Incidences in Three Dimensions and Distinct Distances in the Plane*

[Extended Abstract]

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ABSTRACT

We first describe a reduction from the problem of lower-bounding the number of distinct distances determined by a set S of s points in the plane to an incidence problem between points and a certain class of helices (or parabolas) in three dimensions. We offer conjectures involving the new setup, but are still unable to fully resolve them.

Instead, we adapt the recent new algebraic analysis technique of Guth and Katz [9], as further developed by Elekes et al. [6], to obtain sharp bounds on the number of incidences between these helices or parabolas and points in $\mathbb{R}^3$. Applying these bounds, we obtain, among several other results, the upper bound $O(s^3)$ on the number of rotations (rigid motions) which map (at least) three points of S to three other points of S. In fact, we show that the number of such rotations which map at least $k \geq 3$ points of S to k other points of S is close to $O(s^3/k^{12/7})$.

One of our unresolved conjectures is that this number is $O(s^3/k^2)$, for $k \geq 2$. If true, it would imply the lower bound $\Omega(s/\log s)$ on the number of distinct distances in the plane.

Categories and Subject Descriptors

F.2.2 [**Analysis of algorithms and problem complexity**]: Non-numerical algorithms and problems—*Geometrical problems and computations*; G.2.1 [**Discrete mathematics**]: Combinatorics—*Counting problems*

General Terms

Theory

Keywords

Distinct distances, Incidences, Algebraic Tecniques

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1. THE INFRASTRUCTURE

The motivation for the study reported in this paper comes from the celebrated and long-standing problem, originally posed by Erdős [8] in 1946, of obtaining a sharp lower bound for the number of distinct distances guaranteed to exist in any set S of s points in the plane. Erdős has shown that a section of the integer lattice determines only $O(s/\sqrt{\log s})$ distinct distances, and conjectured this to be a lower bound for any planar point set. In spite of steady progress on this problem, reviewed next, Erdős's conjecture is still open.

L. Moser [12], Chung [4], and Chung *et al.* [5] proved that the number of distinct distances determined by s points in the plane is $\Omega(s^{2/3})$, $\Omega(s^{5/7})$, and $\Omega(s^{4/5}/\text{polylog}(s))$, respectively. Székely [19] managed to get rid of the polylogarithmic factor, while Solymosi and Tóth [17] improved this bound to $\Omega(s^{6/7})$. This was a real breakthrough. Their analysis was subsequently refined by Tardos [21] and then by Katz and Tardos [11], who obtained the current record of $\Omega(s^{(48-14e)/(55-16e)-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(s^{0.8641})$.

In this paper we transform the problem of distinct distances in the plane to an incidence problem between points and a certain kind of curves (helices or parabolas) in three dimensions. As we show, sharp upper bounds on the number of such incidences translate back to sharp lower bounds on the number of distinct distances. Incidence problems in three dimensions between points and curves have been studied in several recent works [2, 6, 16], and a major push in this direction has been made last year, with the breakthrough result of Guth and Katz [9], who have introduced methods from algebraic geometry for studying problems of this kind. This has been picked up by the authors [6], where worst-case tight bounds on the number of incidences between points and lines in three dimensions (under certain restrictions) have been obtained.

The present paper serves two purposes. First, it studies in detail the connection between the distinct distances problem and the corresponding 3-dimensional incidence problem. As it turns out, there is a lot of interesting geometric structure behind this reduction, and the paper (or rather its full version) develops it in detail. We offer several conjectures on the number of incidences, and show how, if true, they yield the almost worst-case tight lower bound $\Omega(s/\log s)$ on the number of distinct distances. Unfortunately, so far we have not succeeded in proving these conjectures. Nevertheless, we have made considerable progress on the incidence problem itself, which is the second purpose of the study in this paper. We show how to adapt the algebraic machinery of [6, 9, 10, 14] to derive sharp bounds for the incidence problem. These bounds are very similar to, and in fact even better than the bounds obtained in [6] for point-line incidences, where they have been shown to be worst-case tight. However, they are not (yet) good enough to yield significant lower bounds for distinct distances. We believe that there is additional geometric structure in the particular problem studied here, which should enable one to further improve the bounds, but so far this remains elusive.

The paper is organized as follows. We first describe the reduction from the planar distinct distances problem to the 3-dimensional incidence problem mentioned above. In doing so, we note and explore several additional geometric connections between the two problems (as manifested, e.g., in the analysis of *special surfaces* given below). We then present the tools from algebraic geometry that are needed to tackle the incidence problem; they are variants of the tools used in [6, 9], adapted to the specific curves that we need to handle. We then go on to bound the number of incidences. We first bound the number of rotations in terms of the number of parabolas, and then bound the number of incidences themselves. The latter task is achieved in two steps. We first use a "purely algebraic" analysis, akin to those in [6, 9], to obtain a weaker bound, which we then refine in the second step, using more traditional space decomposition techniques. The final bound is still not as good as we would like it to be, but it shows that the case studied in this paper "behaves better" than its counterpart involving lines.

Due to severe lack of space, many details are omitted in this version. They can be found in the full version [7].

Distinct distances and incidences with helices. We offer the following novel approach to the problem of distinct distances.

(H1) Notation. Let S be a set of s points in the plane with x distinct distances. Let K denote the set of all quadruples $(a, b, a', b') \in S^4$, such that the pairs (a, b) and (a', b') are distinct (although the points themselves need not be) and $|ab| = |a'b'| > 0$.

Let $\delta_1, \ldots, \delta_x$ denote the x distinct distances in S, and let $E_i = \{(a, b) \in S^2 \mid |ab| = \delta_i\}$. We have

$$|K| = 2\sum_{i=1}^{x}\binom{|E_i|}{2} \geq \sum_{i=1}^{x}(|E_i|-1)^2$$
$$\geq \frac{1}{x}\left[\sum_{i=1}^{x}(|E_i|-1)\right]^2 = \frac{[s(s-1)-x]^2}{x}.$$

(H2) Rotations. We associate each $(a, b, a', b') \in K$ with a (unique) *rotation* (or, rather, a rigid, orientation-preserving transformation of the plane) τ, which maps a to a' and b to b'. A rotation τ, in complex notation, can be written as the transformation $z \mapsto pz + q$, where $p, q \in \mathbb{C}$ and $|p| = 1$. Putting $p = e^{i\theta}$, $q = \xi + i\eta$, we can represent τ by the point $(\xi, \eta, \theta) \in \mathbb{R}^3$. In the planar context, θ is the counterclockwise angle of the rotation, and the center of rotation is $c = q/(1 - e^{i\theta})$, which is defined for $\theta \neq 0$; for $\theta = 0$, τ is a pure translation.

The *multiplicity* $\mu(\tau)$ of a rotation τ (with respect to S) is defined as $|\tau(S) \cap S|$ = the number of pairs $(a, b) \in S^2$ such that $\tau(a) = b$. Clearly, one always has $\mu(\tau) \leq s$, and we will mostly consider only rotations satisfying $\mu(\tau) \geq 2$. As a matter of fact, the bulk of the paper will only consider rotations with multiplicity at least 3. Rotations with multiplicity 2 are harder to analyze.

If $\mu(\tau) = k$ then S contains two congruent and equally oriented copies A, B of some k-element set, such that $\tau(A) = B$. Thus, studying multiplicities of rotations is closely related to analyzing repeated (congruent and equally oriented) patterns in a planar point set; see [3] for a review of many problems of this kind.

(H3) Bounding $|K|$. If $\mu(\tau) = k$ then τ contributes $\binom{k}{2}$ quadruples to K. Let N_k (resp., $N_{\geq k}$) denote the number of rotations with multiplicity exactly k (resp., at least k), for $k \geq 2$. Then

$$|K| = \sum_{k=2}^{s}\binom{k}{2}N_k = \sum_{k=2}^{s}\binom{k}{2}(N_{\geq k} - N_{\geq k+1})$$
$$= N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k}.$$

(H4) The main conjecture.

