, induced from the homomorphism $(G/K \amalg F(T/K)) \to G/K \amalg F(\bar{T}_0)$ given by the identity of G/K and mapping $t^{\bullet g} \mapsto g^{-1} t$, for $t \in \bar{T}_0$ and $g \in G/K$. So $\bar{G}/L^{\bar{G}} = G/K \amalg F(\bar{T}_0)$.

The quotient map $T/K \to (T/K)/(G/K) = T/G$ maps $\bar{T}_0$ homeomorphically onto T/G. Hence we may write $\bar{G}/L^{\bar{G}} \cong G/K \amalg F(T/G)$. $\square$

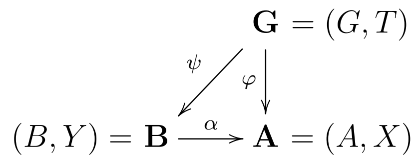
Lemma 7.3. *Assume that there exists a closed set of representatives T_0 of the G-orbits in T. Then $\bar{G} = E \amalg L \amalg F(T_0)$, where $E \le G$ is a free pro-p group isomorphic to $G/\langle G_t \mid t \in T\rangle$.*

Proof. By [2, Theorem 9.5], $G = (\coprod_{t \in T_0} G_t) \amalg E$ for some free pro-p group E. By Lemma 6.12, $\bar{G} = \widetilde{\mathrm{HNN}}(G, T_0, \phi|_{T_0}, L)$. From the presentation of this group (see (6.3)) it follows that $\bar{G} = E \amalg L \amalg F(T_0)$.

As $\langle G_t \mid t \in T\rangle$ is the kernel of the projection $(\coprod_{t\in T_0} G_t) \amalg E \to E$, we have $E \cong G/\langle G_t \mid t \in T\rangle$. $\square$

Lemma 7.4. *Assume that L is finite and let X' be a partition of T. Let A be a finite p-group and let $\varphi\colon G \to A$ be an epimorphism. Then*

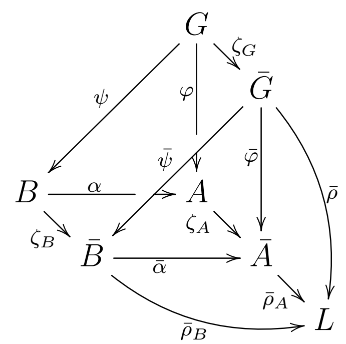
(a) there exists a commutative diagram of piles



(7.3)

in which $\mathbf{A} = (A, X)$ is a finite pile, $\mathbf{B} = (B, Y)$ is a basic pro-p pile (see Example 5.2), and $\alpha\colon \mathbf{B} \to \mathbf{A}$ is a rigid epimorphism. Moreover, $\varphi(T) = X$, and the partition $\{\varphi_T^{-1}(x) \mid x \in X\}$ is finer than X'.

(b) Let $\rho_A\colon A \to L$ be an epimorphism such that $\rho = \rho_A \circ \varphi$ and put $\rho_B = \rho_A \circ \alpha$. Let $\bar{A} = \overline{\mathrm{HNN}}(A, X, \rho_A, L)$ and $\bar{B} = \overline{\mathrm{HNN}}(B, Y, \rho_B, L)$. Then there is a commutative diagram



(7.4)

in which $\bar{\rho}, \bar{\rho}_A, \bar{\rho}_B$ are the HNN-extensions of ρ, ρ_A, ρ_B, respectively.

Proof. Let $\mathcal{G} = \{G_t \mid t \in T\}$. Thus, $\varphi(\mathcal{G}) \subseteq \mathrm{Subgr}(A)$. Let X_0 be a finite set and for every $x \in X_0$ let $A_x \in \varphi(\mathcal{G})$ such that $\varphi(\mathcal{G}) = \{A_x^a \mid x \in X_0,\ a \in A\}$. Let X be the standard A-extension (Construction 4.3) of X_0. Then $\mathbf{A} = (A, X)$ is a finite pile, with $\{A_x \mid x \in X\} = \varphi(\mathcal{G})$.

