A Framework for Decidable Metrical Logics

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Abstract. We propose a framework for defining decidable temporal logics. It is strong enough to define in it all the decidable temporal logics that we found in the literature. We use as semantics the standard model of the positive real line and we use robust logical notions and techniques.

1 Introduction

Obviously the main model for the progression of time is the real line with its natural order and possibly some arithmetical operations. We shall use here the model R^+ of the non negative real numbers that has 0 as its starting moment. The main model for the evolving of a system in time is a function f from R^+ to a finite set of propositions (about the system) where $f(t)$ is the set of propositions that hold at the moment t. This is called a *signal*. Alternatively we may think of every proposition as a one place predicate $P(t)$ which holds or not at time t.

The main tool in Mathematical Logic that deals with signals is the well developed branch of the *Monadic Logic of Order (MLO)*. It is of course possible to code up signals in alternative ways like *Timed Sequences* (provided the signal does not vary to often); it may also be of some advantage to use MLO in some syntactically sugared ways like the different temporal logics; but when doing so one should keep eye contact with the central model and its main tools and be able to justify the deviation. Indeed the classical treatment of pure temporal logics in [5] does it within MLO.

In recent years much effort was put into building formalisms that extend temporal logics to deal with the length of intervals of time (see [3, 17] for surveys). These efforts were not fruitless but we feel that they suffer from the use of mutant models and from ignoring (except for [17, 6]) Monadic Logic. Many formalisms were suggested. Though they are called logics most of them employ ad hoc programming language and automata theory constructs and not robust logical notions, syntax and semantics.

We offer here a novel treatment that does everything within the standard model of real signals, and with standard monadic logic. We propose a logical framework for defining decidable temporal logics. The framework is an expressive decidable monadic logic. It is strong enough to define in it all the decidable temporal logics that we found in the literature. For example the connective $(X\ thenY)_1$ - "X and then Y will occur within one time unit" is straightforwardly formulated in this logic.

The framework has the additional benefit that the discussion applies uniformly to all signals and not only to signals with *finite variability*, i.e - signals that change only finitely often in any bounded interval of time. We benefit from the theory of

Mathematical Logic and all the expressiveness and decidability proofs use Ehrenfeucht's equivalences for MLO.

The work continues [7] where the basic framework to discuss metric extensions of MLO in the standard model was developed.

In Section 2, we shall define and discuss the following sequence of logics:

$$MLO \subseteq MLO_{timer} \subseteq L_1 \subseteq L_2 \subseteq MLO_1$$

MLO is the Monadic Logic of Order, MLO_1 is the Monadic Logic of Order with the $+1$ function. L_1 is a fragment of MLO_1 that uses the $+1$ function only in a naturally restricted way as defined in Section 2.2. L_1 is decidable by [7]. MLO_{timer} is a fragment of L_1 in which the use of the $+1$ is even more systematically restricted (the formulas in first-order timer normal form of Section 2.3). L_2 is introduced here for the first time (see Definition 2). L_2 is much more expressive as exemplified by example 5.1, but it is still decidable.

In Section 3 we recall the general notion of one dimensional temporal logic. In Section 4 we present the main technical theorems for defining decidable temporal logics. Section 5 provides a collection of examples.

It is not easy to compare our work to the extensive efforts made in the field as the models vary from each other. All the previous papers use as a model some kind of an ω-sequence. Such a sequence cannot faithfully represent the real line. Clearly signals with infinite variability cannot be modeled by a sequence. The popular logic $MITL$ of [2] is equivalent to our logic L_1 for finite variability models. A more detailed comparison can be found in [7].

2 Monadic Logic of Order and its Metrical Extensions

2.1 Monadic Logic of Order (MLO)

The syntax of MLO has in its vocabulary *individual (first order) variables* $t_0, t_1, \ldots$, *unary predicate names* $X_0, X_1, \ldots$, and one binary relation $<$ (the order). *Atomic formulas* are of the form $X(t)$, $t_1 < t_2$ and $t_1 = t_2$. *MLO formulas* are obtained from atomic formulas using the Boolean connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$. We will occasionally refer to *Second order* MLO formulas where the unary predicate names are considered as variables and second order quantifiers $\exists X$ and $\forall X$ are allowed. As usual if φ is a formula we may write $\varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$ to indicate that the free variables in φ are among $t_1, \ldots, t_k$ and $X_1, \ldots, X_m$.

