

Proof of $A \rightarrow A$



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5. $A \rightarrow A$

Proof of $A \rightarrow A$

1. $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
Ax2: $A \Rightarrow A, B \Rightarrow (A \rightarrow A), C \Rightarrow A$
2. $(A \rightarrow ((A \rightarrow A) \rightarrow A))$
3. $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
4. $(A \rightarrow (A \rightarrow A))$
5. $A \rightarrow A$

Proof of $A \rightarrow A$

1. $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
Ax2: $A \Rightarrow A, B \Rightarrow (A \rightarrow A), C \Rightarrow A$
2. $(A \rightarrow ((A \rightarrow A) \rightarrow A))$ **Ax1:** $B \Rightarrow (A \rightarrow A)$
3. $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
4. $(A \rightarrow (A \rightarrow A))$
5. $A \rightarrow A$

Proof of $A \rightarrow A$

1. $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
Ax2: $A \Rightarrow A, B \Rightarrow (A \rightarrow A), C \Rightarrow A$
2. $(A \rightarrow ((A \rightarrow A) \rightarrow A))$ **Ax1:** $B \Rightarrow (A \rightarrow A)$
3. $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ **1 and 2 MP**
4. $(A \rightarrow (A \rightarrow A))$
5. $A \rightarrow A$

Proof of $A \rightarrow A$

1. $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
Ax2: $A \Rightarrow A, B \Rightarrow (A \rightarrow A), C \Rightarrow A$
2. $(A \rightarrow ((A \rightarrow A) \rightarrow A))$ **Ax1:** $B \Rightarrow (A \rightarrow A)$
3. $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ **1 and 2 MP**
4. $(A \rightarrow (A \rightarrow A))$ **Ax1:** $B \Rightarrow A$
5. $A \rightarrow A$

Proof of $A \rightarrow A$

1. $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
Ax2: $A \Rightarrow A, B \Rightarrow (A \rightarrow A), C \Rightarrow A$
2. $(A \rightarrow ((A \rightarrow A) \rightarrow A))$ **Ax1:** $B \Rightarrow (A \rightarrow A)$
3. $((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ **1 and 2 MP**
4. $(A \rightarrow (A \rightarrow A))$ **Ax1:** $B \Rightarrow A$
5. $A \rightarrow A$ **3 and 4 MP**

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

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Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
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Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

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and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$

2. $B \rightarrow C$

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$
2. $B \rightarrow C$
3. A

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$
2. $B \rightarrow C$
3. A
4. B

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$
2. $B \rightarrow C$
3. A
4. B
5. C

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$ **Assumption**
2. $B \rightarrow C$
3. A
4. B
5. C

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$ **Assumption**
2. $B \rightarrow C$ **Assumption**
3. A
4. B
5. C

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$ Assumption
2. $B \rightarrow C$ Assumption
3. A Assumption
4. B
5. C

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$ Assumption
2. $B \rightarrow C$ Assumption
3. A Assumption
4. B 1 and 3 MP
5. C

Proof of $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

We show

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

and then by Deduction Theorem conclude
 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.

Proof of $A \rightarrow B, B \rightarrow C, A \vdash C$

1. $A \rightarrow B$ Assumption
2. $B \rightarrow C$ Assumption
3. A Assumption
4. B 1 and 3 MP
5. C 2 and 4 MP

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$



Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$
2. A
3. $(A \rightarrow (\neg B \rightarrow A))$
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$
5. $\neg B \rightarrow A$
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A
3. $(A \rightarrow (\neg B \rightarrow A))$
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$
5. $\neg B \rightarrow A$
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$
5. $\neg B \rightarrow A$
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$
5. $\neg B \rightarrow A$
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$ 1, 4 MP
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$ 1, 4 MP
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$ Ax3
8. $((\neg B \rightarrow A) \rightarrow B)$
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$ 1, 4 MP
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$ Ax3
8. $((\neg B \rightarrow A) \rightarrow B)$ 6, 7 MP
9. B

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$ 1, 4 MP
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$ Ax3
8. $((\neg B \rightarrow A) \rightarrow B)$ 6, 7 MP
9. B 5, 8 MP

Proof of $\vdash (\neg A \rightarrow (A \rightarrow B))$

We first prove $\neg A, A \vdash B$

1. $\neg A$ Assumption
2. A Assumption
3. $(A \rightarrow (\neg B \rightarrow A))$ Ax1
4. $(\neg A \rightarrow (\neg B \rightarrow \neg A))$ Ax1
5. $\neg B \rightarrow A$ 2, 3 MP
6. $\neg B \rightarrow \neg A$ 1, 4 MP
7. $(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$ Ax3
8. $((\neg B \rightarrow A) \rightarrow B)$ 6, 7 MP
9. B 5, 8 MP

Now applying twice the Deduction theorem we obtain
 $\vdash (\neg A \rightarrow (A \rightarrow B))$.