

The Maximum Number of Hamiltonian Paths in Tournaments

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Abstract

Solving an old conjecture of Szele we show that the maximum number of directed Hamiltonian paths in a tournament on n vertices is at most $c \cdot n^{3/2} \cdot \frac{n!}{2^{n-1}}$, where c is a positive constant independent of n .

1 Introduction

A *tournament* T is an oriented complete graph. A *Hamiltonian path* in T is a spanning directed path in it. Let $P(T)$ denote the number of Hamiltonian paths in T . For $n \geq 2$, define $P(n) = \max P(T)$, where T ranges over all tournaments on n vertices. More than forty years ago, Szele [9] showed that

$$P(n) \geq n!/2^{n-1}. \quad (1)$$

His proof is considered to be the first application of the probabilistic method in combinatorics, and is thus mentioned in the beginning of most books and survey-articles on this method, (see, e.g., [3], pp. 11-12, [8], pp. 7-8). This proof is extremely simple; one just has to observe that by linearity of expectation the expected number of Hamiltonian paths in a random tournament on n vertices is $n!/2^{n-1}$. In the same paper, Szele also established an upper bound for $P(n)$ by proving that

$$P(n) \leq c_1 n!/2^{\frac{3}{4}n} \quad (2)$$

where c_1 is a positive constant independent of n .

Notice that the gap between the upper and lower bounds provided by (1) and (2) which have both not been improved since 1943 is exponential in n . In the present note we improve the upper bound and prove the following theorem.

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Theorem 1.1. *The maximum number, $P(n)$, of Hamiltonian paths in a tournament on n vertices satisfies*

$$P(n) \leq c_2 \cdot n^{3/2} \frac{n!}{2^{n-1}} \quad (3)$$

where $c_2 > 0$ is independent of n .

Therefore, $P(n)$ does not exceed the average number of Hamiltonian paths in a tournament on n vertices by more than a small polynomial factor (in n). In particular, (1) and (3) together with Stirling's formula imply that

$$\lim_{n \rightarrow \infty} (P(n)/n!)^{1/n} = \frac{1}{2}.$$

This equality was conjectured by Szele in 1943 [9], see also [6], page 28.

Our short proof of Theorem 1.1 is based on Minc's Conjecture [5] (proved by Brégman [1]), that supplies an upper bound for permanents of $(0,1)$ -matrices. This proof is presented in Section 2. Section 3 contains some concluding remarks and open problems.

2 Permanents and Hamiltonian paths and cycles

A *Hamiltonian cycle* in a tournament T is a spanning directed cycle in T . A *1-factor* of T is a spanning subgraph of T in which every indegree and every outdegree is 1, i.e., a spanning union of vertex disjoint directed cycles. Let $C(T)$ and $F(T)$ denote the number of Hamiltonian cycles and the number of 1-factors of T , respectively. Since every Hamiltonian cycle is also a 1-factor,

$$C(T) \leq F(T) \quad (4)$$

Define $C(n) = \max\{C(T) : T \text{ is a tournament on } n \text{ vertices}\}$ and $F(n) = \max\{F(T) : T \text{ is a tournament on } n \text{ vertices}\}$. By (4)

$$C(n) \leq F(n) \quad (5)$$

The *adjacency-matrix* $A = A_T$ of a tournament $T = (V, E)$ on the set of vertices $V = \{1, 2, \dots, n\}$ is the n by n $(0,1)$ -matrix $A = (a_{ij})$ defined by $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise.

It is easy to check that the permanent $\text{Per } A_T$ of A_T satisfies

$$\text{Per } A_T = F(T) \quad (6)$$

Inequalities 4 and 6 enable us to apply any known upper bound for permanents of $(0,1)$ -matrices to obtain an upper bound for the number of Hamiltonian cycles in tournaments. The following upper bound for permanents was conjectured by Minc [5] and proved by Brégman [1]. See also [7] for an extremely simple (and clever) proof.

Lemma 2.1. *[(Minc's conjecture)] Let A be an n by n $(0,1)$ -matrix, and let r_i denote the number of ones in row i of A , $1 \leq i \leq n$. Then*

$$\text{Per } A \leq \prod_{i=1}^n (r_i!)^{1/r_i} .$$

We also need the following simple but somewhat technical lemma.

Lemma 2.2. *For every two integers a, b satisfying $b \geq a + 2 > a \geq 1$ the inequality*

$$(a!)^{1/a} \cdot (b!)^{1/b} < ((a+1)!)^{1/(a+1)} ((b-1)!)^{1/(b-1)}$$

holds.