A *(canonical real time) model* is the non negative real line R^+ with its natural order and some unary predicates. We shall not repeat the inductive definition saying when is a formula satisfied. Recall that in order to check if the formula $\varphi(t_1, \ldots t_k; X_1, \ldots, X_m)$ is true we need to specify what are the elements $\tau_1, \ldots, \tau_k$ in R^+ and the predicates (subsets) $P_1, \ldots, P_m$ over R^+ which are assigned to the variables $t_1, \ldots, t_k, X_1, \ldots, X_m$. Hence the notation will usually be

$$\langle R^+, \tau_1, \ldots, \tau_k; P_1, \ldots, P_m\rangle \models_{MLO} \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m)$$

which we also abbreviate to $R^+ \models \varphi[\tau_1, \ldots, \tau_k; P_1, \ldots, P_m]$ or even to $R^+ \models \varphi[\overline{\tau}, \overline{P}]$ where the bar denotes a tuple of appropriate length. When we define the semantics of a second order formula or when we deal with validity and satisfiability of a first order formula it is necessary to specify over which predicates should the variables X range. In *full MLO* they range over all unary predicates (i.e. – subsets). A requirement that is often imposed in the literature is that in every bounded time interval a system can change its state only finitely many times. This requirement is called *finite variability* (or non-Zeno) requirement. We consider also finite variability interpretations of first-order and second-order MLO. Under these interpretations monadic predicates range over predicates with finite variability.

2.2 Monadic Logic with a +1 Function - MLO_1

The syntax of MLO_1 extends the syntax of MLO by a one place function $t+1$. Its atomic formulas are those of MLO and formulas $t_i = t_j + 1$. Other formulas are constructed from the atomic formulas exactly as for MLO. The function $t+1$ is interpreted over the reals in the standard way.

We use the standard notation 0 for the minimal element, n for $0 + 1 + \cdots + 1$ (n times) and $t-1$ for 0 when $t < 1$ and for the unique t_1 such that $t_1 + 1 = t$ when $t \geq 1$.

MLO_1 is is too strong and the satisfiability problem can be easily shown to be undecidable (see e.g. [2, 7]). We shall, therefore, identify two decidable fragments of MLO_1.

These fragment will use the function $t+1$ only in a very restricted form. We introduce some syntactical sugar to MLO_1 – the "bounded quantifiers" $(\exists t)^{<t_0+1}_{>t_0}$ and $(\exists t)^{<t_0}_{>t_0-1}$ as follows: if φ is a formula of MLO_1 then we use the shorthand:

$$(\exists t)^{<t_0+1}_{>t_0}\varphi \equiv \exists t(t_0 < t < t_0 + 1 \wedge \varphi)$$

$$(\exists t)^{<t_0}_{>t_0-1}\varphi \equiv \exists t(t_0 - 1 < t < t_0 \wedge \varphi)$$

More general bounded quantifiers $(\exists t)^{<t_0+n}_{>t_0+m}$ for integers n and m are defined similarly: $(\exists t)^{<t_0+n}_{>t_0+m}\varphi \equiv \exists t(t_0 + m < t < t_0 + n \wedge \varphi)$. The quantifiers with weak inequality replacing the strict inequality in one or both ends of the interval with integers $m < n$ are defined similarly.

Definition 1 (The fragment L_1) *L_1 is the fragment of* MLO_1 *which is built from the atomic formulas $t_1 < t_2, t_1 = t_2, X(t)$ using Boolean connectives, first order quantifiers and the following rule:*

if $\varphi(t)$ is a formula of L_1 with t its only first order free variable, m and n integers, and $m < n$, then the formulas $(\exists t)^{<t_0+n}_{>t_0+m}\varphi$, $(\exists t)^{<t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{>t_0+m}\varphi$ are in L_1.

Definition 2 (The fragment L_2) *L_2 is the fragment of* MLO_1 *which is built from the atomic formulas $t_1 < t_2, t_1 = t_2, X(t)$ using Boolean connectives, first order quantifiers and the following rule:*

if $\varphi(t_0, t)$ is a formula of L_2 with t_0, t its only first order free variables, m and n integers, and $m < n$ then the formulas $(\exists t)^{<t_0+n}_{>t_0+m}\varphi$, $(\exists t)^{<t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{\geq t_0+m}\varphi$, $(\exists t)^{\leq t_0+n}_{>t_0+m}\varphi$ are in L_2.

