0366.4118 Percolation on Finite Graphs

Spring Semester 2026

Homework assignment 4

Not for submission/not to be graded

Problem 1. Let G be a graph of minimum degree d, and let $\epsilon > 0$ be a small enough constant. Form a random subgraph G_p by retaining every edge of G with probability $p = \frac{1+\epsilon}{d}$ and independently. Prove: with high probability (as $d \to \infty$) the random subgraph G_p contains a path of length at least $\frac{\epsilon^2 d}{5}$. [**Hint:** mimic the proof given in the class for the likely existence of a long path in $G\left(n, \frac{1+\epsilon}{n}\right)$.]

Problem 2. Let d be a positive integer. Assume a graph $G = (V, E)$ has the followinng properties: the vertex set V is composed of d^2 disjoint sets V_i, each of cardinality d and spanning a clique K_d. For every $1 \leqslant i \leqslant d^2$, every vertex $v \in V_i$ is connected to exactly four vertices in $V \setminus V_i$. Perform edge percolation on G with edge probability $p = \frac{c}{d}$, with $c > 1$ being a constant (while $d \to \infty$), to form a random subgraph G_p. Give a typical lower bound on $|L_{15}|$, the size of the 15-th largest connected component of G_p. Do we typically have $|L_{15}| = O(\log d)$?

Problem 3. Prove: for every $\epsilon > 0$ there is $C = C(\epsilon) > 0$ such that the random cube Q_p^d with $p = \frac{C}{d}$ contains with high probability a matching covering at least $(1-\epsilon)2^d$ vertices. [**Hint:** you can prove first that for $\delta > 0$, there are typically relatively few edges of Q_p^d touching vertices of degree at least $(1+\delta)C$ in Q_p^d; then argue that deleting all vertices of degree at least $(1+\delta)C$ and their incident edges typically leaves a graph with a large matching.]

Problem 4. Let $c \in (0.5, 1]$ be a constant, and set $p = c$. Prove that with high probability in the random cube Q_p^d every vertex $v \in V(Q^d)$ belongs to a cycle. [**Hint:** you can prove for example that whp every $(d-2)$-dimensional subcube of Q^d stays connected in Q_p^d. Or perhaps you can manage with 4-cycles only?]