Critical structures of inner functions II

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Abstract

We study the correspondence proposed in [Ivr21] between inner functions modulo post-compositions with automorphisms of the unit disk and cyclic subspaces of the weighted Bergman space A^2_1. The correspondence sends an inner function I to the invariant subspace $[I']$, and in the opposite direction, assigns to a non-zero function $H \in A^2_1$ the Liouville map I_H associated to the canonical solution of the Gauss curvature equation $\Delta u = |H|^2 e^{2u}$. We prove that I'_H generates the same cyclic subspace as H. Combined with the earlier results in [Ivr21], this shows that these two mappings are inverses of one another, and hence the correspondence $I \to [I']$ is a bijection.

1 Introduction

An *inner function* F is a holomorphic self-map of the unit disk such that for a.e. $\theta \in [0, 2\pi)$, the radial boundary value $F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$ exists and belongs to the unit circle. By a classical result of A. Beurling, one can parametrize inner functions up to post-compositions with rotations by invariant subspaces of H^2. In this paper, we continue the programme started in [Ivr21] of parametrizing inner functions up to post-compositions with Möbius transformations by cyclic subspaces of A^2_1.

The weighted Bergman space A^2_1 consists of holomorphic functions on the unit disk for which

$$\|H\|^2_{A^2_1} = \frac{2}{\pi}\int_{\mathbb{D}} |H(z)|^2(1-|z|^2)\,dA(z) < \infty.$$

We say that a (closed) subspace $X \subset A^2_1$ is *invariant* if $zX \subset X$. For a function $H \in A^2_1$, the *cyclic subspace generated by* H is defined as the minimal invariant subspace of A^2_1 which contains H:

$$[H] = \overline{\{Hp \,:\, p \text{ polynomial}\}}^{A^2_1}.$$

In [Ivr21], the author set out to show that inner functions

$$\mathrm{Inn} \,/\, \mathrm{Aut}(\mathbb{D}) = \{\text{cyclic subspace of } A^2_1\}, \qquad I \to [I'],$$

but only proved that the map is well-defined and injective. Here, we prove surjectivity:

Theorem 1.1. *For every $H \in A^2_1 \setminus \{0\}$, there is an inner function I_H whose derivative generates $[H]$. Moreover, I_H is unique up to post-composition with an automorphism of the unit disk.*

1.1 Canonical solutions

The inner functions I_H are constructed using the Liouville correspondence which provides a bridge between complex analysis and semi-linear elliptic PDE.

For a function $H \in A^2_1 \setminus \{0\}$, we consider the Gauss curvature equation

$$\Delta u = |H|^2 e^{2u}. \tag{GCE$_H$}$$

By Liouville's theorem [KR08, Theorem 3.3], every solution of GCE_H has the form

$$u = \log \frac{1}{|H|} \frac{2|F'|}{1 - |F|^2},$$

where F is a holomorphic self-map of the unit disk which has the same critical points as H, counted with multiplicity. Furthermore, F is unique up to post-composition with an automorphism of the unit disk. We refer to F as the *Liouville map* of u.

By the theory of semi-linear elliptic PDEs, for every $h \in L^\infty(\partial\mathbb{D})$, the boundary value problem

$$\begin{cases} \Delta u = |H|^2 e^{2u}, & \text{in } \mathbb{D}, \\ u = h, & \text{on } \partial\mathbb{D}, \end{cases} \tag{1.1}$$

has a unique solution. Here, the boundary values are understood in the sense of boundary traces: as $r \to 1$, the measures $u(re^{i\theta})d\theta \to hd\theta$ converge in the weak-$*$ topology. Furthermore, if $h_1 \le h_2$ pointwise a.e. then, $u_{H,h_1} \le u_{H,h_2}$.

