The Expressive Power of Temporal and First-Order Metric Logics

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Abstract. The First-Order Monadic Logic of Order ($FO[<]$) is a prominent logic for the specification of properties of systems evolving in time. The celebrated result of Kamp [14] states that a temporal logic with just two modalities Until and Since has the same expressive power as $FO[<]$ over the standard discrete time of naturals and continuous time of reals. An influential consequence of Kamp's theorem is that this temporal logic has emerged as the canonical Linear Time Temporal Logic (LTL). Neither LTL nor $FO[<]$ can express over the reals properties like P holds exactly after one unit of time. Such local metric properties are easily expressible in $FO[<, +1]$ - the extension of $FO[<]$ by $+1$ function. Hirshfeld and Rabinovich [10] proved that no temporal logic with a finite set of modalities has the same expressive power as $FO[<, +1]$.
$FO[<, +1]$ lacks expressive power to specify a natural global metric property "the current moment is an integer." Surprisingly, we show that the extension of $FO[<, +1]$ by a monadic predicate "x is an integer" is equivalent to a temporal logic with a finite set of modalities.

1 Introduction

1.1 Temporal Logics and Kamp's Theorem

Temporal Logics were introduced to Computer Science by Pnueli in [18]. They provide a convenient framework for reasoning about "reactive" systems. This made temporal logics a popular subject in the Computer Science community, enjoying extensive research in the past 30 years.

In a temporal logic we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are expressed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1, \ldots, \varphi_k$ possibly on 'past' or 'future' points to a statement $M(\varphi_1, \ldots, \varphi_k)$ on the 'present' point. The rule to determine the truth of a statement $M(\varphi_1, \ldots, \varphi_k)$ is called a *truth table*. The choice of particular modalities with their truth tables yields different temporal logics.

A basic modality is $\Diamond$ - eventually: $\Diamond P$ says: "P holds some time in the future." It is formalized by a formula $\varphi(z_0, P) := (\exists z > z_0)P(z)$ with one free variable z_0 (for the current moment). This is a formula of the First-Order Monadic

Logic of Order ($FO[<]$) - a fundamental formalism in Mathematical Logic where formulas are built from atomic monadic formulas $P(z)$ and atomic order formulas $z_1 = z_2$, $z_1 < z_2$, by Boolean connectives and first-order quantifiers $\exists z$ and $\forall z$. Most modalities used in the literature are defined by such first-order truth tables, and, as a result, every temporal formula translates directly into an equivalent first-order formula. Thus, the different temporal logics may be considered a convenient way to present fragments of first-order logic. A first-order logic can also serve as a yardstick by which one can check the strength of temporal logics. A temporal logic is *expressively complete* for a fragment L of a predicate logic if every formula of L with a single free variable is equivalent to a temporal formula.

Actually, the notion of expressive completeness is with respect to the type of the underlying model since the question whether two formulas are equivalent depends on the domain over which they are evaluated. The standard linear time intended models are the Naturals $\langle \mathbb{N}, < \rangle$ for discrete time and the Reals $\langle \mathbb{R}, < \rangle$ for continuous time.

A major result concerning temporal logics is Kamp's theorem [14,5] which states that the temporal logic with two modalities "P Until Q" and "P Since Q" is expressively complete for $FO[<]$ over the above two linear time canonical models.

LTL (Linear Time Temporal Logic) is the temporal logic with two modalities Until and Since. An influential consequence of Kamp's result is that *LTL* has emerged as the canonical temporal logic.

1.2 Expressing Metrical properties

The choice between $FO[<]$ and *LTL* is merely a matter of personal preference, as far as only the expressive power is concerned. For discrete time these logics suffice. Properties like "Every P will be followed promptly enough by a Q" can be explicitly written once a number k is chosen, and "promptly enough" is interpreted as: "within k steps."

LTL and $FO[<]$ are expressively equivalent whether the system evolves in discrete or in continuous time. However, for continuous time both logics lack the power to express properties of the kind just described, and we must strengthen their expressive power.