Replacing X with the union of suitably many copies of X, if necessary, we may assume that $|\{x \in X \mid A_x = A'\}|$ is sufficiently large, for every $A' \in \varphi(\mathcal{G})$. Thus, by Corollary 5.6, we can complete φ (by a continuous map $\varphi_T\colon T \to X$) to a morphism of piles $\varphi\colon \mathbf{G} \to \mathbf{A}$ such that the partition $\{\varphi_T^{-1}(x) \mid x \in X\}$ is finer than X'. Replace X by $\varphi_T(T)$ to assume that φ_T is surjective.

Let P_B be a free pro-p group with an epimorphism $\alpha'\colon P_B \to A$ and let B_x be a copy of A_x, for each $x \in X_0$. Form the free pro-p

product $B = P_B \amalg (\coprod_{x\in X_0} A_x)$, and let $\alpha\colon B \to A$ be the epimorphism that extends α' and is identity on each A_x. Let Y be the standard B-extension of X_0. Then $\mathbf{B} = (B, Y)$ is a pile and the identity of X_0 extends the homomorphism $\alpha\colon B \to A$ to a rigid epimorphism $\alpha\colon \mathbf{B} \to \mathbf{A}$ of piles.

Recall that $\mathbf{G}$ is a projective pile. By Proposition 5.4 there is a morphism $\psi\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \psi = \varphi$.

(b) By assumption $\rho_A = \rho \circ \varphi$ and $\rho_B = \rho_A \circ \alpha$. Hence also

$$\rho = \bar\rho \circ \zeta_G = \rho_A \circ \varphi = \rho_A \circ \alpha \circ \psi = \rho_B \circ \psi.$$

The existence of $\bar\varphi$, $\bar\alpha$, and $\bar\psi$ follows from this by Lemma 6.9, with the pile morphism φ (resp. α, ψ) and the identity of L. The commutativity of (7.3) ensures that (7.4) is commutative. □

Lemma 7.5. *Assume that L is finite. Then the HNN-map $\zeta_G\colon G \to \bar G$ is injective.*

Proof. First assume that $\mathbf{G}$ is basic (Example 5.2). By Lemma 6.12, $\bar G$ is an HNN'-extension of G, that is, an HNN-extension of $G \amalg L$. Extend $\rho\colon G \to L$ by the identity of L to a homomorphism $\rho_L\colon G \amalg L \to L$. Notice that $G \amalg L = \mathrm{Ker}(\rho_L) \rtimes L$. By Lemma 5.13, $\mathrm{Ker}(\rho_L)$ is a free pro-p group. Thus, ζ_G is injective, by [10, Lemma A.1].

In the general case it suffices to show that $\mathrm{Ker}(\zeta_G)$ is contained in every open normal subgroup K of G.

Put $A = G/K$ and let $\varphi\colon G \to A$ be the quotient map. Recall that $F = \mathrm{Ker}(\rho)$ is an open normal subgroup of G, because L is finite. Hence we may replace K with $K \cap F$ to assume that $K \le F$, and hence there is an epimorphism $\pi\colon A \to L$ such that $\rho = \pi \circ \varphi$. By Lemma 7.4 there are commutative diagrams (7.3) and (7.4).

By the first paragraph of this proof ζ_B is injective, hence by (7.4)

$$\mathrm{Ker}\,\zeta_G \le \mathrm{Ker}\,\bar\psi \circ \zeta_G = \mathrm{Ker}\,\zeta_B \circ \psi = \mathrm{Ker}\,\psi.$$

By (7.3), $\mathrm{Ker}\,\psi \le \mathrm{Ker}\,\varphi = K$. Thus $\mathrm{Ker}\,\zeta_G \le K$. □

Lemma 7.6. *Assume that L is finite. Then*

$$\bar G = L \amalg E, \tag{7.5}$$

where $E \cong G/\langle G_t \mid t \in T\rangle \amalg F(T/G)$.

Proof. If $\mathbf{G}$ is basic, this is Lemma 7.3.

Claim 1. $\tilde G$ is a special HNN-extension (Definition 6.1) with base group L, say,

$$\tilde G = \mathrm{HNN}(L, S, \{L_s \mid s \in S\}) \tag{7.6}$$

for some continuous family $\{L_s \mid s \in S\}$ of subgroups of L.