Note that the only difference between L_2 and L_1 is that after a bounded quantifier in L_1 only a formula with one free variable may appear, whereas in L_2 it can contain two variables one of which must appear in the bounds of the quantifier. L_2 is much more expressive than L_1. Note that if we allow another free variable t_1 in $(\exists t)^{\leq t_0+1}_{>t_0}\varphi(t_0, t, t_1)$ then $t_1 = t_0 + 1$ is expressible and we obtain the full power of MLO_1

2.3 Decidability

Notation. For every n we define the formula:

$$\mathrm{Timer}_n(X_1, \ldots X_n, Y_1, \ldots, Y_n) \equiv \bigwedge_i \forall t(Y_i(t) \longleftrightarrow (\forall t_1)^{<t}_{>t-1} X_i(t_1))$$

i.e. each Y_i is a timer that measures if X_i persisted for at least one unit of time.

A formula is said to be in first (second) order **timer normal form** if it has the form

$$\exists \bar{Z}.\ \mathrm{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \wedge \phi,$$

where ϕ is a first (second) order monadic formula and $\bar{Z}$ is a list of monadic variables that may contain $X_1, \ldots, X_n, Y_1, \ldots, Y_n$; all these monadic variables may occur in ϕ.

Theorem 3. *[7]*

1. *The satisfiability of the formulas in first-order timer normal form is decidable.*
2. *The satisfiability of the formulas in the second-order timed normal form under the finite variability semantics is decidable.*

Note: Unless explicitly otherwise stated as in Theorem 3(2) all the decidability and expressiveness results in this paper, like Theorem 3(1), hold both for the unrestricted canonical semantics and for finite variability canonical semantics. This is so since the property "X has finite variability" is expressible in pure first-order monadic logic [10], and because the transformations used in the analysis of the expressiveness apply uniformly to both semantics.

We will show (see Section 4)

Theorem 4. *The satisfiability and validity problems for L_2 formulas are decidable.*

This Theorem extends the result of [7], where we have proved that L_1 is decidable.

3 Temporal Logic (TL) and its Metric Extensions

The syntax of Temporal Logic (TL) has in its vocabulary *Predicate variables* $X_1, X_2, \ldots$ and some *(modal) connective names* with prescribed arity $O_1^{(k_1)} \ldots O_n^{(k_n)}$ (the arity notation is usually omitted). For example the "sometime in the future" connective $\diamondsuit$ has arity 1. The "_until_" connective has arity 2.

Atomic formulas are just variables X_i and *temporal formulas* are obtained from the atoms using Boolean connectives and applying the modal connectives: if $\varphi_1, \ldots, \varphi_{k_i}$ are temporal formulas then so is $O^{(k_i)}(\varphi_1, \ldots, \varphi_{k_i})$. As usual we write φ *until* ψ instead of *until* (φ, ψ).

Every connective $O^{(k)}$ is interpreted in R^+ as an operator $O_M^{(k)} : [P(R^+)]^k \to P(R^+)$ which assigns "the set of points where $O[P_1, \ldots, P_k]$ holds" to the k-tuple $\langle P_1, \ldots, P_k \rangle$. In the usual way an operator OP_φ is assigned to every temporal formula φ by structural induction. Satisfaction relation $\models$ is defined as usual

$$(R^+,\ P_1, \ldots, P_n, \tau) \models \varphi \text{ iff } \tau \in OP_\varphi(P_1, \ldots, P_n)$$

Let φ be a formula of some logic L. Assume that $\varphi(Y, X_1, \ldots, X_n)$ has the property that for every $P_1, \ldots, P_n$ there is a unique Q such that $R^+ \models \varphi(Q, P_1, \ldots, P_n)$; then the connective O is **defined** by φ if $Q = O_M(P_1, \ldots, P_n) \Longleftrightarrow R^+ \models \varphi(Q, P_1, \ldots, P_n)$.

Assume that for every connective $O^{(k)}$ there is a formula $\overline{O}(t_0, X_1, \ldots, X_k)$ of some logic L such that:

$$O_M^{(k)}(P_1, \ldots, P_k) = \{\tau \mid R^+ \models \overline{O}[\tau; P_1, \ldots, P_k]\} \ .$$

This formula is called a *truth table* of $O^{(k)}$. For example the connective $\diamondminus X$, "X has happened before" is defined by the truth table $\varphi(t_0, X) \equiv \exists t < t_0.\ X(t)$.