CONJECTURE 1. *For any $2 \leq k \leq s$, we have*

$$N_{\geq k} = O\left(s^3/k^2\right).$$

Suppose that the conjecture were true. Then we would have

$$\frac{[s(s-1)-x]^2}{x} \leq |K| = O(s^3)\cdot\left[1 + \sum_{k\geq 3}\frac{1}{k}\right] = O(s^3\log s),$$

which would have implied that $x = \Omega(s/\log s)$. This would have almost settled the problem of obtaining a tight bound for the minimum number of distinct distances guaranteed to exist in any set of s points in the plane, since, as mentioned above, the upper bound for this quantity is $O(s/\sqrt{\log s})$ [8].

We note that Conjecture 1 is rather deep; even the simple instance $k = 2$, asserting that there are only $O(s^3)$ rotations which map (at least) two points of S to two other points of S (at the same distance apart), seems quite difficult. In this paper we establish a variety of upper bounds on the number of rotations and on the sum of their multiplicities. In particular, these results provide a partial positive answer, showing that $N_{\geq 3} = O(s^3)$; that is, the number of rotations which map a (degenerate or non-degenerate) triangle determined by S to another congruent (and equally oriented) such triangle, is $O(s^3)$. Bounding N_2 by $O(s^3)$ is still an open problem. See Section 5 for a simple proof of the weaker bound $N_{\geq 2} = O(s^{10/3})$.

Lower bound. In the full version [7] we present a construction (suggested by Haim Kaplan) which shows:

LEMMA 2. *There exist sets S in the plane of arbitrarily large cardinality, which determine $\Theta(|S|^3)$ distinct rotations, each mapping a triple of points of S to another triple of points of S.*

A "weakness" of this construction is that all these rotations map a *collinear* triple of points of S to another collinear triple. (In the terminology to follow, these will be called *flat* rotations.) We do not know whether the number of rotations which map a *non-collinear* triple of points of S to another non-collinear triple can be $\Omega(|S|^3)$. We tend to conjecture that this is indeed the case.

We also do not know whether Conjecture 1 is worst-case tight (if true). That is, do there exist sets S, with $s = |S|$ arbitrarily large, so that there are $\Omega(s^3/k^2)$ distinct rotations, each mapping at least k points of S to k other points of S?

(H5) Helices. To estimate $N_{\geq k}$, we reduce the problem of analyzing rotations and their interaction with S to an incidence problem in three dimensions, as follows.

With each pair $(a, b) \in S^2$, we associate the curve $h_{a,b}$, in a 3-dimensional space parametrized by (ξ, η, θ), which is the locus of all rotations which map a to b. That is, the equation of $h_{a,b}$ is given by

$$h_{a,b} = \{(\xi, \eta, \theta) \mid b = ae^{i\theta} + (\xi, \eta)\}.$$

Putting $a = (a_1, a_2)$, $b = (b_1, b_2)$, this becomes

$$\xi = b_1 - (a_1\cos\theta - a_2\sin\theta), \qquad (1)$$
$$\eta = b_2 - (a_1\sin\theta + a_2\cos\theta).$$

This is a *helix* in $\mathbb{R}^3$, having four degrees of freeedom, which we parametrize by (a_1, a_2, b_1, b_2). It extends from the plane $\theta = 0$ to the plane $\theta = 2\pi$; its two endpoints lie vertically above each other, and it completes exactly one revolution between them.

(H6) Helices, rotations, and incidences. Let P be a set of rotations, represented by points in $\mathbb{R}^3$, as above, and let H denote the set of all s^2 helices $h_{a,b}$, for $(a,b) \in S^2$ (note that $a = b$ is permitted). Let $I(P,H)$ denote the number of incidences between P and H. Then we have

$$I(P,H) = \sum_{\tau \in P} \mu(\tau).$$

Rotations τ with $\mu(\tau) = 1$ are not interesting, because each of them only contributes 1 to the count $I(P,H)$, and we will mostly ignore them. For the same reason, rotations with $\mu(\tau) = 2$ are also not interesting for estimating $I(P,H)$, but they need to be included in the analysis of $N_{\geq 2}$. Unfortunately, as already noted, we do not yet have a good upper bound (i.e., cubic in s) on the number of such rotations.

(H7) Incidences and the second conjecture.

CONJECTURE 3. *For any P and H as above, we have*

$$I(P,H) = O\left(|P|^{1/2}|H|^{3/4} + |P| + |H|\right).$$

Suppose that Conjecture 3 were true. Let $P_{\geq k}$ denote the set of all rotations with multiplicity at least k (with respect to S). We then have

$$kN_{\geq k} = k|P_{\geq k}| \leq I(P_{\geq k}, H) = O\left(N_{\geq k}^{1/2}|H|^{3/4} + N_{\geq k} + |H|\right),$$

from which we obtain

$$N_{\geq k} = O\left(\frac{s^3}{k^2} + \frac{s^2}{k}\right) = O\left(\frac{s^3}{k^2}\right),$$

thus establishing Conjecture 1, and therefore also the lower bound for x (the number of distinct distances) derived above from this conjecture.

Remark. Conjecture 3 can also be formulated for an *arbitrary* subset H of all possible helices.

Note that two helices $h_{a,b}$ and $h_{c,d}$ intersect in at most one point—this is the unique rotation which maps a to b and c to d (if it exists at all, namely if $|ac| = |bd|$). Hence, combining this fact with a standard cutting-based decomposition technique, similar to what has been noted in [16], say, yields the weaker bound

$$I(P,H) = O\left(|P|^{2/3}|H|^{2/3} + |P| + |H|\right), \tag{2}$$

which, alas, only yields the much weaker bound $N_{\geq k} = O\left(s^4/k^3\right)$, which is completely useless for deriving any lower bound on x. (We will use this bound, though, in Section 6.)

(H8) From helices to parabolas. The helices $h_{a,b}$ are non-algebraic curves, because of the use of the angle θ as a parameter. This can be easily remedied, in the following standard manner. Assume that θ ranges from $-\pi$ to π, and substitute, in the equations (1), $Z = \tan(\theta/2)$, $X = \xi(1+Z^2)$, and $Y = \eta(1+Z^2)$, to obtain

$$\begin{aligned} X &= (a_1+b_1)Z^2 + 2a_2 Z + (b_1 - a_1) \\ Y &= (a_2+b_2)Z^2 - 2a_1 Z + (b_2 - a_2), \end{aligned} \tag{3}$$

which are the equations of a *planar parabola* in the (X,Y,Z)-space. (The parabola degenerates to a line if $b = -a$, a situation that we will rule out by choosing an appropriate generic coordinate frame in the original xy-plane.) We denote the parabola corresponding to the helix $h_{a,b}$ as $h^*_{a,b}$, and refer to it as an *h-parabola*.

(H9) Joint and flat rotations. A rotation $\tau \in P$ is called a *joint* of H if τ is incident to at least three helices of H whose tangent lines at τ are non-coplanar. Otherwise, still assuming that τ is incident to at least three helices of H, τ is called *flat*.

Let $\tau = (\xi, \eta, \theta) \in P$ be a rotation, incident to three distinct helices $h_{a,b}$, $h_{c,d}$, $h_{e,f}$. From their equations, as given in (1), the directions of the tangents to these helices at τ are

$$\begin{aligned} &(a_1 \sin\theta + a_2\cos\theta,\ -a_1\cos\theta + a_2\sin\theta,\ 1) \\ &(c_1 \sin\theta + c_2\cos\theta,\ -c_1\cos\theta + c_2\sin\theta,\ 1) \\ &(e_1 \sin\theta + e_2\cos\theta,\ -e_1\cos\theta + e_2\sin\theta,\ 1). \end{aligned}$$

Put $p = \cos\theta$ and $q = \sin\theta$. Then the three tangents are coplanar if and only if

$$\begin{vmatrix} a_1q + a_2p & -a_1p + a_2q & 1 \\ c_1q + c_2p & -c_1p + c_2q & 1 \\ e_1q + e_2p & -e_1p + e_2q & 1 \end{vmatrix} = 0.$$

Simplifying the determinant, and recalling that $p^2 + q^2 = 1$, the condition is equivalent to

$$\begin{vmatrix} a_1 & a_2 & 1 \\ c_1 & c_2 & 1 \\ e_1 & e_2 & 1 \end{vmatrix} = 0.$$

In other words, the three helices $h_{a,b}$, $h_{c,d}$, $h_{e,f}$ form a joint at τ if and only if the three points a, c, e (and thus also b, d, f) are non-collinear.