For $n \in \mathbb{Z}$, we write $u_{H,n}$ for the solution of (GCE_H) with constant boundary values n. By the monotonicity of solutions, $u_{H,m} \le u_{H,n}$ if $m \le n$. It is explained in [Ivr21] that the pointwise limit $u_{H,\infty} = \lim u_{H,n}$ is also solution, called the *canonical solution*.

We write I_H for the Liouville map associated to $u_{H,\infty}$. In [Ivr21], the following observations were made regarding I_H:

- I_H is an inner function.
- $I'_H \in [H]$.
- If $[H_1] = [H_2]$ then $I_{H_1} = I_{H_2}$.
- $[I'] = [(m \circ I)']$ for any $m \in \mathrm{Aut}(\mathbb{D})$.
- If F is an inner function, then $I_{F'} = F$.

What is missing is that I'_H generates $[H]$. This is the content of Theorem 1.1.

2 Preliminaries

In this section, we gather some miscellaneous lemmas that will be used throughout this paper. We start with the following standard lemma:

Lemma 2.1. *Let X be an invariant subspace of A^2_1. If $f \in X$ and $b \in H^\infty$, then $fb \in X$.*

The lemma can be proved by first approximating $b(z)$ in A^2_1 by its dilates $b(rz)$ with $0 < r < 1$, and then approximating each dilate uniformly on the closed unit disk by polynomials.

Lemma 2.2. *For every holomorphic self-map J of the unit disk and $0 \leq \beta < 2$, we have*

$$\int_{\mathbb{D}} \frac{|J'(z)|^2}{(1-|J(z)|^2)^\beta}(1-|z|^2)dA(z) < \infty.$$

Proof. Since increasing β increases the integrand, we may assume that $1 < \beta < 2$. Consider the function

$$v(z) = 1 - (1-|J(z)|^2)^{2-\beta}.$$

Differentiating, we get

$$\Delta v \geq 4(2-\beta)(\beta-1) \cdot |J'|^2(1-|J|^2)^{-\beta}. \tag{2.1}$$

Thus, v is a subharmonic function on the unit disk which takes values between 0 and 1. Applying the Poisson-Jensen formula for subharmonic functions [Ran95, Theorem 4.5.1] on $B(0,r)$ and taking $r \to 1$ yields

$$\frac{1}{2\pi}\int_{\mathbb{D}} \Delta v(z) \log\frac{1}{|z|}\, dA(z) \leq 1.$$

The lemma follows after substituting the lower bound for Δv from (2.1) into the preceding formula and using $1 - |z|^2 \leq 2\log(1/|z|)$. □

Remark. From the Poisson-Jensen formula, it follows that when $\beta = 2$, the above integral is finite if and only if

$$\int_{\partial\mathbb{D}} \log\frac{1}{1-|J(\zeta)|^2}\, d\zeta < \infty.$$

By a result of de Leeuw and Rudin, this occurs precisely when J is a non-extreme point of the unit ball of H^∞, see [dLR58, Section 5] or [Hof62, Chapter 9] for details.

The following lemma is a simple consequence of the Schwarz lemma:

Lemma 2.3. *For every $R > 0$, there exists a constant $C = C(R) > 0$ such that for any holomorphic self-map J of the unit disk,*

$$1/C \leq \frac{1-|J(z_1)|^2}{1-|J(z_2)|^2} \leq C, \qquad z_1, z_2 \in \mathbb{D}, \qquad d_{\mathbb{D}}(z_1,z_2) < R.$$

Let $I \subset \partial\mathbb{D}$ be an arc on the unit circle of length $< 1/10$. The Carleson square with base I is defined as

$$Q_I = \big\{ z \in \mathbb{D} : |z| > 1 - |I|,\ z/|z| \in I \big\}.$$

Let ξ_I be the midpoint of the arc I. We denote the center of Q_I by $z_{Q_I} = (1-|I|/2)\xi_I$ and the side length of Q_I by $\ell(Q_I) = |I|$.