It is clear that if $\varphi(t_0, X)$ is a truth table of an operator O then O is defined by the formula $\forall t.\ Y(t) \leftrightarrow \varphi(t, X)$.

Definition 5 *We say that a temporal logic T is expressively complete with respect to a fragment L of first order logic iff (1) every formula of T has a truth table in L and (2) every formula of L with one free variable is a truth table for a formula of T.*

3.1 Propositional Temporal Logic (PTL)

Propositional temporal logics are temporal logics with connectives defined in MLO. There are such logics which are expressively complete with respect to MLO. In particular we shall denote by PTL (Propositional Temporal Logic) the temporal logic that was proven expressively complete in [8], with the two connectives X *until* Y and X *since* Y.

- The modality X *until* Y has a truth table
  $\psi(t_0, X, Y) \equiv \exists t_1(t_0 < t_1 \wedge Y(t_1) \wedge \forall t(t_0 < t < t_1 \to X(t)))$.
- The modality X *since* Y has a truth table
  $\psi(t_0, X, Y) \equiv \exists t_1(t_0 > t_1 \wedge Y(t_1) \wedge \forall t(t_1 < t < t_0 \to X(t)))$.

3.2 Quantitative Temporal Logic (QTL)

We identified in [7] a temporal logic which is expressively complete for the Monadic Logic L_1: QTL is the temporal logic constructed from an expressively complete set of modalities for MLO *and* two new modalities $\diamondsuit_1 X$ - "X will happen within one unit of time" and $\diamondminus_1 X$ - "X happened during the previous unit of time" defined by the tables (in t_0): $\diamondsuit_1 X : \quad (\exists t)_{>t_0}^{<t_0+1} X(t)$ and $\diamondminus_1 X : \quad (\exists t)_{>t_0-1}^{<t_0} X(t)$.

Proposition 6 *[7] QTL is expressively complete for the Monadic Logic L_1 and is decidable.*

It is clear that $\diamondsuit_1 X$ and $\overleftarrow{\diamondsuit}_1 X$ are the most basic metrical modalities. They are expressible in all formalisms for reasoning about real time, in particular in the popular MITL [2]. MITL was interpreted on timed state sequences and trace interval sequences and was shown to be decidable in these models [2]. Trace interval sequences encode finite variability predicates and from the decidability of MITL on trace interval sequences it is easy to derive the decidability of QTL and MITL under finite variability interpretation. On the other hand MITL modalities are expressible in L_1 [7]. Therefore from the decidability of L_1 under arbitrary canonical interpretation it follows that MITL is also decidable under arbitrary canonical interpretation.

3.3 Stronger Decidable Temporal Logics

Stronger decidable temporal logics maybe obtained by adding to QTL some more temporal connectives which are defined in the Monadic Logic L_2. Some examples will be discussed in Section 5 but none seems to be as natural as the $\diamondsuit_1$ connective. We also conjecture that the Logic L_2 has no finite base, i.e there is no finite set of connectives defined in L_2 which is expressively complete, but this is the subject of future work.

Definition 7 *A formula $(\exists t)^{\prec t_0+n}_{\succ t_0+m}\phi$ of L_2 is called* unnested *if the formula ϕ is in pure MLO, $m < n$ and $\prec \in \{\leq, <\}$ and $\succ \in \{\geq, >\}$.*

Definition 8 *A logic T' is effectively more expressive than a logic T if for every formula p of T one can effectively construct a formula p' of T' such that p is equivalent to p'.*

Proposition 9 *Let T be a temporal logic. Assume that every connective has a table definable in L_2. Then there is an effectively more expressive temporal logic T' such that every connective of T' has a table definable by an unnested L_2 formula.*

Proposition 10 *Let L be a temporal logic. Assume that every connective has an operator that is defined by a formula in first-order (second-order) timer normal form. Then the operator of every L formula is effectively definable by a formula in first-order (respectively second-order) timer normal form.*

Note that Proposition 10 does not hold if we write everywhere "defined by a truth table" instead of "defined by a formula". This is the reason why we generalized the notion of truth table to that of a defining formula.