CLAIM 4. *A rotation τ is a joint of H if and only if τ maps a non-degenerate triangle determined by S to another (congruent and equally oriented) non-degenerate triangle determined by S. A rotation τ is a flat rotation if and only if τ maps at least three collinear points of S to another collinear triple of points of S, but does not map any point of S outside the line containing the triple to another point of S.*

The preceding analysis also shows that, for any fixed rotation τ, the directions of the tangents at τ to the helices incident to τ are all distinct. This will be important in the algebraic analysis given below.

(H10) Special surfaces. In preparation for the forthcoming algebraic analysis, we need the following property of our helices.

Let τ be a flat rotation, with multiplicity $k \geq 3$, and let ℓ and ℓ' be the corresponding lines in the plane, such that there exist k points $a_1, \ldots, a_k \in S \cap \ell$ and k points $b_1, \ldots, b_k \in S \cap \ell'$, such that τ maps a_i to b_i for each i (and in particular maps ℓ to ℓ'). By definition, τ is incident to the k helices h_{a_i,b_i}, for $i = 1, \ldots, k$.

Let u and v denote unit vectors in the direction of ℓ and ℓ', respectively. Clearly, there exist two reference points $a \in \ell$ and $b \in \ell'$, such that for each i there is a real number t_i such that $a_i = a + t_i u$ and $b_i = b + t_i v$. As a matter of fact, for each real t, τ maps $a + tu$ to $b + tv$, so it is incident to $h_{a+tu,b+tv}$. Note that a and b are not uniquely defined: we can take a to be any point on ℓ, and shift b accordingly along ℓ'.

Let $H(a,b;u,v)$ denote the set of these helices. Since a pair of helices can meet in at most one point, all the helices in $H(a,b;u,v)$ pass through τ but are otherwise pairwise disjoint. Using the reparametrization $(\xi,\eta,\theta) \mapsto (X,Y,Z)$, we denote by $\Sigma = \Sigma(a,b;u,v)$ the surface which is the union of all the h-parabolas that are the images of the helices in $H(a,b;u,v)$. We refer to such a surface Σ as a *special surface*.

An important comment is that most of the ongoing analysis also applies when only two helices are incident to τ; they suffice to determine the four parameters a, b, u, v that define the surface Σ.

We also remark that, although we started the definition of $\Sigma(a,b;u,v)$ with a flat rotation τ, the definition only depends on the parameters a,b,u, and v. If τ is not flat it may determine many special surfaces, one for each line that contains two or more points of S which τ maps to other (also collinear) points of S. Also, as we will shortly see, the same surface can be obtained from a different set (in fact, many such sets) of parameters a', b', u', and v' (or, alternatively, from different flat rotations τ'). An "intrinsic" definition of special surfaces will be given shortly.

The surface $\Sigma(a,b;u,v)$ is a cubic algebraic surface, whose equation, worked out in detail in the full version [7], is

$$E_2(Z)X - E_1(Z)Y + K(Z) = 0, \quad \text{where} \tag{4}$$

$$\begin{aligned} E_1(Z) &= (u_1+v_1)Z + (u_2+v_2) \\ E_2(Z) &= (u_2+v_2)Z - (u_1+v_1), \end{aligned} \tag{5}$$

and $K(Z)$ is the cubic polynomial

$$\Big((u_1+v_1)Z + (u_2+v_2)\Big)\Big((a_2+b_2)Z^2 - 2a_1Z + (b_2-a_2)\Big) -$$
$$\Big((u_2+v_2)Z - (u_1+v_1)\Big)\Big((a_1+b_1)Z^2 + 2a_2Z + (b_1-a_1)\Big).$$

We refer to the cubic polynomial in the left-hand side of (4) as a *special polynomial*. Thus a special surface is the zero set of a special polynomial.

(H11) The geometry of special surfaces. Special surfaces pose a technical challenge to the analysis. Specifically, each special surface Σ captures a certain underlying pattern in the ground set S, which may result in many incidences between rotations and h-parabolas, all contained in Σ. The next step of the analysis studies this pattern in detail.

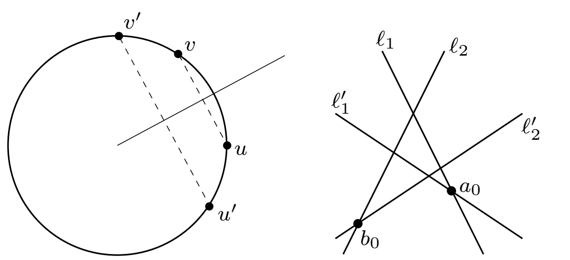

Figure 1: Left: The configuration of u, v, u', v'. Right: The structure of τ and τ' on a common special surface Σ.

Consider first a simple instance of this situation, in which two special surfaces Σ, Σ', generated by two distinct flat rotations τ, τ', coincide. More precisely, there exist four parameters a, b, u, v such that τ maps the line $\ell_1 = a + tu$ to the line $\ell_2 = b + tv$ (so that points with the same parameter t are mapped to one another), and four other parameters a', b', u', v' such that τ' maps (in a similar manner) the line $\ell'_1 = a' + tu'$ to the line $\ell'_2 = b' + tv'$, and $\Sigma(a,b;u,v) = \Sigma(a',b';u',v')$. Denote this common surface by Σ. Since the surfaces coincide, simple algebraic calculations, detailed in [7], show that the angle bisector between u and v must coincide with that between u' and v', as depicted in Figure 1(left). Moreover, as is easily checked, if we let a_0 be the intersection point of ℓ_1 and ℓ'_1, and let b_0 be the intersection point of ℓ_2 and ℓ'_2, then both τ and τ' map a_0 to b_0, and $h^*_{a_0,b_0}$ is contained in Σ. (See Figure 1(right) and [7] for details.)

Since the preceding analysis applies to any pair of distinct rotations on a common special surface Σ, it follows that we can associate with Σ a common direction w and a common shift δ, so that for each $\tau \in \Sigma$ there exist two lines ℓ, ℓ', where τ maps ℓ to ℓ', so that the angle bisector between these lines is in direction w, and τ is the unique rigid motion, obtained by rotating ℓ to ℓ' around their intersection point $\ell \cap \ell'$, and then shifting ℓ' along itself by a distance whose projection in direction w is δ. See Figure 1(right).

Let Σ be a special surface, generated by $H(a,b;u,v)$; that is, Σ is the union of all parabolas of the form $h^*_{a+tu,b+tv}$, for $t \in \mathbb{R}$. Let τ_0 be the common rotation to all these parabolas, so it maps the line $\ell_0 = \{a + tu \mid t \in \mathbb{R}\}$ to the line $\ell'_0 = \{b + tv \mid t \in \mathbb{R}\}$, so that every point $a + tu$ is mapped to $b + tv$.

Let $h^*_{c,d}$ be a parabola contained in Σ but not passing through τ_0. Take any pair of distinct rotations τ_1, τ_2 on $h^*_{c,d}$. Then there exist two respective real numbers t_1, t_2, such that $\tau_i \in h^*_{a+t_iu,b+t_iv}$, for $i = 1, 2$. Thus τ_i is the unique rotation which maps c to d and $a_i = a + t_iu$ to $b_i = b + t_iv$. In particular, we have $|a + t_iu - c| = |b + t_iv - d|$. This in turn implies that the triangles a_1a_2c and b_1b_2d are congruent; see Figure 2.

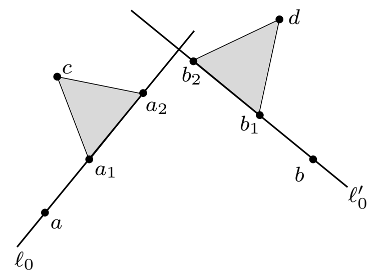

Figure 2: The geometric configuration corresponding to a parabola $h^*_{c,d}$ contained in Σ.

Given c, this determines d, up to a reflection about ℓ'_0. We claim that d has to be on the "other side" of ℓ'_0, namely, be such that the triangles a_1a_2c and b_1b_2d are oppositely oriented. Indeed, if they were equally oriented, then τ_0 would have mapped c to d, and then $h^*_{c,d}$ would have passed through τ_0, contrary to assumption.