Proof. Define $f(a) = (a!)^{1/a} / ((a+1)!)^{1/(a+1)}$. Then the assertion is simply that $f(a) < f(b-1)$. Since $b-1 > a$ it suffices to show that for every integer $x \geq 2$, $f(x-1) < f(x)$. Substituting the expressions for f and raising both sides to the power $x(x-1)(x+1)$ we conclude that it suffices to show that for $x \geq 2$

$$((x-1)!)^{x(x+1)} \cdot ((x+1)!)^{x(x-1)} < (x!)^{2(x^2-1)} ,$$

i.e.,

$$(x!)^2 (x+1)^{x^2-x} < x^{x^2+x}$$

or

$$\left(\frac{x^x}{x!}\right)^2 > \left(\frac{x+1}{x}\right)^{x(x-1)} .$$

The last inequality holds for $x = 2$, since $2^2 > (3/2)^2$. For $x \geq 3$, we use the facts that $4^x > e^{x+1}$ and that $x! < (\frac{x+1}{2})^x$ to conclude that

$$\left(\frac{x^x}{x!}\right)^2 > \frac{2^{2x}}{(\frac{x+1}{2})^{2x}} > \frac{4^x}{e^2} > \frac{e^{x+1}}{e^2} = e^{x-1} > \left[\frac{x+1}{x}\right]^{x(x-1)} .$$

This completes the proof. □

Corollary 2.3. *Define $g(x) = (x!)^{1/x}$. For every integer $S \geq n$ the maximum of the function $\prod_{i=1}^n g(x_i)$ subject to the constraints $\sum_{i=1}^n x_i = S$ and $x_i \geq 1$ are integers, is obtained iff the variables x_i are as equal as possible, (i.e. iff each x_i is either $\lfloor S/n \rfloor$ or $\lceil S/n \rceil$).*

Proof. If there are i and j such that $x_i \geq x_j + 2$ then, by Lemma 2.2, if $x'_i = x_i - 1$, $x'_j = x_j + 1$ and $x'_e = x_e$ for all $e \neq i, j$ then $\prod_{i=1}^n g(x_i) < \prod_{i=1}^n g(x'_i)$. The desired result follows. □

Returning to tournaments, let T be a tournament on the $n \geq 3$ vertices $\{1, 2 \dots n\}$ and let $A = A_T = (a_{ij})$ be its adjacency matrix. Let r_i be the number of ones in row i of A , i.e., the outdegree of the vertex i in T . Clearly $\sum_{i=1}^n r_i = \binom{n}{2} \geq n$ and by (6), $F(T) = \text{Per } A$. If at least one r_i is 0 then $F(T) = \text{Per } A = 0$. Otherwise, by Lemma 2.1 and Corollary 2.3, $F(T)$ is at most the value of the function $\prod_{i=1}^n (r_i!)^{1/r_i}$, where the integral variables r_i satisfy $\sum_{i=1}^n r_i = \binom{n}{2}$, and they are as equal as possible. For odd n , $n = 2k + 1$, this value is

$$(k!)^{\frac{2k+1}{k}} = (1 + o(1)) \frac{k}{e} \cdot (k!)^2 = (1 + o(1)) \frac{k}{e} \frac{(2k)!}{2^{2k}} \sqrt{\pi k} = (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2}e} n^{3/2} \frac{(n-1)!}{2^n}.$$

For even n , $n = 2k$, this value is

$$\begin{aligned} (k!)^{k/k} \cdot (k-1)!^{k/(k-1)} &= (1 + o(1)) \left(\frac{1}{e}\right) (k!)^2 = (1 + o(1)) \frac{1}{e} \cdot \frac{(2k)!}{2^{2k}} \sqrt{\pi k} \\ &= (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2}e} n^{3/2} \frac{(n-1)!}{2^n}. \end{aligned}$$

(In both cases we applied Stirling's formula, which gives

$$(k!)^{1/k} = (1 + o(1)) \frac{k}{e} \quad \text{and} \quad \frac{(2k)!}{(k!)^2} = (1 + o(1)) \frac{2^{2k}}{\sqrt{\pi k}}.)$$

We have thus proved the following theorem, which supplies an upper bound for $F(n)$ and $C(n)$.

Theorem 2.4. *For every tournament T on n vertices*

$$C(T) \leq F(T) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2}e} n^{3/2} \frac{(n-1)!}{2^n}.$$

Notice that $\frac{(n-1)!}{2^n}$ is the expected number of Hamiltonian cycles in a random tournament on n vertices and hence the result above shows that the maximum possible number of Hamiltonian cycles is not very far from the average one.