4 Main Technical Results

Theorem 11. *1. Let $\varphi(t_0, t)$ be a formula of first-order MLO and $m < n$ be integers. The operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0, t)$ is definable by a formula in first-order timer normal form, which is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.*

2. *Let $\varphi(t_0,\ t)$ be a formula of second-order MLO and $m < n$ be integers. The operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t)$ is definable by a formula ψ in second-order timer normal form, moreover, ψ is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.*
3. *Let $\varphi(t_0,\ t)$ be a formula of second-order MLO and $m < n$ be integers. For the semantics of finite variability predicates the operator with truth table $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t)$ is definable by a formula ψ in second-order timer normal form, moreover, ψ is effectively constructed from φ. Similarly for the tables with one or two weak inequalities replacing the strict ones.*

About the proof: Given an interval I and monadic predicates on I we denote by $Th^n(I)$ the set of all sentences with quantifier depth n that are true in I with these predicates. The key observation for (1) is that Th^n of a lexicographical sum of two intervals is definable from the Th^n theory of the summands [4]. For (2) and (3) we use a similar theorem for monadic second order logic [14].

Theorem 12. *1. Assume that every connective of a temporal logic T has a truth table of the form $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t)$, where $\varphi(t_0,\ t)$ is a formula of first-order MLO and $m < n$ are integers. Then T is decidable. Similarly for the tables with one or two weak inequalities replacing the strict ones.*

2. *Assume that every connective of a temporal logic T has a a truth table definable in L_2. Then T is decidable.*
3. *Assume that every connective of a temporal logic T has a truth table of the form $(\exists t)^{<t_0+n}_{>t_0+m}\varphi(t_0,\ t)$, where $\varphi(t_0,\ t)$ is a formula of second-order MLO and $m < n$ are integers. Then T is decidable under finite variability semantics. Similarly for the tables with one or two weak inequalities replacing the strict ones.*

Proof. (Sketch) (1) and (3) follows from Theorem 11, Proposition 10 and Theorem 3. (2) follows from (1) and Proposition 9.

Observe that satisfiability of L_2 formulas is effectively reducible to the satisfiability of L_2 formulas with one free variable. This observation together with Theorem 12 implies Theorem 4 (Section 2) that states that L_2 is decidable.

5 Decidable Extension of QTL

5.1 Example

We add to QTL a new temporal connective $(\diamond X_1, \diamond X_2)_1$: "$X_1$ and then X_2 will both happen within one unit of time". The table for this connective is given by the MLO_1 formula $\exists t_1 t_2.\ (t_0 < t_1 < t_2 < t_0 + 1 \wedge X_1(t_1) \wedge X_2(t_2))$. This can be easily reformulated as a L_2 formula: $(\exists t)^{<t_0+1}_{>t_0}.\exists t_1 t_2(t_0 < t_1 < t_2 < t \wedge X_1(t_1) \wedge X_2(t_2))$. Therefore the extended logic is decidable by Theorem 12. The following proposition shows that this is a *proper* extension of QTL

Proposition 13 *The connective $(\diamond X_1, \diamond X_2)$ is not expressible in QTL (or equivalently in $MITL$).*

Proof. (Sketch) Let $X(t)$ be a predicate that is true exactly at the points $n \times \frac{2}{3}$ for all natural numbers n. It can be shown by structural induction that every QTL formula concerning X is equivalent from some point t_0 on to either X or to $\neg X$ or to *true* or to *false*. On the other hand the formula $(\Diamond X, \Diamond X)_1$ does not have this property.

Similarly we can obtain a proper increasing hierarchy of connectives $(\Diamond X_1, \ldots, \Diamond X_n)_1$ which are all definable in L_2 by formulas

$$(\exists t)^{\leq t_0+1}_{>t_0}.\exists t_1 t_2 \ldots t_n (t_0 < t_1 < \ldots < t_n < t \wedge \bigwedge_i X_i(t_i)).$$

In contrast one can show

Proposition 14 *TL extended by the connective which states that the interval (t_0, t_0+1) can be partitioned into three subintervals such that X_1 holds on the first subinterval, X_2 holds on the second and X_3 holds on the third is undecidable.*

In the next subsection we provide some definitions about finite variability signals and thereafter define a variety of decidable connectives to illustrate the expressive power of L_2.