Now form the two sets

$$\begin{aligned} A &= \{p \in S \mid \text{there exists } q \in S \text{ such that } h^*_{p,q} \subset \Sigma\} \\ B &= \{q \in S \mid \text{there exists } p \in S \text{ such that } h^*_{p,q} \subset \Sigma\}. \end{aligned} \tag{6}$$

The preceding discussion implies that A and B are congruent and oppositely oriented.

To recap, each rotation $\tau \in \Sigma$, incident to $k \geq 2$ parabolas contained in Σ, corresponds to a pair of lines ℓ, ℓ' with the above properties, so that τ maps k points of $S \cap \ell$ (rather, of $A \cap \ell$) to k points of $S \cap \ell'$ (that is, of $B \cap \ell'$). If τ is flat, its entire multiplicity comes from points of S on ℓ (these are the points of $A \cap \ell$) which are mapped by τ to points of S on ℓ' (these are points of $B \cap \ell'$), and all the corresponding parabolas are contained in Σ. If τ is a joint then, for any other point $p \in S$ outside ℓ which is mapped by τ to a point $q \in S$ outside ℓ', the parabola $h^*_{p,q}$ is not contained in Σ, and crosses it transversally at the unique rotation τ.

Note also that any pair of parabolas $h^*_{c_1,d_1}$ and $h^*_{c_2,d_2}$ which are contained in Σ intersect, necessarily at the unique rotation which maps c_1 to d_1 and c_2 to d_2. This holds because $|c_1c_2| = |d_1d_2|$, as follows from the preceding discussion.

Special surfaces are anti-rotations. Let Σ be a special surface, and let A, B be the subsets of S associated with Σ, as in (6). Then there exists a single *anti-rotation* (a rigid, orientation-reversiong motion of the plane) which maps A to B. Conversely, any anti-rotation can be associated with a unique special surface in this manner. However, the number of incidences within a special surface may be larger than the incidence count of the anti-rotation with the appropriate variants of the h-parabolas: the former counts incidences between the points of A (or of B) and the lines that they determine, while the latter only counts the size of A (or of B).

Parabolas on special surfaces.

LEMMA 5. *A special surface can contain at most s h-parabolas.*

Proof: Let Ξ be the given special surface. We claim that for each $a \in S$ there can be at most one point $b \in S$ such that $h^*_{a,b} \subset \Xi$. Indeed, suppose that there exist two such points $b_1, b_2 \in S$. Since any pair of h-parabolas on Ξ intersect, h^*_{a,b_1} and h^*_{a,b_2} meet at a rotation τ, which maps a to both b_1 and b_2, an impossibility which completes the proof. □

2. TOOLS FROM ALGEBRAIC GEOMETRY

We review in this section (without proofs) the basic tools from algebraic geometry that have been used in [6, 9]. We state here the variants that arise in the context of incidences between points and our h-parabolas. The proofs can be found in the full version of the paper [7].

So let C be a set of $n \le s^2$ h-parabolas in $\mathbb{R}^3$. Recalling the definitions in (H9), we say that a point (rotation) a is a *joint* of C if it is incident to three parabolas of C whose tangents at a are non-coplanar. Let $J = J_C$ denote the set of joints of C. We will also consider points a that are incident to three or more parabolas of C, so that the tangents to all these parabolas are coplanar, and refer to such points as *flat* points of C. We recall (see (H9)) that any pair of distinct h-parabolas which meet at a point have there distinct tangents.

First, we note that, using a trivial application of Bézout's theorem [15], a trivariate polynomial p of degree d which vanishes at $2d+1$ points that lie on a common h-parabola $h^* \in C$ must vanish identically on h^*.

Critical points and parabolas. A point a is *critical* (or *singular*) for a trivariate polynomial p if $p(a) = 0$ and $\nabla p(a) = 0$; any other point a in the zero set of p is called *regular*. A parabola h^* is *critical* if all its points are critical.

Another application of Bézout's theorem implies the following.

PROPOSITION 6. *Let C be as above. Then any trivariate square-free polynomial p of degree d can have at most $d(d-1)$ critical parabolas in C.*

For regular points of p, we have the following easy observation.

PROPOSITION 7. *Let a be a regular point of p, so that $p \equiv 0$ on three parabolas of C passing through a. Then these parabolas must have coplanar tangents at a.*

Hence, a point a incident to three parabolas of C whose tangent lines at a are non-coplanar, so that $p \equiv 0$ on each of these parabolas, must be a critical point of p.

The main ingredient in the algebraic approach to incidence problems is the following, fairly easy (and rather well-known) result.

PROPOSITION 8. *Given a set S of m points in 3-space, there exists a nontrivial trivariate polynomial $p(x, y, z)$ which vanishes at all the points of S, of degree d, for any d satisfying $\binom{d+3}{3} > m$.*

Proof: (See [6, 7, 9].) A trivariate polynomial of degree d has $\binom{d+3}{3}$ monomials, and requiring it to vanish at m points yields these many homogeneous equations in the coefficients of these monomials. Such an underdetermined system always has a nontrivial solution. □

Flat points and parabolas. Call a regular point τ of a trivariate polynomial p *geometrically flat* if it is incident to three distinct parabolas of C (with necessarily coplanar tangent lines at τ, no pair of which are collinear) on which p vanishes identically.

Handling geometrically flat points in our analysis is somewhat trickier than handling critical points, and involves the second-order partial derivatives of p. The analysis, detailed in [7] (and similar to those in [6, 9]) leads to the following properties.

PROPOSITION 9. *Let p be a trivariate polynomial, and define*

$$\Pi(p) = p_Y^2 p_{XX} - 2p_X p_Y p_{XY} + p_X^2 p_{YY}.$$

Then, if τ is a regular geometrically flat point of p (with respect to three parabolas of C) then $\Pi(p)(\tau) = 0$.

Remarks. (1) $\Pi(p)$ is one of the polynomials that form the *second fundamental form* of p; see [6, 7, 9, 13] for details.

(2) Although most details are suppressed, it is important to note that for Proposition 9 to hold it is crucial that τ be incident to (at least) three parabolas of C. This is why we can only handle rotations of multiplicity at least 3.

In particular, if the degree of p is d then the degree of $\Pi(p)$ is at most $(d-1) + (d-1) + (d-2) = 3d - 4$.

In what follows, we call a point τ *flat* for p if $\Pi(p)(\tau) = 0$. Call an h-parabola $h^* \in C$ *flat* for p if all the points of h^* are flat points of p (with the possible exception of a discrete subset). Arguing as in the case of critical points, if h^* contains more than $2(3d-4)$ flat points then h^* is a flat parabola.

The next proposition shows that, in general, trivariate polynomials do not have too many flat parabolas. The proof is based on Bézout's theorem, as does the proof of Proposition 6.

PROPOSITION 10. *Let p be any trivariate square-free polynomial of degree d with no special polynomial factors. Then p can have at most $d(3d-4)$ flat h-parabolas in C.*

3. THE NUMBER OF ROTATIONS

In this section we extend the recent algebraic machinery of Guth and Katz [9], as further developed by Elekes et al. [6], using the algebraic tools set forth in the preceding section, to establish the bound $O(n^{3/2}) = O(s^3)$ on the number of rotations with multiplicity at least 3 in a collection of n h-parabolas.

THEOREM 11. *Let C be a set of at most n h-parabolas in $\mathbb{R}^3$, and let P be a set of m rotations, each of which is incident to at least three parabolas of C. Suppose further that no special surface contains more than q parabolas of C. Then $m = O(n^{3/2} + nq)$.*

Remarks. (1) The recent results of [10, 14] imply that the number of joints in a set of n h-parabolas is $O(n^{3/2})$. The proofs in [10, 14] are much simpler than the proof given below, but they do not apply to flat points (rotations) as does Theorem 11. Since flat rotations are an integral part of the setup considered in this paper, we need to count them too, using the stronger Theorem 11. Moreover, even if we were to consider only joint rotations, the analysis of their incidences with the h-parabolas will turn some of them into flat rotations (by pruning some of the parabolas), so we will need to face flat rotations, no matter what.

(2) By Lemma 5, we always have $q \le s$, and we also have $n^{1/2} \le s$, so the "worst-case" bound on m is $O(ns)$.