Proof of Claim 1. Claim 1 will follow from (7.2) by Theorem 6.4, provided we show that every finite subgroup L_0 of $\tilde{G}$ is conjugate to a subgroup of L.

Let $G_L = G \amalg L$. By [10, Lemma A.1], the map $G_L \to \tilde{G}$ is injective, hence, by [8, Theorem 7.1.2], L_0 is contained in a conjugate of G_L. Hence, by [5, Theorem A], L_0 is contained in a conjugate of either L or G. In the latter case L_0 is contained in a conjugate of G_t, for some $t \in T$, by [4, Proposition 5.4.3], and thus, as $G_t^t \le L$, again contained in a conjugate of L.

By Remark 6.3 we may assume that $\tilde{F} = \langle S \rangle^{\tilde{G}}$.

Throughout the rest of the proof we use the following notation:

$$\begin{aligned} T' &= \{t \in T \mid G_t \neq 1\}, \\ S' &= \{s \in S \mid L_s \neq 1\}, \quad S'' = S \smallsetminus S' \\ R_0 &= \langle (t^{\bullet g})^{-1} g^{-1} t \rho(g) \mid t \in T',\, g \in G \rangle^{\tilde{G}}, \\ R &= \langle C_{\tilde{F}}(G_t) \mid t \in T' \rangle^{\tilde{G}}. \\ R' &= \langle C_{\tilde{F}}(L_s)) \mid s \in S' \rangle^{\tilde{G}}. \end{aligned}$$

Claim 2. $R_0 = R = R' = \langle S' \rangle^{\tilde{G}}$.

Proof of Claim 2.

Let $t \in T'$ and $g \in G$. Let $g_t \in G_t$. Then $g_t^g \in G_t^g = G_{t^{\bullet g}}$. By (6.1), $\rho(g_t) = g_t^t$ and $\rho(g_t^g) = (g_t^g)^{t^{\bullet g}}$, hence

$$g_t^{g\, t^{\bullet g}} = \rho(g_t^g) = \rho(g_t)^{\rho(g)} = g_t^{t\, \rho(g)},$$

whence $(t^{\bullet g})^{-1} g^{-1} t \rho(g) \in \tilde{F}$ centralizes G_t, and therefore is in R. As $R \lhd \tilde{G}$, this proves $R_0 \le R$.

By Lemma 6.5(c), there is an epimorphism $\tilde{G} \to L \amalg F(S'')$ with kernel $\langle S' \rangle^{\tilde{G}}$, which is R', by Lemma 6.5(b). This epimorphism is injective on L, and hence on every G_t, since G_t is conjugate to a subgroup of L in $\tilde{G}$. Thus the image of G_t is a non-trivial finite subgroup of L. The centralizer in $L \amalg F(S'')$ of any non-trivial element of L is contained in L, by [5, Theorem B], and hence trivially intersects $\langle S'' \rangle^{L \amalg F(S'')}$, the image of $\tilde{F} = \langle S \rangle^{\tilde{G}}$ in $L \amalg F(S'')$. Hence $C_{\tilde{F}}(G_t) \le R'$ for every $t \in T'$, whence $R \le R'$.

To show that $R' \le R_0$ – as R_0 is, by Remark 6.11, the kernel of the canonical epimorphism $\theta_G \colon \tilde{G} \to \bar{G}$ – we have to prove that $\theta_G(R') = 1$, that is, that $\theta_G(C_{\tilde{F}}(L_s)) = 1$, for every $s \in S'$. Since $\theta_G(\tilde{F}) = \bar{F}$ and since θ_G is the identity on L, it suffices to show that $C_{\bar{F}}(L_s) = 1$, for every $s \in S'$.

Fix $s \in S'$. We will only use that L_s is a finite nontrivial subgroup of L.

If $\mathbf{G}$ is basic, then, by Lemma 7.3, $\bar{G}$ is the free pro-p product of L with a free pro-p group. As L_s is finite, by [5, Theorem A] $L_s \leq L^{\bar{g}}$ for some $\bar{g} \in \bar{G}$, and, as $L_s \neq 1$, by [5, Theorem B] $C_{\bar{G}}(L_s) \leq L^{\bar{g}}$. But $L \cap \bar{F} = 1$, hence $L^{\bar{g}} \cap \bar{F} = 1$, whence $C_{\bar{F}}(L_s) = 1$, as contended.