5.2 Finite Variability Signals and their Traces

A string $l_0 l_1 \ldots l_n$ is **stuttering free** iff it is non empty and no two adjacent symbols in it are equal.

Recall that a function η from a bounded subinterval $(a,\ b)$ of the reals into a finite set Σ has *finite variability* if there exists a finite increasing sequence $a = a_0 < a_1 \ldots < a_n = b$ such that η is constant on every interval (a_i, a_{i+1}).

A function f from R^+ to Σ has finite variability if for every bounded interval I the restriction of f to I has finite variability.

The following lemma is straightforward.

Lemma 15 *Suppose that $\eta : \ (a,\ b) \to \Sigma$ has finite variability, then there exists a unique increasing sequence $a = a_0 < a_1 < a_2 \ldots < a_n = b$ such that*

1. *η is almost constant on every interval $(a_i,\ a_{i+1})$, i.e., for every i there is $l_i \in \Sigma$ such that the set $\{x \in (a_i,\ a_{i+1})\ :\ \eta(x) \neq l_i\}$ is finite.*
2. *For every $i < n-1$, the value of η on (a_i, a_{i+1}) differs from the value of η on (a_{i+1}, a_{i+2}).*

Definition 16 *(Trace of a finite variability function.) Let η be a finite variability function over $(a,\ b)$ and let $a_0, \ldots, a_n$ be as in Lemma 15. For $i < n$ let l_i be the values of η on (a_i, a_{i+1}). The trace of η (denoted by $\mathbf{trace}(\eta)$) is the stuttering free string $l_0 l_1 \ldots l_{n-1}$.*

Trace of a tuple: Let $\langle \eta_1, \ldots, \eta_n \rangle$ be an n-tuple of finite variability functions from $(a,\ b)$ into $\{0,\ 1\}$. With this n-tuple we associate a function η from $(a,\ b)$ into $\{0, 1, \ldots, 2^n-1\}$ defined as $\eta(t) = i$ if $\langle \eta_1(t), \ldots, \eta_n(t) \rangle$ is the binary representation of i. The above mapping defines a one-to-one correspondence between the set of n-tuple of finite variability functions from $(a,\ b)$ into $\{0, 1\}$ and finite variability functions from $(a,\ b)$ into $\{0, \ldots, 2^n - 1\}$. The trace of an n-tuple (denoted by $\mathbf{trace}(\eta_1, \ldots, \eta_n)$) is defined as the trace of the corresponding function.

5.3 Automata Modality

Let R be a regular expression over the alphabet $\{0,\ 1\}^n$ and let $k < m$ be integers. Define the n-ary connective $R_{k,m}$ as follows: $R_{k,m}(X_1, \ldots, X_n)$ holds at t_0 if there is $t' \in (t_0 + k,\ t_0 + m)$ such that $\mathbf{trace}(Y_1, \ldots, Y_n) \in R$, where Y_i is the restriction of X_i to $(t_0,\ t')$.

Proposition 17 *The extension of QTL by the connectives $R_{k,m}$ is decidable for finite variability semantics.*

Proof. (Sketch) It is straightforward to show that for every regular expression R there is a formula $\phi_R(t_0,\ t, \bar{X})$ in pure monadic second order logic such that $\phi_R(t_0,\ t, \bar{X})$ holds iff $\mathbf{trace}(Y_1, \ldots, Y_n) \in R$, where Y_i is the restriction of X_i to $(t_0,\ t)$. Hence, by Theorem 12, the logic is decidable.

5.4 Decidable Modalities about a Whole Interval

Let s be a stuttering free string over the alphabet $\{0,\ 1\}^n$ and let $k < m$ be integers.

Define an n-ary connective s_m as follows: $s_m(X_1, \ldots, X_n)$ holds at t_0 if $\mathbf{trace}(Y_1, \ldots, Y_n) = s$, where Y_i is the restriction of X_i to $(t_0,\ t_0 + m)$.

Proposition 18 *The extension of QTL by the connectives s_m is decidable.*

Proof. (Sketch) We will show that the connective s_m is definable in L_2 and therefore, the extension is decidable. It is easy to write a a first order monadic formula $\phi_1(\bar{X}, t_0, t)$ which says that the trace of $\overline{X}$ on the interval $(t_0,\ t)$ is s; it is also easy to write a first order monadic formula $\phi_2(\bar{X}, t_0, t)$ which says that the trace of $\overline{X}$ on the interval $(t_0,\ t)$ has length greater than the length of s. The desirable formula is:

$$(\exists t)_{>t_0}^{<t_0+n} \phi_1 \wedge \neg(\exists t)_{>t_0}^{<t_0+n} \phi_2.$$

Observe that the s_1 connective specifies what will happen on the entire interval of length one and it is still decidable unlike the connectives discussed in Proposition 14.