(3) Note that the parameter n in the statement of the theorem is arbitrary, not necessarily the maximum number s^2. When n attains its maximum possible value s^2, the bound becomes $m = O(n^{3/2}) = O(s^3)$.

The proof of Theorem 11 uses the proof technique of [6], properly adapted to the present, somewhat more involved context of h-parabolas and rotations.

Proof. We first prove the theorem under the additional assumption that $q = n^{1/2}$. The proof proceeds by induction on n, and shows that $m \le An^{3/2}$, where A is a sufficiently large constant whose choice will be dictated by the forthcoming analysis. The statement holds for all $n \le n_0$, for some constant n_0, if we choose A to be sufficiently large. Fix $n > n_0$, and suppose that the claim holds for all $n' < n$. Let C and P be as in the statement of the theorem, with $|C| = n$, and suppose to the contrary that $|P| > An^{3/2}$.

We first apply the following iterative pruning process to C. As long as there exists a parabola $h^* \in C$ incident to fewer than $cn^{1/2}$ rotations of P, for some constant $1 \le c \ll A$ that we will fix later, we remove h^* from C, remove its incident rotations from P, and repeat this step with respect to the reduced set of rotations. In this process we delete at most $cn^{3/2}$ rotations. We are thus left with a subset of at least $(A-c)n^{3/2}$ of the original parabolas, each incident to at least $cn^{1/2}$ surviving rotations, and each surviving rotation is incident to at least three surviving parabolas. For simplicity, continue to denote these sets as C and P.

Choose a random sample C^s of parabolas from C, by picking each parabola independently with probability t, where t is a small constant that we will fix later.

The expected number of parabolas that we choose is $tn_1 \le tn$, where n_1 is the number of parabolas remaining after the pruning. We have $n_1 = \Omega(n^{1/2})$, because each surviving parabola is incident to at least $cn^{1/2}$ surviving rotations, each incident to at least two other surviving parabolas; since all these parabolas are distinct (recall that a pair of parabolas can meet in at most one rotation point), we have $n_1 \ge 2cn^{1/2}$. Hence, using Chernoff's bound, as in [6] (see, e.g., [1]), we obtain that, with positive probability, (a) $|C^s| \le 2tn$. (b) Each parabola $h^* \in C$ contains at least $\frac{1}{2}ctn^{1/2}$ rotations that lie on parabolas of C^s. (To see (b), take a parabola $h^* \in C$ and a rotation $\tau \in P \cap h^*$. Note that τ will be incident to a parabola of C^s with probability at least t, so the expected number of rotations in $P \cap h^*$ which lie on parabolas of C^s is at least $ctn^{1/2}$. This, combined with Chernoff's bound, implies (b).)

We assume that C^s does indeed satisfy (a) and (b), and then (recalling that $c \ge 1$) choose $n^{1/2}$ arbitrary rotations on each parabola in C^s, to obtain a set S of at most $2tn^{3/2}$ rotations.

Applying Proposition 8, we obtain a nontrivial trivariate polynomial $p(X,Y,Z)$ which vanishes at all the rotations of S, whose degree is at most the smallest integer d satisfying $\binom{d+3}{3} \ge |S|+1$, so

$$d \le \lceil (6|S|)^{1/3} \rceil \le (12t)^{1/3} n^{1/2} + 1 \le 2(12t)^{1/3} n^{1/2},$$

for n (i.e., n_0) sufficiently large. Without loss of generality, we may assume that p is square-free—by removing repeated factors, we get a square-free polynomial which vanishes on the same set as the original p, with the same upper bound on its degree.

The polynomial p vanishes on $n^{1/2}$ points on each parabola in C^s. This number is larger than $2d$, if we choose t sufficiently small so as to satisfy $4(12t)^{1/3} < 1$. Hence p vanishes identically on all these parabolas. Any other parabola of C meets at least $\frac{1}{2}ctn^{1/2}$ parabolas of C^s, at distinct points, and we can make this number also larger than $2d$, with an appropriate choice of t and c (we need to ensure that $ct > 8(12t)^{1/3}$). Hence, p vanishes identically on each parabola of C.

We will also later need the property that each parabola of C contains at least $9d$ points of P; that is, we require that $cn^{1/2} > 9d$, which will hold if $c > 18(12t)^{1/3}$.

To recap, the preceding paragraphs impose several inequalities on c and t, and a couple of additional similar inequalities will be imposed later on. All these inequalities are easy to satisfy by choosing $t < 1$ to be a sufficiently small positive constant, and c a sufficiently large constant. (These choices will also affect the choice of A—see below.)

We note that p can have at most $d/3$ special polynomial factors (since each of them is a cubic polynomial); i.e., p can vanish identically on at most $d/3$ respective special surfaces $\Xi_1, \ldots, \Xi_k$, for $k \le d/3$. We factor out all these special polynomial factors from p, and let $\tilde{p}$ denote the resulting polynomial, which is a square-free polynomial without any special polynomial factors, of degree at most d.

Consider one of the special surfaces Ξ_i, and let t_i denote the number of parabolas contained in Ξ_i. Then any rotation on Ξ_i is either an intersection point of (at least) two of these parabolas, or it lies on at most one of them. The number of rotations of the first kind is $O(t_i^2)$. Any rotation τ of the second kind is incident to at least one parabola of C which crosses Ξ_i transversally at τ. We note that each h-parabola h^* can cross Ξ_i in at most three points. Indeed, substituting the equations of h^* into the equation $E_2(Z)X - E_1(Z)Y + K(Z) = 0$ of Ξ_i (see (4)) yields a cubic equation in Z, with at most three roots. Hence, the number of rotations of the second kind is $O(n)$, and the overall number of rotations on Ξ_i is $O(t_i^2 + n) = O(n)$, since we have assumed in the present version of the proof that $t_i \le n^{1/2}$.

Summing the bounds over all surfaces Ξ_i, we conclude that altogether they contain $O(nd)$ rotations, which we bound by $bn^{3/2}$, for some absolute constant b.

We remove all these vanishing special surfaces, together with the rotations and the parabolas which are fully contained in them, and let $C_1 \subseteq C$ and $P_1 \subseteq P$ denote, respectively, the set of those parabolas of C (rotations of P) which are not contained in any of the vanishing surfaces Ξ_i.

Note that there are still at least three parabolas of C_1 incident to any remaining rotation in P_1, since none of the rotations of P_1 lie in any surface Ξ_i, so all parabolas incident to such a rotation are still in C_1.

Clearly, $\tilde{p}$ vanishes identically on every $h^* \in C_1$. Furthermore, every $h^* \in C_1$ contains at most d points in the surfaces Ξ_i, because, as just argued, it crosses each surface Ξ_i in at most three points.

Hence, each $h^* \in C_1$ contains at least $8d$ rotations of P_1. Since each of these rotations is incident to at least three parabolas in C_1, each of these rotations is either critical or geometrically flat for $\tilde{p}$.

Consider a parabola $h^* \in C_1$. If h^* contains more than $2d$ critical rotations then h^* is a critical parabola for $\tilde{p}$. By Proposition 6, the number of such parabolas is at most $d(d-1)$. Any other parabola $h^* \in C_1$ contains more than $6d$ geometrically flat points and hence h^* must be a flat parabola for $\tilde{p}$. By Proposition 10, the number of such parabolas is at most $d(3d-4)$. Summing up we obtain

$$|C_1| \le d(d-1) + d(3d-4) < 4d^2.$$

We require that $4d^2 < n/2$; that is, $32(12t)^{2/3} < 1$, which can be guaranteed by choosing t sufficiently small.