In the general case we will show that $C_{\bar{F}}(L_s)$ is contained in every open normal subgroup $\bar{M}$ of $\bar{G}$. Let $\bar{\lambda}\colon \bar{G} \to C := \bar{G}/\bar{M}$ be the quotient map; we show that $\bar{\lambda}(C_{\bar{F}}(L_s)) = 1$.

As $\bar{F}$ is open in $\bar{G}$, we may assume without loss of generality that $\bar{M} \leq \bar{F}$, that is, $\operatorname{Ker}\bar{\lambda} \leq \operatorname{Ker}\bar{\rho}$. Thus there is an epimorphism $\pi\colon C \to L$ such that $\pi \circ \bar{\lambda} = \bar{\rho}$. Let $\varphi\colon G \to A$ be the epimorphism $\varphi = \bar{\lambda} \circ \zeta_G$, where $A := \varphi(G) \leq C$.

Let $\rho_A = \pi|_A\colon A \to L$ and $\rho_B = \rho_A \circ \alpha\colon B \to L$. Then

$$\rho = \bar{\rho} \circ \zeta_G = \pi \circ \bar{\lambda} \circ \zeta_G = \rho_A \circ \varphi = \rho_A \circ \alpha \circ \psi = \rho_B \circ \psi.$$

By Lemma 7.4 we can complete A to a finite pile $\mathbf{A} = (A, X)$ and complete φ (by a continuous surjective map $\varphi_T\colon T \to X$) to a morphism of piles $\varphi\colon \mathbf{G} \to \mathbf{A}$ and obtain commutative diagrams (7.3) and (7.4) with a basic pile $\mathbf{B}$. Moreover, the partition $\{\varphi_T^{-1}(x) \mid x \in X\}$ is finer than the partition induced by $\bar{\lambda} \circ \zeta_T\colon T \to C$, whence there is a map $\mu\colon X \to C$ such that $\mu \circ \varphi_T = \bar{\lambda} \circ \zeta_T$.

Since $\bar{A}$ satisfies the same defining relations as $\bar{G}$, the homomorphism $\hat{\mu}\colon A \amalg L \amalg F(X) \to C$, which is the inclusion $A \to C$ on A, $\bar{\lambda}|_L$ on L, and μ on X, induces a homomorphism $\bar{\mu}\colon \bar{A} \to C$, such that $\bar{\mu} \circ \bar{\varphi} = \bar{\lambda}$.

Thus we have the following commutative diagram

$$\begin{array}{ccccccc}
\bar{G} & = & \bar{G} & = & \bar{G} & & \\
\downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \bar{\lambda}} & \searrow^{\bar{\rho}} & \\
\bar{B} & \xrightarrow{\bar{\alpha}} & \bar{A} & \xrightarrow{\bar{\mu}} & C & \xrightarrow{\pi} & L
\end{array}$$

In particular, $\bar{\lambda}$ factors through $\bar{\psi}$. So, to show $\bar{\lambda}(C_{\bar{F}}(L_s)) = 1$, it suffices to show that $\bar{\psi}(C_{\bar{F}}(L_s)) = 1$. As $\bar{\lambda}$ is injective on L, so is $\bar{\psi}$, hence $D := \bar{\psi}(L_s)$ is a finite nontrivial subgroup of $\bar{B}$.

Let $\bar{\rho}_B\colon \bar{B} \to L$ be the HNN-extension of $\rho_B = \rho_A \circ \alpha\colon B \to L$ and let $\bar{F}_B = \operatorname{Ker}(\bar{\rho}_B)$. Then $\bar{\rho} = \bar{\rho}_B \circ \bar{\psi}$, whence $\bar{\psi}(\bar{F}) \leq \bar{F}_B$. Thus $\bar{\psi}(C_{\bar{F}}(L_s)) \leq C_{\bar{F}_B}(D)$. But $C_{\bar{F}_B}(D) = 1$, as shown above in the case of a basic pile. This finishes the proof of $R' \leq R_0$.

Finally, by Lemma 6.5(b), $R' = \langle S' \rangle^{\tilde{G}}$.