Lemma 2.4. *For every holomorphic self-map J of the unit disk and Carleson square $Q \subset \mathbb{D}$,*

$$\frac{1}{\ell(Q)} \int_Q |J'(z)|^2 (1 - |z|^2)\, dA(z) \le C(1 - |J(z_Q)|^2), \tag{2.2}$$

where $C > 0$ is an absolute constant.

Remark. From the inclusion $H^\infty \subset \mathrm{BMO}$, it follows that the left hand side is bounded by an absolute constant. The lemma says that the normalized J-energy of Q is small if $|J(z_Q)|$ is close to 1.

Proof. By the Poisson-Jensen formula, we have

$$\begin{aligned} |J(w)|^2 &= \frac{1}{2\pi} \int_{\partial\mathbb{D}} |J(\zeta)|^2 P(w,\zeta) |d\zeta| - \frac{1}{2\pi} \int_{\mathbb{D}} \Delta |J(z)|^2 G(w,z) dA(z) \\ &\le 1 - \frac{2}{\pi} \int_{\mathbb{D}} |J'(z)|^2 G(w,z) dA(z). \end{aligned}$$

Rearranging, we get

$$\int_{\mathbb{D}} |J'(z)|^2 G(w,z) dA(z) \lesssim 1 - |J(w)|^2.$$

Using the interpretation of the Green's function as the occupation density of Brownian motion, it is not difficult to see that

$$G(w,z) \asymp \frac{1 - |z|^2}{\ell(Q)}, \qquad w = (1 - 2\ell(Q))\xi_I, \qquad z \in Q.$$

Consequently,

$$\frac{1}{\ell(Q)} \int_Q |J'(z)|^2 (1 - |z|^2)\, dA(z) \lesssim 1 - |J(w)|^2.$$

Since the hyperbolic distance $d_{\mathbb{D}}(w, z_Q) = O(1)$, by Lemma 2.3, the right hand side is comparable to $1 - |J(z_Q)|^2$. $\square$

3 A homotopy argument

We will deduce Theorem 1.1 from the following result:

Theorem 3.1. *Let J, F be non-constant holomorphic self-maps of the unit disk which have the same critical points, counting multiplicity. Suppose that their quotient $q = F'/J'$ satisfies*

$$|q(z)| \le \frac{C}{1 - |J(z)|^2}. \tag{3.1}$$

Then, $F' \in [J']$.

Remark. (i) Let $\mathrm{Inv}(H^\infty)$ be the collection of bounded analytic functions on the unit disk with bounded inverses. The proof below will show that there exists a sequence of functions $a_k \in \mathrm{Inv}(H^\infty)$ such that $J'a_k \to F'$ in A^2_1.

(ii) Since the unit disk is simply-connected and q is zero-free, it admits a holomorphic logarithm. We fix one such branch and define the fractional powers of q by $q^t = e^{t \log q}$, $0 \le t \le 1$.

We consider the path

$$h_t = J'q^t, \qquad 0 \le t \le 1,$$

which connects $h_0 = J'$ with $h_1 = F'$. Since $F' = J'q$, we have

$$|h_t|^2 = |J'|^{2(1-t)}|F'|^{2t}.$$

By the weighted AM-GM inequality,

$$|h_t|^2 \;\le\; (1-t)|J'|^2 + t|F'|^2 \;\le\; |J'|^2 + |F'|^2.$$

Thus, $h_t \in A^2_1$ for every $0 \le t \le 1$ and

$$\|h_t\|^2_{A^2_1} \le \|J'\|^2_{A^2_1} + \|F'\|^2_{A^2_1}.$$

As $t \to 1$, the functions h_t converge pointwise to h_1. By Lebesgue's dominated convergence theorem, $h_t \to h_1$ in A^2_1 as $t \to 1$. Since $[J']$ is closed, if we can prove

$$h_t \in [J'], \qquad \text{for every } 0 < t < 1,$$

it would follow that h_1 is also in $[J']$.