We next want to apply the induction hypothesis to C_1, with the parameter $4d^2$ (which dominates the size of C_1). For this, we first need to argue that each special surface contains at most $(4d^2)^{1/2} = 2d$ parabolas of C_1. Indeed, let Ξ be a special surface. Using (4), eliminate, say, Y from the equation of Ξ and substitute the resulting expression into the equation of $\tilde{p}$, to obtain a bivariate polynomial $\tilde{p}_0(X, Z)$. Let h^* be a parabola of C_1 contained in Ξ. We represent h^* by its X-equation of the form $X = Q(Z)$, and observe that $\tilde{p}_0(X, Z)$ vanishes on the zero set of $X - Q(Z)$. Hence $\tilde{p}_0$ must be divisible by $X - Q(Z)$. Note that, in a generic coordinate frame in the xy-plane, two different parabolas cannot have the same equation $X = Q(Z)$, because this equation uniquely determines a_1, b_1, and a_2, and then, in a generic frame, b_2 is also uniquely determined. Note also that the degree of $\tilde{p}_0$ is at most $3d$, and that the degree of each factor $X - Q(Z)$ is 2, implying that Σ can contain at most $3d/2$ parabolas of C_1.

Hence, the maximum number of parabolas of C_1 contained in a special surface is at most $3d/2 \leq (4d^2)^{1/2}$, so, by the induction hypothesis, the number of points in P_1 is at most

$$A(4d^2)^{3/2} \leq \frac{A}{2^{3/2}} n^{3/2}.$$

Adding up the bounds on the number of points on parabolas removed during the pruning process and on the special surfaces Ξ_i (which correspond to the special polynomial factors of p), we obtain

$$|P| \leq \frac{A}{2^{3/2}} n^{3/2} + (b + c)n^{3/2} \leq An^{3/2} ,$$

with an appropriate, final choice of t, c, and A. This contradicts the assumption that $|P| > An^{3/2}$, and thus establishes the induction step for n, and, consequently, completes the proof of the restricted version of the theorem.

Proof of the general version: The proof proceeds almost exactly as the proof of the restricted version, except for the analysis of the number of rotations on the special surfaces Ξ_i, which, using the preceding notations, is bounded by

$$O\left(\sum_i (t_i^2 + n)\right) = O\left(q \cdot \sum_i t_i + nd\right) = O(n^{3/2} + nq).$$

See [7]. □

We summarize the remarks following Theorem 11, combined with Lemma 2, in the following corollary.

COROLLARY 12. *Let S be a set of s points in the plane. Then there are at most $O(s^3)$ rotations which map some (degenerate or non-degenerate) triangle spanned by S to another (congruent and equally oriented) such triangle. This bound is tight in the worst case.*

4. INCIDENCES BETWEEN PARABOLAS AND ROTATIONS

In this section we further adapt the machinery of [6] to derive an upper bound on the number of incidences between m rotations and n h-parabolas in $\mathbb{R}^3$, where each rotation is incident to at least three parabolas.

THEOREM 13. *For an underlying ground set S of s points in the plane, let C be a set of at most $n \leq s^2$ h-parabolas defined on S, and let P be a set of m rotations with multiplicity at least 3 (with respect to S). Then*

$$I(P, C) = O\left(m^{1/3}n + m^{2/3}n^{1/3}s^{1/3}\right).$$

Remark. As easily checked, the first term dominates the second term when $m \leq n^2/s$, and the second term dominates when $n^2/s < m \leq ns$. In particular, the first term dominates when $n = s^2$, because we have $m = O(s^3) = O(n^2/s)$

Proof: The proof of Theorem 13 proceeds in two steps. We first establish a bound which is independent of m, and then apply it to obtain the m-dependent bound asserted in the theorem. Due to lack of space, we only sketch the proof for the first step, given in

THEOREM 14. *Let C be a set of at most $n \leq s^2$ h-parabolas defined on S, and let P be a set of rotations with multiplicity at least 3 with respect to S, such that no special surface contains more than $n^{1/2}$ parabolas of C. Then the number of incidences between P and C is $O(n^{3/2})$.*

Proof. Write $I = I(P, C)$ for short, and put $m = |P|$. We will establish the upper bound $I \leq Bn^{3/2}$, for some sufficiently large absolute constant B, whose specific choice will be dictated by the various steps of the proof. Suppose then to the contrary that $I > Bn^{3/2}$ for the given C and P.

For $h^* \in C$, let $\nu(h^*)$, the *multiplicity* of h^*, denote the number of rotations incident to h^*. We have $\sum_{h^* \in C} \nu(h^*) = I$; the average multiplicity of a parabola h^* is I/n.

We begin by applying the following pruning process. Put $\nu = I/(6n)$. As long as there exists a parabola $h^* \in C$ whose multiplicity is smaller than ν, we remove h^* from C, but do not remove any rotation incident to h^*. We keep repeating this step (without changing ν), until each of the surviving parabolas has multiplicity at least ν. Moreover, if, during the pruning process, some rotation τ loses $\lfloor \mu(\tau)/2 \rfloor$ incident parabolas, we remove τ from P. This decreases the multiplicity of some parabolas, and we use the new multiplicities in the test for pruning further parabolas, but we keep using the original threshold ν.

When we delete a parabola h^*, we lose at most ν incidences with surviving rotations. When a rotation τ is removed, the number of current incidences with τ is smaller than or equal to twice the number of incidences with τ that have already been removed. Hence, the total number of incidences that were lost during the pruning process is a most $3n\nu = I/2$. Thus, we are left with a subset P_1 of the rotations and with a subset C_1 of the parabolas, so that each $h^* \in C_1$ is incident to at least $\nu = I/(6n)$ rotations of P_1, and each rotation $\tau \in P_1$ is incident to at least three parabolas of C_1 (the latter is an immediate consequence of the rule for pruning a rotation). Moreover, we have $I(P_1, C_1) \geq I/2$. It therefore suffices to bound $I(P_1, C_1)$.

Let $n_1 = |C_1|$. Since at least three parabolas in C_1 are incident to each rotation in P_1, it follows that each parabola in C_1 is incident to at most $n_1/2$ rotations of P_1, and therefore $I(P_1, C_1) \leq n_1^2/2$. Combining this with the fact that $I(P_1, C_1) \geq I/2$, we get that $n_1 \geq B^{1/2}n^{3/4}$.

We fix the following parameters

$$x = \frac{n_1}{n^{1/2}} \qquad \text{and} \qquad t = \delta \frac{n_1}{n},$$

for an appropriate absolute constant $\delta < 1$, whose value will be fixed shortly. Clearly, $t < 1$, and we can also ensure that $x < \nu$, i.e., that $I > 6n_1n^{1/2}$, by choosing $B > 6$. Furthermore, since $n_1 \geq B^{1/2}n^{3/4}$ we have $x \geq B^{1/2}n^{1/4}$.

We construct a random sample C_1^s of parabolas of C_1 by choosing each parabola independently at random with probability t; the expected size of C_1^s is tn_1. Now take x (arbitrary) rotations of P_1 on each parabola of C_1^s (which can always be done since $x < \nu$), to form a sample S of rotations in P_1, of expected size at most txn_1.

For any parabola $h^* \in C_1$, the expected number of rotations of $P_1 \cap h^*$ which lie on parabolas of C_1^s is at least $t\nu$ (each of the at least ν rotations $a \in P_1 \cap h^*$ is incident to at least one other parabola of C_1, and the probability of this parabola to be chosen in C_1^s is t). We assume that B is large enough so that $t\nu = \delta \frac{n_1}{n} \frac{I}{6n} \geq \frac{\delta B}{6} \frac{n_1}{n^{1/2}}$ is larger than $2x$ (it suffices to choose $B > 12/\delta$). Since $t\nu > 2x = \Omega(n^{1/4})$, and the expected size of C_1^s is $tn_1 = \frac{\delta n_1^2}{n} \geq B\delta n^{1/2}$, we can use Chernoff's bound, to show that there exists a sample C_1^s such that (i) $|C_1^s| \leq 2tn_1$, and (ii) each parabola $h^* \in C_1$ contains at least $\frac{1}{2}t\nu > x$ rotations of P_1 which lie on parabolas of C_1^s. In what follows, we assume that C_1^s satisfies these properties. In this case, we have $|S| \leq 2txn_1$.

Now construct, using Proposition 8, a nontrivial suqare-free trivariate polynomial p which vanishes on S, of smallest degree d satisfying $\binom{d+3}{3} \geq |S| + 1$, so

$$\begin{aligned} d &\leq \lceil (6|S|)^{1/3} \rceil \leq (12txn_1)^{1/3} + 1 = (12\delta)^{1/3} \frac{n_1}{n^{1/2}} + 1 \\ &\leq 2(12\delta)^{1/3} \frac{n_1}{n^{1/2}} \end{aligned}$$

for n sufficiently large (for small values of n we ensure the bound by choosing B sufficiently large, as before).