Claim 3. $\bar{G} = L \coprod F(S'')$.

Indeed, by Remark 6.11, $\bar{G} = \tilde{G}/R_0$, so, by Claim 2, $\bar{G} = \tilde{G}/\langle S' \rangle^{\tilde{G}}$. By Lemma 6.5(c), $\bar{G} = L \amalg F(S'')$.

Finally, by Claim 3, $\bar{G}/L^{\bar{G}} \cong F(S'')$, so, by Lemma 7.2, $F(S'') \cong G/K \amalg F(T/G)$. □

Theorem 7.7.

(i) $\tilde{G}$ is proper, i.e., the base map $\xi\colon G \amalg L \to \tilde{G}$ is injective. Hence also $\eta_G\colon G \to \tilde{G}$ is injective.
(ii) $\zeta_G\colon G \to \bar{G}$ is injective.
(iii) $\bar{G} = L \amalg E$, where $E \cong G/\langle G_t \mid t \in T\rangle \amalg F(T/G)$.

Proof. Let $\mathcal{U}$ be the set of open normal subgroups of L. For every $U \in \mathcal{U}$ let $N_U = \rho^{-1}(U)$. Since T is closed under G-conjugation, the subgroup $\tilde{N}_U = \langle N_U \cap G_t \mid t \in T\rangle$ is normal in G.

Claim. $\bigcap_{U\in\mathcal{U}} \tilde{N}_U = \{1\}$.

Indeed, let V be an open subgroup of G. As $\{G_t \mid t \in T\}$ is a continuous family, $\bigcup_{t\in T} G_t$ is a closed subset of G (see Lemma 3.1(d)). So is the kernel F of ρ. Since $\rho_{|G_t}$ is injective for every $t \in T$, we have $F \cap (\bigcup_{t\in T} G_t) = \{1\}$. Also $F = \bigcap_{U\in\mathcal{U}} N_U$, so $\bigcap_{U\in\mathcal{U}} \big(N_U \cap (\bigcup_{t\in T} G_t)\big) = \{1\} \subseteq V$. By the compactness of G there is U such that $N_U \cap (\bigcup_{t\in T} G_t) \subseteq V$. In particular, $N_U \cap G_t \subseteq V$ for every $t \in T$. Therefore $\tilde{N}_U = \langle G_t \cap N_U \mid t \in T\rangle \subseteq V$. Since V is an arbitrary open subgroup of G, this implies that $\bigcap_{U\in\mathcal{U}} \tilde{N}_U = \{1\}$.

For every $U \in \mathcal{U}$ set $L_U = L/U$, $G_U = G/\tilde{N}_U$, $T_U = T/\tilde{N}_U$, $\mathbf{G}_U = (G_U, T_U)$, and let $\rho_U\colon G_U \to L_U$ be the homomorphism induced from ρ. Then the G_U-stabilizer of t_U is $(G_U)_{t_U} = G_t\tilde{N}_U/\tilde{N}_U$, where $t \in T$ is a representative of $t_U \in T_U$.

It follows from the Claim that $\rho = \varprojlim_{U\in\mathcal{U}} \rho_U$. Clearly, $\operatorname{Ker}\rho_U = N_U/\tilde{N}_U$. Let $t \in T$ and let t_U be its image in T_U. By the definition of $\tilde{N}_U$ we have $G_t \cap \tilde{N}_U = G_t \cap N_U$, hence $N_U \cap G_t\tilde{N}_U = \tilde{N}_U$, whence ρ_U is injective on $(G_U)_{t_U}$. Thus we may define

$$\tilde{G}_U = \widetilde{\mathrm{HNN}}(G_U, T_U, \rho_U, L_U), \tag{7.7}$$

$$\bar{G}_U = \overline{\mathrm{HNN}}(G_U, T_U, \rho_U, L_U). \tag{7.8}$$

By Lemma 5.9, $\mathbf{G}_U$ is projective and by Corollary 5.10, $N_U/\tilde{N}_U$ is free pro-p. Therefore, by [10, Lemma A.1], (7.7) is a proper HNN-extension, i.e., the map $\xi_U\colon G_U \amalg L \to \tilde{G}_U$ is injective.