5.5 Duration Calculus Connectives

The Duration Calculus (DC for brevity) [16] is a formalism for the specification of real time systems. The Duration Calculus is based on interval logic and uses real numbers to model time. DC has been successfully applied in case studies of software embedded systems, e.g., a gas burner [13], a railway crossing [15] and has been used to define the real time semantics of other languages.

DC formulas specify properties of intervals. For the lack of space we will not define the syntax and the semantics of DC. Let us note here that the basic semantical relation is whether a formula D holds on an interval I.

It was shown in [12, 11] that the propositional fragment of DC is a fragment of first-order monadic logic of order. Hence for every formula D of Propositional Duration Calculus and integers $k < m$ the following connectives are definable in L_2:

1. $D_{k,m}$ holds at t_0 iff there is $t \in (t_0 + k,\ t_0 + m)$ such that D holds on the interval (t_0, t).
2. D_k holds at t_0 iff there is $t \in (t_0,\ t_0 + k)$ such that D holds on the interval (t_0, t).
3. D_k^{sub} holds at t_0 iff there is a subinterval I of $(t_0,\ t_0 + k)$ such that D holds on I.

5.6 More Connectives

Note that all connectives mentioned above specify only non metrical properties of subintervals of $(t_0,\ t_0 + k)$. Connectives that specify metrical properties can be also easily formulated in L_2. For example, it is straightforward to say in L_2 that interval $(t_0,\ t_0+5)$ contains a subinterval of length at least one where X holds and a subinterval of length at least three where Y is dense.

A formula $\varphi(t_0, t)$ is fictious outside $(t_0,\ t)$ if it depends only on the values of X inside $(t_0,\ t)$. All connectives defined above used only such formulas inside bounded quantifiers. It is conceivable that interesting connectives may rise from the use of a formula $\varphi(t_0, t)$ which is not fictious outside $(t_0,\ t)$.

6 Conclusion

We developed a logical framework for defining decidable temporal logics. It is strong enough to define in it all the decidable temporal logics that we found in the literature. We use as semantics the standard model of the positive real line with all the signals or with finite variability signals.

In the survey [3] sixteen models for real time logics are provided (see Section 2.2 there). The canonical model and the finite variability canonical model are not among these models; the sixteen models are all ω-sequences either of points on the time line (*"state sequences"*) or of time intervals (*"interval trace sequences"*). These models do not faithfully reflect the real line. They were probably chosen in order to prove decidability by reduction to Alur-Dill timed automata [1] which in turn are reduced to ω-automata. Our proof of decidability is automata free and works for general predicates as well as finite variability predicates. A more detailed comparison of the canonical model to sequence models and of QTL to the other formalisms can be found in [7].

For finite variability models the complexity of the satisfiability problem for pure first-order monadic logic is non-elementary while it is PSPACE complete for pure TL. In [7] we described a polynomial reduction of QTL to pure temporal logic (as far as satisfiability is concerned). The proof uses pure logical equivalences which are valid in all canonical models. By a simple analysis of the complexity of the translation of $MITL$ into QTL we obtain easily the complexity results about $MITL$ [2] without tailoring special automata as in the previous proofs. The analysis of the complexity of the temporal logics with L_2 definable modalities is left as future endeavor.

It is easy to modify the theory to deal with more general time lines. For the rational line Rabin [9] proved that pure *second order* monadic logic is decidable and we obtain a stronger version of Theorem 12. For more general orders that do not have a natural $+1$ function the function should be replaced by a binary relation $R(t_1, t_2)$ satisfying some simple axioms that can be interpreted as expressing a notion of "t_2 is not too far ahead of t_1". A closer analysis will be needed to find out under what conditions on the order do we have Theorem 12.

Wilke [17] uses monadic logic to obtain stronger decidable logics. His main recipe for defining decidable connectives is like Proposition 17 but for the state sequence model. His work very much influenced ours both conceptually and technically.

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