We will choose $\delta < 1/6144$, so $x > 4d$.

As above, and without loss of generality, we may assume that p is square-free: factoring out repeated factors only lowers the degree of p and does not change its zero set.

The following properties hold: (a) Since $x > 2d$, p vanishes at more than $2d$ rotations on each parabola of C_1^s, and therefore, as already argued, it vanishes identically on each of these parabolas. (b) Each parabola $h^* \in C_1$ contains at least $\frac{1}{2}t\nu > x > 2d$ rotations which lie on parabolas of C_1^s. Since, as just argued, p vanishes at these rotations, it must vanish identically on h^*. Thus, $p \equiv 0$ on every parabola of C_1.

Before proceeding, we enforce the inequality $d^2 < \frac{1}{8}n_1$ which will hold if we choose δ so that $(12\delta)^{2/3} < 1/32$. Similarly, an appropriate choice of δ (or B) also ensures that $\nu > 9d$.

We next consider all the special polynomial factors of p, and factor them out, to obtain a square-free polynomial $\tilde{p}$, of degree at most d, with no special polynomial factors. As in the previous analysis, p can have at most $d/3$ special polynomial factors, so it can vanish identically on at most $d/3$ special surfaces $\Xi_1, \ldots, \Xi_k$, for $k \leq d/3$. Let $C_2 \subseteq C_1$ denote the set of those parabolas of C_1 which are not contained in any of the vanishing surfaces Ξ_i. For each parabola $h^* \in C_2$, $\tilde{p}$ vanishes identically on h^*, and (as argued above) at most d rotations in $P_1 \cap h^*$ lie in the surfaces Ξ_i. Hence, h^* contains at least $8d$ remaining rotations, each of which is either critical or flat for $\tilde{p}$, because each such point is incident to at least three parabolas (necessarily of C_2) on which $\tilde{p} \equiv 0$.

Hence, either at least $2d$ of these rotations are critical, and then h^* is a critical parabola for $\tilde{p}$, or at least $6d$ of these rotations are flat, and then h^* is a flat parabola for $\tilde{p}$. Applying Propositions 6 and 10, the overall number of parabolas in C_2 is therefore at most

$$d(d-1) + d(3d-4) < 4d^2 < \frac{1}{2}n_1.$$

On the other hand, by assumption, each vanishing special surface Ξ_i contains at most $n^{1/2}$ parabolas of C, so the number of parabolas contained in the special vanishing surfaces is at most $n^{1/2}d < \frac{1}{4}n^{1/2}x \leq \frac{1}{4}n_1$, with our choice of δ.

Hence, the overall number of parabolas in C_1 is smaller than $\frac{1}{2}n_1 + \frac{1}{4}n_1 < n_1$, a contradiction that completes the proof of Theorem 14. $\square$

Proof of Theorem 13. Write $I = I(P, C)$ for short. Set $\nu = cm^{1/3}$ and $\mu = cn/m^{2/3}$, for some sufficiently large constant c whose value will be determined later, and apply the following pruning process. As long as there exists a parabola $h^* \in C$ whose multiplicity is smaller than ν, we remove h^* from C, but do not remove any rotation incident to h^*. Similarly, as long as there exists a rotation $\tau \in P$ whose multiplicity is smaller than μ, we remove τ from P. Of course, these removals may reduce the multiplicity of some surviving rotations or parabolas, making additional rotations and parabolas eligible for removal. We keep repeating this step (without changing the initial thresholds ν and μ), until each of the surviving parabolas has multiplicity at least ν and each of the surviving rotations has multiplicity at least μ. We may assume that $\mu \geq 3$, by choosing c suficiently large and using Theorem 11(i).

When we delete a parabola h^*, we lose at most ν incidences with surviving rotations. When a rotation τ is removed, we lose at most μ incidences with surviving parabolas. All in all, we lose at most $n\nu + m\mu = 2cm^{1/3}n$ incidences, and are left with a subset P_1 of P and with a subset C_1 of C, so that each parabola of C_1 is incident to at least ν rotations of P_1, and each rotation of P_1 is incident to at least μ parabolas of C_1 (these subsets might be empty). Put $n_1 = |C_1|$ and $m_1 = |P_1|$. We have $I \leq I(P_1, C_1) + 2cm^{1/3}n$, so it remains to bound $I(P_1, C_1)$, which we do as follows.

We fix some sufficiently small positive parameter $t < 1$, and construct a random sample $P_1^s \subset P_1$ by choosing each rotation of P_1 independently with probability t. The expected size of P_1^s is $m_1 t$, and the expected number of points of P_1^s on any parabola of C_1 is at least $\nu t = ctm^{1/3}$. Chernoff's bound implies that, with positive probability, $|P_1^s| \leq 2m_1 t$, and $|P_1^s \cap h^*| \geq \frac{1}{2}ctm^{1/3}$ for every $h^* \in C_1$, and we assume that P_1^s does satisfy all these inequalities. (For the bound to apply, m_1 (and m) must be at least some sufficiently large constant; if this is not the case, we turn the trivial bound $m_1 n$ (or mn) on I into the bound $O(m_1^{1/3}n)$ (or $O(m^{1/3}n)$) by choosing the constant of proportionality sufficiently large.)

Construct, using Proposition 8, a nontrivial square-free trivariate polynomial p which vanishes on P_1^s, whose degree is at most the smallest integer d satisfying $\binom{d+3}{3} \geq 2tm_1 + 1$, so

$$d \leq \lceil (12tm_1)^{1/3} \rceil \leq 3t^{1/3} m_1^{1/3},$$

assuming (as above) that m_1 is sufficiently large.

Choosing c to be large enough, we may assume that $\nu t > 18d$. (This will hold if we ensure that $ct > 54t^{1/3}$.) This implies that p vanishes at more than $9d$ points on each parabola $h^* \in C_1$, and therefore it vanishes identically on each of these parabolas.

As in the previous analysis, we factor out the special polynomial factors of p, obtaining a square-free polynomial $\tilde{p}$, of degree at most d, with no special polynomial factors. Let $\Xi_1, \ldots, \Xi_k$ denote the special surfaces on which p vanishes identically (the zero sets of the special polynomial factors of p), for some $k \leq d/3$.

Let $C_2 \subseteq C_1$ (resp., $P_2 \subseteq P_1$) denote the set of those parabolas of C_1 (resp., rotations of P_1) which are not contained in any of the vanishing surfaces Ξ_i. Put $C_2' = C_1 \setminus C_2$ and $P_2' = P_1 \setminus P_2$.

For each parabola $h^* \in C_2$, $\tilde{p}$ vanishes identically on h^*, and, as argued in the proof of Theorem 11, at most d rotations of $P_1 \cap h^*$ lie in the surfaces Ξ_i. Hence, h^* contains more than $8d$ rotations of P_2, and, arguing as in the preceding proof, each of these rotations is either critical or flat for $\tilde{p}$. Hence, either more than $2d$ of these rotations are critical, and then h^* is a critical parabola for $\tilde{p}$, or more than $6d$ of these rotations are flat, and then h^* is a flat parabola for $\tilde{p}$. Applying Propositions 6 and 10, the overall number of parabolas in C_2 is therefore at most

$$d(d-1) + d(3d-4) < 4d^2.$$

We now apply Theorem 14 to C_2 and P_2, with the bound $4d^2$ on the size of C_2. Arguing as before, the conditions of this theorem are easily seen to hold for these sets. Theorem 14 then implies that the number of incidences between P_2 and C_2, which is also equal to the number of incidences between P_2 and C_1, is

$$I(P_2, C_1) = I(P_2, C_2) = O((4d^2)^{3/2}) = O(d^3) = O(m) .$$

Moreover, since each parabola of C_2 contains at least eight times more rotations of P_2 than of P_2', this bound also applies to the number of incidences between P_2' and C_2.

It therefore remains to bound the number of incidences between P_2' and C_2', namely, between the rotations and parabolas contained in the vanishing special surfaces Ξ_i. To do so, we iterate over the surfaces, say, in the order $\Xi_1, \ldots, \Xi_k$. For each surface Ξ_i in turn, we process the rotations and parabolas contained in Ξ_i and then remove them from further processing on subsequent surfaces.