We have commutative diagrams

$$\begin{array}{ccc} G \amalg L & \xrightarrow{\xi} & \tilde{G} \\ \downarrow & & \downarrow \\ G_U \amalg L & \xrightarrow{\xi_U} & \tilde{G}_U, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\zeta_G} & \bar{G} \\ \downarrow & & \downarrow \\ G_U & \xrightarrow{\zeta_{G_U}} & \bar{G}_U, \end{array} \tag{7.9}$$

where the vertical maps on the right are induced from the quotient maps $G \to G_U$ and $T \to T_U$.

(i) Since $\xi = \varprojlim_{U \in \mathcal{U}} \xi_U$, we deduce from Corollary 6.10 that ξ is injective.

(ii) By Lemma 7.5 the HNN-map $\zeta_{G_U} : G_U \to \bar{G}_U$ is injective. Since in the above diagram $\zeta_G = \varprojlim_{U \in \mathcal{U}} \zeta_{G_U}$, we deduce that ζ_G is injective.

(iii) Put $K = \langle G_t \mid t \in T \rangle$ and $K_U = \langle (G_U)_{t_U} \mid t_U \in T_U \rangle$, for every U. By Lemma 7.6, $\bar{G}_U = L_U \amalg E_U$, where $E_U \cong G_U/K_U \amalg F(T_U/G_U)$. By Lemma 5.11, G_U/K_U, and hence also E_U, is free pro-p. But $G = \varprojlim_U G_U$, $L = \varprojlim_U L_U$, and $\bar{G} = \varprojlim_U \bar{G}_U$, hence by Lemma 6.13 there is a free pro-p group E such that $\bar{G} = L \amalg E$.

It follows that $E \cong \bar{G}/L^{\bar{G}}$, hence by Lemma 7.2, $E \cong G/K \amalg F(T/G)$. □

In the next lemma we identify G with its image in $\bar{G}$ via the embedding ζ_G.

Lemma 7.8. *Let $\sigma \in \bar{G}$ such that $L^\sigma \cap G \neq 1$. Then there is a unique $t \in T$ such that $L^\sigma \cap G = G_t$ in $\bar{G}$. It satisfies $\sigma t \in L$. Moreover,*

$$\{L^\sigma \cap G \mid \sigma \in \bar{G}\} \smallsetminus \{1\} = \{G_t \mid t \in T\} \smallsetminus \{1\}. \tag{7.10}$$

Proof.

Claim 1. $L^{t^{-1}} \cap G \geq G_t$ in $\bar{G}$.

Indeed, $G_t^t = \rho(G_t) \leq L \leq \bar{G}$, so

$$L^{t^{-1}} \cap G \geq L^{t^{-1}} \cap G_t = (L \cap \rho(G_t))^{t^{-1}} = \rho(G_t)^{t^{-1}} = G_t.$$

Claim 2. If $L^\sigma \cap G \leq G_t$, then $\sigma t \in L$ and $L^\sigma \cap G = G_t$.

Indeed, by the assumption, $L^\sigma \cap G = L^\sigma \cap G_t$, so

$$1 \neq (L^\sigma \cap G)^t = (L^\sigma \cap G_t)^t = L^{\sigma t} \cap \rho(G_t) = (L^{\sigma t} \cap L) \cap \rho(G_t).$$

In particular, $L^{\sigma t} \cap L \neq 1$ in $\bar{G} = L \amalg E$ (Theorem 7.7(iii)). By [9, Theorem 9.1.12] this implies that $\sigma t \in L$. Therefore $L^\sigma = L^{t^{-1}}$, hence $L^\sigma \cap G = G_t$, by Claim 1. Thus, Claim 2 is established.

We are now going to verify the hypothesis in Claim 2.

If L is finite, then $L^\sigma \cap G$ is a finite subgroup of G. By [4, Proposition 5.4.3] there is $t \in T$ such that $L^\sigma \cap G \leq G_t$. So $L^\sigma \cap G = G_t$ by Claim 2. By Proposition 5.7(b) such t is unique.