Suppose we know that $h_t \in [J']$ and we want to show that $h_{t+s} \in [J']$ for some $s > 0$. Unfortunately, we cannot simply multiply h_t by q^s since q may be unbounded. Instead, we define the auxiliary functions

$$a_{r,s}(z) = q(rz)^s, \qquad 0 < r < 1.$$

Since q is non-vanishing, each function $a_{r,s}$ is bounded with bounded inverse. We will show that

$$h_t a_{r,s} \to h_{t+s}, \qquad \text{in } A^2_1, \tag{3.2}$$

as $r \to 1$, provided that

$$2s < 1 - t. \tag{3.3}$$

In view of Lemma 2.1, this allows us to advance along the path h_t in sufficiently small steps. Given any $t < 1$, we can choose a finite partition

$$0 \,=\, t_0 \,<\, t_1 \,<\, \dots \,<\, t_n \,=\, t$$

such that $2(t_{j+1} - t_j) < 1 - t_j$ for every j. After n steps, we obtain that $h_t \in [J']$.

Let us examine the statement (3.2) more closely. Expanding definitions, we need to show that

$$J'(z)q(z)^t q(rz)^s \to J'(z)q(z)^{s+t} \qquad \text{in } A^2_1.$$

The convergence is uniform on compact subsets of the unit disk. To upgrade this to convergence in the A^2_1 norm, we split the unit disk into the ball $B(0, r)$ and the annulus $A(0; r, 1)$.

3.1 Estimate on the ball $B(0, r)$

We now show that

$$\left\| \chi_{B(0,r)}(h_t a_{r,s} - h_{t+s}) \right\|_{A^2_1} \to 0, \tag{3.4}$$

as $r \to 1$.

For $z \in B(0, r)$, the hyperbolic distance between z and rz is uniformly bounded. Hence by Lemma 2.3,

$$1 - |J(rz)|^2 \asymp 1 - |J(z)|^2.$$

Using the estimate (3.1), we obtain

$$|J'(z)q(z)^t q(rz)^s|^2(1-|z|^2) \lesssim |J'(z)|^2(1-|J(z)|^2)^{-2(t+s)}(1-|z|^2)$$

and

$$|J'(z)q(z)^{s+t}|^2(1-|z|^2) \lesssim |J'(z)|^2(1-|J(z)|^2)^{-2(t+s)}(1-|z|^2),$$

for $z \in B(0,r)$. The elementary estimate $|u-v|^2 \le 2|u|^2 + 2|v|^2$ shows

$$\left|J'(z)q(z)^t q(rz)^s - J'(z)q(z)^{s+t}\right|^2(1-|z|^2) \lesssim |J'(z)|^2(1-|J(z)|^2)^{-2(t+s)}(1-|z|^2),$$

for $z \in B(0,r)$. Notice that the right hand side is independent of $0 < r < 1$. By Lemma 2.2 and the bound on the step size (3.3), the right hand side is integrable. From here, (3.4) follows from the dominated convergence theorem.

3.2 Estimate on the annulus $A(0;r,1)$

We cut the annulus $A(0,r,1)$ into Carleson squares Q_j of side length $h = 1-r$. Set

$$D_J = 1-|J|^2, \qquad D_F = 1-|F|^2,$$

and

$$m_j = \int_{Q_j} |h_t(z)|^2(1-|z|^2)\, dA(z).$$

Applying Hölder's inequality on each Carleson square and using Lemma 2.4, we get

$$\begin{aligned} m_j &\le \left(\int_{Q_j} |J'(z)|^2(1-|z|^2)dA(z)\right)^{1-t}\left(\int_{Q_j} |F'(z)|^2(1-|z|^2)dA(z)\right)^t \\ &\lesssim h\, D_J(z_{Q_j})^{1-t}\, D_F(z_{Q_j})^t \\ &\lesssim h\, D_J(z_{Q_j})^{1-t}. \end{aligned}$$