Let us then consider a special surface Ξ_i. Let m_i and n_i denote respectively the number of rotations and parabolas contained in Ξ_i, which were not yet removed when processing previous surfaces. The number of incidences between these rotations and parabolas can be bounded by the classical Szemerédi-Trotter incidence bound [20] (see also (2)), which is $O(m_i^{2/3} n_i^{2/3} + m_i + n_i)$. Summing these bounds over all the special surfaces Ξ_i, and using Hölder's inequality and the fact, established in Lemma 5, that $n_i \le s$, we get an overall bound of

$$O\left(s^{1/3} \sum_i m_i^{2/3} n_i^{1/3} + \sum_i (m_i + n_i)\right) =$$
$$O\left(m^{2/3} n^{1/3} s^{1/3} + m + n\right),$$

where we use the facts that $\sum_i m_i \le m$ and $\sum_i n_i \le n$, which follow since in this analysis each parabola and rotation is processed at most once. The two linear terms satisfy $n = O(m^{1/3} n)$ (the bound obtained in the pruning process), and $m = O(m^{2/3} n^{1/3} s^{1/3})$ since $m = O(ns)$; see Remark (2) following Theorem 11.

We are not done yet, because each rotation of P_2' is processed only once, within the first surface Ξ_i containing it. This, however, can be handled as in [6]. That is, let τ be a rotation which was processed within the first surface Ξ_i containing it. Suppose that τ also lies on some later surface Ξ_j, with $j > i$, and let h^* be a parabola contained in Ξ_j, which has not been removed yet; in particular, h^* is not contained in Ξ_i, and thus meets it transversally, so the incidence between h^* and τ can be regarded as one of the transversal incidences in Ξ_i, which we have been ignoring so far. To count them, we simply recall that each parabola, whether of C_2' or of C_2, has at most three transversal intersections with a surface Ξ_i, for a total of at most d crossings with all the vanishing surfaces. Since each of these parabolas contains at least $9d$ rotations of P_1, those "transversal incidences" are only a fraction of the total number of incidences, and we simply ignore them altogether.

To recap, we obtain the following bound on the number of incidences between P_1 and C_1:

$$I(P_1, C_1) = O\left(m^{1/3} n + m^{2/3} n^{1/3} s^{1/3}\right).$$

Adding the bound $2cm^{1/3}n$ on the incidences lost during the pruning process, we get the asserted bound. □

5. FURTHER IMPROVEMENTS

In this section we further improve the bound in Theorem 13 using more standard space decomposition techniques. We show:

THEOREM 15. *The number of incidences between m arbitrary rotations and n h-parabolas, defined for a planar ground set with s points, is*

$$O^*\left(m^{5/12} n^{5/6} s^{1/12} + m^{2/3} n^{1/3} s^{1/3} + n\right),$$

where the $O^*(\cdot)$ notation hides polylogarithmic factors. In particular, when all $n = s^2$ h-parabolas are considered, the bound is

$$O^*\left(m^{5/12} s^{7/4} + s^2\right).$$

Brief sketch of the proof: We dualize the problem, mapping each h-parabola into a point in parametric 4-space, so that each rotation becomes a 2-plane. We project the dual points and planes onto some generic 3-space, and bound the number of incidences between these n points and m planes. We do this using a $(1/r)$-cutting of the arrangement of these planes, for an appropriate parameter r, and use the bound of Theorem 13 within each cell of the cutting. We need to pay special attention to situations where many points lie on a line which is contained in many planes (which is always a problematic issue in analyzing point-plane incidences). Fortunately, the special geometric structure in our setup allows us to control this situation, and get the bound asserted in the theorem. See [7] for full details. □

COROLLARY 16. *Let C be a set of n h-parabolas and P a set of rotations, with respect to a planar ground set S of s points. Then, for any $k \ge 3$, the number $M_{\ge k}$ of rotations of P incident to at least k parabolas of C satisfies*

$$M_{\ge k} = O^*\left(\frac{n^{10/7} s^{1/7}}{k^{12/7}} + \frac{ns}{k^3} + \frac{n}{k}\right).$$

For $n = s^2$, the bound becomes

$$M_{\ge k} = O^*\left(\frac{s^3}{k^{12/7}}\right).$$

Proof: We have $I(P, C) \ge k M_{\ge k}$. Comparing this bound with the upper bounds in Theorem 15 yields the asserted bounds. □

6. CONCLUSION

In this paper we have reduced the problem of obtaining a near-linear lower bound for the number of distinct distances in the plane to a problem involving incidences between points and a special class of parabolas (or helices) in three dimensions. We have made significant progress in obtaining upper bounds for the number of such incidences, but we are still short of tightening these bounds to meet the conjectures on these bounds made in the introduction.

To see how far we still have to go, consider the bound in Corollary 16, for the case $n = s^2$, which then becomes $O^*(s^3/k^{12/7})$. (Here $M_{\ge k}$ coincides with $N_{\ge k}$ as defined in (H3).) Moreover, we also have the Szemerédi-Trotter bound $O(s^4/k^3)$, which is smaller than the previous bound for $k \ge s^{7/9}$. Substituting these bounds in the analysis of (H3) and (H4), we get

$$\frac{[s(s-1) - x]^2}{x} \le |K| =$$
$$N_{\ge 2} + \sum_{k \ge 3} (k-1) N_{\ge k} =$$
$$N_{\ge 2} + O(s^3) \cdot \left[1 + \sum_{k=3}^{s^{7/9}} \frac{1}{k^{5/7}} + \sum_{k > s^{7/9}} \frac{s}{k^2}\right] =$$
$$N_{\ge 2} + O(s^{29/9}).$$

It is fairly easy to show that $N_{\geq 2}$ is $O(s^{10/3})$, by noting that $N_{\geq 2}$ can be upper bounded by $O\left(\sum_i |E_i|^2\right)$, where E_i is as defined in (H1). Using the upper bound $|E_i| = O(s^{4/3})$ [18], we get

$$N_{\geq 2} = O\left(\sum_i |E_i|^2\right) = O(s^{4/3}) \cdot O\left(\sum_i |E_i|\right) = O(s^{10/3}).$$

Thus, at the moment, $N_{\geq 2}$ is the bottleneck in the above bound, and we only get the (weak) lower bound $\Omega(s^{2/3})$ on the number of distinct distances. Showing that $N_{\geq 2} = O(s^{29/9})$ too (hopefully, a rather modest goal) would improve the lower bound to $\Omega(s^{7/9})$, still a rather weak lower bound.

Nevertheless, we feel that the reduction to incidences in three dimensions is fruitful, because
(i) It sheds new light on the geometry of planar point sets, related to the distinct distances problem.
(ii) It gave us a new, and considerably more involved setup in which the new algebraic technique of Guth and Katz could be applied. As such, the analysis in this paper might prove useful for obtaining improved incidence bounds for points and other classes of curves in three dimensions. The case of points and circles is an immediate next challenge.

Another comment is in order. Our work can be regarded as a special variant of the complex version of the Szemerédi-Trotter theorem on point-line incidences [20]. In the complex plane, the equation of a line (in complex notation) is $w = pz + q$. Interpreting this equation as a transformation of the real plane, we get a *homothetic map*, i.e., a rigid motion followed by a scaling. We can therefore rephrase the complex version of the Szemerédi-Trotter theorem as follows. We are given a set P of m pairs of points in the (real) plane, and a set M of n homothetic maps, and we seek an upper bound on the number of times a map $\tau \in M$ and a pair $(a, b) \in P$ "coincide", in the sense that $\tau(a) = b$. In our work we only consider "complex lines" whose "slope" p has absolute value 1 (these are our rotations), and the set P is simply $S \times S$.

The main open problems raised by this work are:
(a) Obtain a cubic upper bound for the number of rotations which map only two points of the given ground planar set S to another pair of points of S. Any upper bound smaller than $O(s^{3.1358})$ would already be a significant step towards improving the current lower bound of $\Omega(s^{0.8641})$ on distinct distances [11].
(b) Improve further the upper bound on the number of incidences between rotations and h-parabolas. Ideally, establish Conjectures 1 and 2.

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