In the general case we have $G = \varprojlim_U G_U$, $T = \varprojlim_U T_U$, $L = \varprojlim_U L_U$, and $\bar{G} = \varprojlim_U \bar{G}_U$, with finite L_U, as in Theorem 7.7. Let $\pi_U \colon \bar{G} \to \bar{G}_U$ and $\pi_{U,V} \colon \bar{G}_V \to \bar{G}_U$, for $V \leq U$, be the maps of the inverse system. We may assume that $\pi_U(L^\sigma \cap G) \neq 1$ for every U. Since $\pi_U(L^\sigma \cap G) \leq L_U^{\pi_U(\sigma)} \cap G_U$, we have $L_U^{\pi_U(\sigma)} \cap G_U \neq 1$. So by the preceding case there is a unique $t_U \in T_U$ such that $L_U^{\pi_U(\sigma)} \cap G_U = (G_U)_{t_U}$ for every U.

If $V \leq U$ and $\pi = \pi_{U,V}$, then $L_V^{\pi_V(\sigma)} \cap G_V = (G_V)_{t_V}$ and $\pi(L_V^{\pi_V(\sigma)}) = L_U^{\pi_U(\sigma)}$, $\pi(G_V) = G_U$, so $(G_U)_{\pi(t_V)} = \pi((G_V)_{t_V}) \leq (G_U)_{t_U}$. By Proposition 5.7(b), $\pi(t_V) = t_U$.

It follows that there is a unique $t \in T$ such that $\pi_U(t) = t_U$ for every U. Therefore $\pi_U(L^\sigma \cap G) \leq \pi_U(G_t)$. Hence $L^\sigma \cap G \leq G_t$. By Claim 2, $L^\sigma \cap G = G_t$. As $\pi_U(\sigma t) \in L_U = \pi_U(L)$ for every U, we have $\sigma t \in L$.

The above proves that the left-handed side of (7.10) is contained in the right-handed side. Conversely, let $t \in T$ such that $G_t \neq 1$. By Claim 1, $G_t \leq L^{t^{-1}} \cap G$. In particular $L^{t^{-1}} \cap G \neq 1$. So there is a unique $s \in T$ such that $L^{t^{-1}} \cap G = G_s$. Thus, $G_t \leq G_s$. By Proposition 5.7(b), $s = t$. Hence $L^{t^{-1}} \cap G = G_t$. ☐

Proof of Theorem 1.2. By Proposition 5.7 and Remark 5.8, $\mathbf{G} = (G, T)$ is a projective pile. By Theorem 7.7, $\bar{G} = L \amalg F$, where F is free, and $\zeta = \zeta_G \colon G \to \bar{G}$ is injective. The rest is (7.10). ☐

Theorem 1.1 simply follows from Theorem 1.2 if one puts ρ to be the identity map of G to its copy L.

Theorem 1.3 follows from Theorem 7.7 observing that $\zeta_G(G_t)$ in its proof is conjugate to $\rho(G_t)$ for every $t \in T$.

Proof of Theorem 1.4. (i) $\Leftrightarrow$ (ii): By Proposition 5.7 and Remark 5.8.

(ii) $\Rightarrow$ (iii): Let $\zeta \colon G \to \bar{G} := L \amalg F$ be the embedding of Theorem 1.1. Put $\hat{T} = \{L^\sigma \mid \sigma \in \bar{G}\}$ and let $\bar{G}$ act on $\hat{T}$ by conjugation. Then $\bar{G}_{L^\sigma} = L^\sigma$ and if $L^\sigma \neq L^\tau$, then $L^\sigma \cap L^\tau = 1$ ([4, Lemma 3.1.10]). Hence the action of G on $\hat{T}$ via ζ is such that $G_{L^\sigma} = L^\sigma \cap \zeta(G)$ and, by Theorem 1.1, it satisfies (a) for $\hat{T}$.

Write $L = \varprojlim L_i$, $F = \varprojlim F_i$ as inverse limits of finite quotient groups of L and finitely generated free quotients of F, respectively. Put $\bar{G}_i = L_i \amalg F_i$ and let G_i be the image of $\zeta(G)$ in $\bar{G}_i$. Then $\bar{G}_i$, and hence also G_i, is countably generated. Put $T_i = \{L_i^\sigma \mid \sigma \in \bar{G}_i\}$, for every i, then $(G, \hat{T}) = \varprojlim_i (G_i, T_i)$.