Some measure of length of time needs to be included, and the language must be adapted to it. This is done by assuming that there is a basic unit of length; let's call it "length 1." For predicate logic it is a standard procedure to extend the language by a name for the "+1" function, or for a corresponding relation. It will then be the question which fragment of the extended language $FO[<, +1]$ suits our needs.

Burgess and Gurevich [4] proved that $FO[<]$ is decidable over the reals. Unfortunately, $FO[<, +1]$ is undecidable over the reals. Much research was carried out to find decidable temporal logics which can specify some metric properties. Extending temporal logic, without relating it to a corresponding predicate logic, has led to a veritable babel of metric temporal logics over the reals [15,3,2,16,5,21,1,7,8,9]. The most popular among decidable temporal logics is

MITL (Metric Interval Temporal Logic) introduced by Alur, Feder and Henzinger [1]. *MITL* uses infinitely many modalities. However, it has the same expressive power as QTL (Quantitative Temporal Logic [9]), which has besides the modalities $Until$ and $Since$ two metric modalities: $\Diamond_{(0,1)}P$ and $\Diamond_{(-1,0)}P$. The first one states that P will happen (at least once) within the next unit of time, and the second says that P happened within the last unit of time.

Adding the power to say "P will be true (at least once) within the next unit of time" is natural and necessary. There is, however, no reason to believe that this gives us the required expressive power. Is it enough, or do we need additional modalities? If we must add more modalities, which ones should we choose? A. Pnueli was the first to address these questions.

In previous work we have defined the *counting modalities* $C_n(P)$ and $\overleftarrow{C}_n(P)$ for $n \in Nat$. $C_n(P)$ says "P will hold at least at n points within the next unit of time" and its dual $\overleftarrow{C}_n(P)$ says "P was true at least at n points within the previous unit of time" [9,10].

TLC (Temporal Logic with Counting) is the extension of *LTL* by all counting modalities. For $n \in \mathbb{N}$, a fragment TLC_n of *TLC* has only finitely many modalities: Until, Since and C_k, $\overleftarrow{C}_k$ for $k \leq n$. In particular, TLC_1 is exactly QTL and has the same expressive power as *MITL*.

We proved in [9,10,11] the following:

1. *TLC* is decidable and equivalent to a natural fragment of $FO[<,+1]$.
2. TLC_n is strictly less expressive than TLC_{n+1}, so this is a strict hierarchy.
3. If the expressive power of a temporal logic $\mathcal{L}$ is between *TLC* and $FO[<,+1]$, then $\mathcal{L}$ has infinitely many modalities.

As a consequence of (3), and in contrast to Kamp's theorem, no temporal logic with a finite set of modalities is expressively equivalent to $FO[<,+1]$ over the reals.

1.3 Kamp's Theorem in Metric Setting

Over the reals, $FO[<,+1]$ still lacks expressive power to specify a natural global metric property "the current moment is an integer."

This paper is concerned with the expressive power of $FO[<,+1]$ over the expansion $\mathbb{R}_{\mathbb{Z}}$ of the reals by a monadic predicate interpreted as the set of integers. We prove that $FO[<,+1]$ has the same expressive power as a temporal logic with a finite set of modalities, hence an analog of Kamp's theorem holds.

More specifically, *MTL* (Metric Temporal Logic [15]) in addition to four modalities of QTL has two more modalities: $\Diamond_{=1}$ and $\Diamond_{=-1}$; $\Diamond_{=1}(P)$ says: "P is true exactly after one unit of time" its dual $\Diamond_{=-1}(P)$ says "P was true exactly before one unit of time."

Our main result states that $FO[<,+1]$ has the same expressive power as *MTL* over $\mathbb{R}_{\mathbb{Z}}$.

The paper is organized as follows. Section 2 provides definitions of the first-order monadic logics and of temporal logics. In Section 3, Kamp's theorem and our main result are stated. Sections 4 outlines a proof of the main theorem. The structure of the proof of expressive completeness is similar to the simplified proof of Kamp's theorem [20]. We recall the relevant notions and propositions from [6,20] used in the proof of Kamp's theorem. Then, we generalize these propositions to the metric setting and prove expressive equivalence of *MTL* and $FO[<,+1]$ over $\mathbb{R}_{\mathbb{Z}}$. Sections 5-7 contain the proof of main technical lemmas, which uses some ideas from [17,19]. The last section presents conclusion and discusses related works.

