0366.4118 Percolation on Finite Graphs

Spring Semester 2026

Homework assignment 3

Not for submission/not to be graded

[The first three questions are aimed to get you more familiar with the binary cube and its (deterministic) properties.]

Problem 1. Prove that for every positive integer d, the binary cube Q^d is d-connected.

Problem 2. Let $\bar{x}, \bar{y}$ be vertices of different parity in the cube Q^d. Show that Q^d contains a Hamilton path with endpoints $\bar{x}$ and $\bar{y}$. [**Hint:** use induction on d.]

Problem 3 (Gilbert-Varshamov bound)**.** For integers $0 \leqslant k \leqslant d$ denote by $b(d,k)$ the volume of a ball of radius k in Q^d, i.e., $b(d,k) = \sum_{i=0}^{k} \binom{d}{i}$. Prove that there is a set $C \subseteq V(Q^d)$ of cardinality $|C| \geqslant \frac{2^d}{b(d,k)}$ such that every pair of distinct vertices $\bar{x}, \bar{y} \in C$ are at distance more than k in Q^d.

Problem 4. Let $c > 0$ be a constant, and set $p = \frac{c}{d}$. Let Q_p^d be a random subgraph of Q^d formed by retaining every edge of Q^d independently and with probability p.
(a) Prove that there is a constant $c_1 = c_1(c) > 0$ such that with high probability Q_p^d contains a path of length at least $\frac{c_1 d}{\ln d}$. [**Hint:** you can use the claim from Problem 2 here.]
(b)* Prove that in fact there is a constant $c_2 = c_2(c) > 0$ such that with high probability Q_p^d contains a path of length at least $c_2 d$. [**Hint:** you can use the claim from Problem 3 here.]

Problem 5. Let $c > 0$ be a constant, and set $p = \frac{c}{d}$. Prove that with high probability the random cube Q_p^d is non-planar. [**Hint:** do **not** use the approaches from HW1 here.]

Problem 6. Find (as accurately as you can) the threshold $p(d)$ for the disappearance of isolated vertices in Q_p^d.