By the pro-p version of the Kurosh Subgroup Theorem [2, Theorem 9.7] applied to the subgroup G_i of $\bar{G}_i = L_i \amalg F_i$, G_i is a free pro-p product $G_i = \coprod_{x_i \in X_i}(G_{x_i}) \amalg F_i'$, for some closed set X_i of representatives of the G_i-orbits in T_i and some free pro-p group F_i'.

(iii) $\Rightarrow$ (i): Put $\mathbf{G} = (G, \hat{T})$ and $\mathbf{G}_i = (G_i, T_i)$, for every i. We have to solve a finite embedding problem (5.1) for $\mathbf{G}$. But φ factors via some $\mathbf{G}_i$. This means that there exist $\nu_i \colon \mathbf{G} \to \mathbf{G}_i$, $\varphi_i \colon \mathbf{G}_i \to \mathbf{A}$, such that $\varphi = \varphi_i \circ \nu_i$. By [4, Proposition 5.4.2], G_i is $\{(G_i)_t \mid t \in T_i\}$-projective, so, by Proposition 5.7, $\mathbf{G}_i$ is projective. Therefore there exists a solution ψ_i of the embedding problem (α, φ_i). Then $\psi_i \circ \nu_i$ is a solution of (5.1).

(i) $\Rightarrow$ (iv): Let again $\zeta \colon G \to \bar{G} := L \amalg F$ be the embedding of Theorem 1.1. Let X be a basis of F, that is, a profinite subspace of F such that $F = F(X)$. Then $\bar{G}$ can be viewed as the special HNN-extension $\bar{G} = \mathrm{HNN}(L, X, \mathcal{L})$, where $\mathcal{L}$ is a family of copies L_x of 1, for every $x \in X$. Thus $\bar{G}$ is the fundamental group of a bouquet of groups Γ, with L attached to the unique vertex and X the space of edges, with 1 attached to each of them ([8, Example 6.2.3(e)]).

Let S be the standard pro-p graph of Γ ([8, Section 6.3]). By [8, Corollary 6.3.6], S is a pro-p tree. By [8, Lemma 6.3.2(b)] $\bar{G}$ acts on S, with trivial edge stabilizers and vertex stabilizers being conjugates of L. Then the restriction of this action to $\zeta(G)$ gives the required action.

(iv) $\Rightarrow$ (i): Let G be a pro-p group that acts on a pro-p tree Γ with trivial edge stabilizers. Let V be its set of vertices and let U be an open normal subgroup of G. Then $\tilde{U} = \langle U \cap G_v \mid v \in V\rangle$ is a normal subgroup of G and $G_U = G/\tilde{U}$ acts on $\Gamma_U = \Gamma/\tilde{U}$ (with vertex stabilizers of order bounded by the index $(G : U)$), hence by [8, Proposition 4.1.1] Γ_U is a pro-p tree. Moreover, $U/\tilde{U}$ acts freely on Γ_U and hence is free pro-p [8, Theorem 4.1.2]. So G_U is virtually free pro-p and every nontrivial torsion element $g \in G_U$ belongs to some vertex stabilizer $(G_U)_v$ by [8, Theorem 4.1.8].

In fact, even $C_{G_U}(g) \leq (G_U)_v$, since for any $c \in C_{G_U}(g)$, $g(cv) = cv$, so $1 \neq g \in (G_U)_v \cap (G_U)_{cv}$ and since edge stabilizers are trivial, by [8, Corollary 4.1.6] we have $v = cv$, that is, $c \in (G_U)_v$.

We can now apply [10, Corollary 6.3] to deduce the existence of an embedding $\zeta_U \colon G_U \to H = G/U \amalg F$ for some free pro-p group F. Then by 5.12(d) G_U is $\mathcal{G}_U = \{(G_U)_v \mid v \in V\} = \{G \cap (G/U)^h \mid h \in H\}$-projective. Since $(G, \mathcal{G}) = \varprojlim_U (G_U, \mathcal{G}_U)$, G is $\mathcal{G}$-projective, because every finite embedding problem factors via some $(G_U, \mathcal{G}_U)$. $\square$

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