2 Logics

In this section we recall definitions of the first-order monadic logics and of temporal logics.

Fix a set Σ of *atoms*. We use $P, R, S \dots$ to denote members of Σ. The syntax and semantics of both logics are defined below with respect to such Σ.

2.1 First-order monadic logics

In the context of first-order logics, the atoms of Σ are considered as *unary predicate symbols*.

The signature of $FO[<]$ (first-order monadic logic of order) in addition to Σ contains two binary relation symbols: $<$ and $=$. We use $x, y, z, \dots$ for (first-order) variables. The formulas are defined by the following grammar:

$$atomic := \ x < y \ \mid \ x = y \ \mid \ P(x) \ \ (\text{where} \ \ P \in \Sigma)$$

$$\varphi := atomic \ \mid \ \neg\varphi \ \mid \ \varphi \vee \varphi \ \mid \ \varphi \wedge \varphi \ \mid \ \exists x\varphi \ \mid \ \forall x\varphi$$

We will also use the standard abbreviated notation for *bounded quantifiers*, e.g., $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$, and $(\forall x)^{<z}(\dots)$ denotes $\forall x((x < z) \rightarrow (\dots))$, and $((\forall x)^{<z_2}_{>z_1}(\dots)$ denotes $\forall x((z_1 < x < z_2) \rightarrow (\dots))$, etc.

A Σ-structure (or just structure) $\mathcal{M}$ for $FO[<]$ is a tuple $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{T}$ is a set - the *domain* of $\mathcal{M}$, $<$ is a linear order relation on $\mathcal{T}$, and $\mathcal{I} : \Sigma \rightarrow \mathcal{P}(\mathcal{T})$ is the *interpretation* of Σ (where $\mathcal{P}$ is the powerset notation).

$FO[<,+1]$ is the extension of $FO[<]$ by a unary $+1$ functional symbol. We mostly will be interested in the interpretations of $FO[<,+1]$ over the reals. Under such interpretations, the domain of $\mathcal{M}$ is the set $\mathbb{R}$ of reals, $<$ and $+1$ are interpreted in the standard way, and unary predicate symbols from Σ are interpreted as unary predicates on the reals. We call such structures $\mathbb{R}$-structures. If, in addition, Σ contains a predicate name Int, interpreted as the set $\mathbb{Z}$ of integers, a structure is called an $\mathbb{R}_{\mathbb{Z}}$ structure.

It will be convenient for us to use another first-order language which is equivalent to $FO[<,+1]$ over $\mathbb{R}_{\mathbb{Z}}$ structures. This is the extension of $FO[<]$ by a unary function symbol $\lfloor x \rfloor$ - interpreted as the integer part of x, and by the unary functions $+c$ for $c \in \mathbb{Z}$. Its terms are defined by the grammar $t := x \mid \lfloor t \rfloor \mid t + c$ for $c \in \mathbb{Z}$. A standard term is a term of the form $\lfloor x \rfloor + c$ or $x + c$. It is clear that every term is equivalent to a standard term. By abusing notations, this logic will be also denoted by $FO[<,+1]$.

We use the standard notation $\mathcal{M}, a_1, a_2, \ldots a_n \models \varphi(x_1, x_2, \ldots x_n)$ to indicate that the formula φ with free variables among $x_1, \ldots, x_n$ is satisfiable in $\mathcal{M}$ when x_i are interpreted as elements a_i of $\mathcal{M}$.

2.2 Temporal Logics

In the context of temporal logics the atoms of Σ are used as *atomic propositions* (also called *propositional atoms*). Formulas are built using these atoms, and a set (finite or infinite) B of *modality names*, where a non-negative integer *arity* is associated with each modality $\mathsf{M} \in B$.