It is not too difficult to deduce Theorem 1.1, which gives an upper bound for the number of Hamiltonian paths in tournaments, from Theorem 2.4. This is done in the following proposition.

Proposition 2.5. *For every tournament $T = (V, E)$ on n vertices there is a tournament T' on $n + 1$ vertices satisfying;*

$$C(T') \geq P(T) \cdot \frac{\lfloor n^2/4 \rfloor}{2^{\binom{n}{2}}} \geq \frac{1}{4} P(T).$$

Therefore

$$P(n) \leq 4C(n+1) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2}e} n^{3/2} \frac{n!}{2^{n-1}}.$$

Proof. Let T' be the random tournament obtained from T by adding to T a new vertex y , choosing a random subset $V_1 \subset V$ of cardinality $|V_1| = \lfloor n/2 \rfloor$ and adding the directed edges $\{(y, v_1) : v_1 \in V_1\} \cup \{(v_2, y) : v_2 \in V \setminus V_1\}$. The expected number of Hamiltonian paths in T whose first vertex lies in V_1 and whose last vertex lies in $V \setminus V_1$ is $\frac{\lfloor n^2/4 \rfloor}{2 \binom{n}{2}} \cdot P(T)$, and each such path corresponds to a Hamiltonian cycle in T' . Thus, the expected number of Hamiltonian cycles in T' is $P(T) \cdot \frac{\lfloor n^2/4 \rfloor}{2 \binom{n}{2}}$, and the existence of the desired T' follows. $\square$

3 Concluding remarks and open problems

- By Theorem 2.4 the maximum number $F(n)$ of 1-factors in a tournament on n vertices is at most $O(\sqrt{n} \frac{n!}{2^n})$. Note that the expected number of 1-factors in such a tournament is not much smaller, as it is $(e^{-3/2} + o(1))n!/2^n$ (where the $e^{-3/2} + o(1)$ factor appears since we are counting only the permutations which have no cycles of length 1 or 2). It is worth noting that *every* regular tournament, i.e., every tournament in which each indegree and each outdegree is $(n-1)/2$, where n is odd, has a slightly larger number of 1-factors. This is because if T is such a tournament then A_T has precisely $\frac{n-1}{2}$ ones in each row and each column. Thus $\frac{2}{n-1}A_T$ is doubly stochastic and hence, by Van der Waerden's Conjecture (proved in [4] and [2])

$$F(T) = \text{Per}(A_T) \geq \left(\frac{n-1}{2}\right)^n \cdot \frac{n!}{n^n} = (1 + o(1)) \cdot \frac{1}{e} \cdot \frac{n!}{2^n}.$$

Thus T has at least as many 1-factors as $(1 + o(1)) \cdot \sqrt{e}$ times the average number of 1-factors in a tournament on n vertices.

- Szele's lower bound for the maximum number $P(n)$ of Hamiltonian paths in a tournament on n vertices, stated in inequality (1), is probabilistic. It seems interesting to describe explicitly tournaments with many Hamiltonian paths. Moser (cf. [6]) gave a construction of a tournament with more than $n!/3^n$ Hamiltonian paths. In fact, it is not too difficult to give an explicit construction of tournaments T_n on n vertices satisfying $P(n) \geq n!/(2 + o(1))^n$. Indeed, let T_n be the tournament on the set of vertices $Z_n = \{0, 1, \dots, n-1\}$, in which (i, j) is a directed edge for all $i, j \in Z_n$ satisfying $(i - j) \pmod{n} < n/2$. (For even n , orient the edges connecting i and $i + \frac{n}{2}$, $(0 \leq i < \frac{n}{2})$, arbitrarily). We can show that

$$P(T_n) \geq nC(T_n) > C(T_n) \geq n!/(2 + o(1))^n. \quad (7)$$

In particular, $\lim_{n \rightarrow \infty} (P(T_n)/n!)^{1/n} = \lim_{n \rightarrow \infty} (P(n)/n!)^{1/n} = \frac{1}{2}$. We omit the detailed proof of (7). It seems plausible that $C(T_n) = (1 + o(1))C(n)$, but at the moment we are unable to prove or disprove this statement.

- The gap between our upper and lower bounds for $P(n)$ is only $O(n^{3/2})$. It would be extremely interesting to close this gap and determine $P(n)$ up to a constant factor.

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