Thus, if $p = \frac{2s}{1-t}$, then

$$m_j^p \lesssim h^p\, D_J(z_{Q_j})^{2s} \qquad \text{and hence} \qquad D_J(z_{Q_j})^{-2s} m_j \lesssim h^p\, m_j^{1-p}. \tag{3.5}$$

By Lemma 2.3 and (3.1), we have

$$|q(rz)|^{2s} \lesssim D_J(z_{Q_j})^{-2s}, \qquad z \in Q_j.$$

Summing over the Carleson squares that make the annulus $A(0, r, 1)$, we get

$$\begin{aligned}\int_{A_r} |h_t(z)|^2 |q(rz)|^{2s}(1-|z|^2)dA(z) &\lesssim \sum_j D_J(z_{Q_j})^{-2s} m_j \\ &\lesssim \sum_j h^p\, m_j^{1-p} \\ &\lesssim \Big(\sum_j h\Big)^p \Big(\sum_j m_j\Big)^{1-p}.\end{aligned}$$

Here, we have used Hölder's inequality for sums with exponents $1/p$ and $1/(1-p)$.

Since the lengths of the sides of the Carleson squares add up to the circumference of the circle, the first term is bounded. Consequently,

$$\int_{A_r} |h_t(z)|^2 |q(rz)|^{2s}(1-|z|^2)dA(z) \lesssim \Big(\int_{A_r} |h_t(z)|^2(1-|z|^2)dA(z)\Big)^{1-p}.$$

As $h_t \in A^2_1$, the last integral tends to 0 as $r \to 1$.

Since $h_{t+s} \in A^2_1$, the absolute continuity of the integral gives

$$\int_{A_r} |h_{t+s}(z)|^2(1-|z|^2)dA(z) \to 0, \qquad \text{as } r \to 1.$$

By the elementary estimate $|u-v|^2 \le 2|u|^2 + 2|v|^2$,

$$\begin{aligned}\int_{A_r} |h_t a_{r,s} - h_{t+s}|^2(1-|z|^2)dA(z) \le 2\int_{A_r} |h_t(z)|^2 |q(rz)|^{2s}(1-|z|^2)dA(z) \\ +2\int_{A_r} |h_{t+s}(z)|^2(1-|z|^2)dA(z).\end{aligned}$$

Since both terms on the right tend to 0,

$$\big\|\chi_{A(0;r,1)}(h_t a_{r,s} - h_{t+s})\big\|_{A^2_1} \to 0,$$

as $r \to 1$. Together with (3.4), this shows (3.2) as desired.

3.3 Application to invariant subspaces

We now deduce Theorem 1.1 from Theorem 3.1:

Proof of Theorem 1.1. Let I_0 and I be Liouville maps for $u_{H,0}$ and $u_{H,\infty}$ respectively. From [Ivr21, Theorem 3.1], we know that $I' \in [H]$ while I_0' generates $[H]$. As $u_{H,0} \le u_{H,\infty}$,

$$\frac{|I_0'|}{1-|I_0|^2} \le \frac{|I'|}{1-|I|^2}.$$

Since I_0 and I have the same critical set as H, the quotient $q = I_0'/I'$ is holomorphic and non-vanishing. Rearranging, we get

$$|q| \le \frac{1-|I_0|^2}{1-|I|^2} \le \frac{1}{1-|I|^2}.$$

By Theorem 3.1, $I_0' \in [I']$. Hence, $[H] = [I_0'] = [I']$.

Finally, we address uniqueness. Suppose F and G are inner functions with $[F'] = [G'] = [H]$. By the third bullet in Section 1.1, $I_{F'} = I_{G'}$, while by the fifth, $I_{F'} = F$ and $I_{G'} = G$. Therefore, F and G agree up to post-composition with an automorphism of the unit disk. □

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Declaration on the use of AI Tools

The proof of Theorem 1.1 was found by ChatGPT-6 Astra. The role of the human author is purely expository.

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