LTL (Linear Time Temporal Logic) has two modalities *strict*-Until and *strict*-Since. *LTL* formulas are defined by the following grammar:

$$F := \quad P \ \mid\ \neg F \ \mid\ F \vee F \ \mid\ F \wedge F \ \mid\ F \ \mathsf{Until}\ F \ \mid\ F \ \mathsf{Since}\ F, \text{ where } P \in \Sigma.$$

MTL (Metric Temporal Logic) has four additional unary modalities: *MTL* syntax extends the syntax of *LTL* by the following rules: If F is a formula, then $\Diamond_{(0,1)}F$, $\Diamond_{(-1,0)}F$, $\Diamond_{=1}F$ and $\Diamond_{=-1}F$ are formulas.

QTL (Quantitative Temporal Logic) is the fragment of *MTL* which uses only the modalities Until, Since, $\Diamond_{(0,1)}$ and $\Diamond_{(-1,0)}$.

Semantics. The semantics defines when a temporal formula holds at a *time-point* (or *moment* or element of the domain) in a structure $\mathcal{M}$.

The semantics is defined inductively: given a structure $\mathcal{M}$ with a domain $\mathcal{T}$ and $a \in \mathcal{T}$, define when a formula F *holds* in $\mathcal{M}$ at a - notation: $\mathcal{M}, a \models F$ - as follows:

- $\mathcal{M}, a \models P$ iff $a \in \mathcal{I}(P)$ for any atom $P \in \Sigma$.
- $\mathcal{M}, a \models F \vee G$ iff $\mathcal{M}, a \models F$ or $\mathcal{M}, a \models G$; similarly ("pointwise") for $\wedge$, $\neg$.
- $\mathcal{M}, a \models F \ \mathsf{Until}\ G$ iff there is $a' > a$ such that $\mathcal{M}, a' \models G$ and $\mathcal{M}, b \models F$ for every b in an open interval (a, a').
- $\mathcal{M}, a \models F \ \mathsf{Since}\ G$ iff there is $a' < a$ such that $\mathcal{M}, a' \models G$ and $\mathcal{M}, b \models F$ for every b in an open interval (a', a).

MTL is interpreted over the reals with the standard interpretation of $+1$ and -1 funcional symbols. It has four additional semantical clauses for modalities: $\Diamond_{(0,1)}$ - within the next unit of time, $\Diamond_{(-1,0)}$ - within the last unit of time, $\Diamond_{=1}$ - exactly after one unit of time, and $\Diamond_{=-1}$ - exactly before one unit of time.

- $\mathcal{M}, a \models \Diamond_{=1}F$ iff $\mathcal{M}, a+1 \models F$.
- $\mathcal{M}, a \models \Diamond_{=-1}F$ iff $\mathcal{M}, a-1 \models F$.
- $\mathcal{M}, a \models \Diamond_{(0,1)}F$ iff there is $a' \in (a, a+1)$ such that $\mathcal{M}, a' \models F$.
- $\mathcal{M}, a \models \Diamond_{(-1,0)}F$ iff there is $a' \in (a-1, a)$ such that $\mathcal{M}, a' \models F$.

In $\mathbb{R}_{\mathbb{Z}}$ structures Σ contains a symbol Int, interpreted as the set $\mathbb{Z}$ of integers, and

- $\mathcal{M}, a \models \mathsf{Int}$ iff a is an integer.

We conclude this section by recalling a definition of a temporal logic TLC with an infinite sets of modalities. Thought TLC is not used directly in our technical results, it is useful to to explain the role of $\mathbb{Z}$ in expressing the modality of TLC by MTL formulas.

TLC (Temporal Logic with Counting) is the extension of LTL by an infinite set of modalitues C_n and $\overleftarrow{C}_n$ for $n \in \mathbb{N}$ - counting modalities. The TLC syntax extends the syntax of LTL by the following rules: if F is a formula, then $C_n(F)$ and $\overleftarrow{C}_n(F)$ are formulas. The semantical clauses for modalities: $C_n(P)$ - "P will hold at least at n points within the next unit of time," and $\overleftarrow{C}_n(P)$ - "P was true at least at n points within the previous unit of time" are:

- $\mathcal{M}, a \models C_n(F)$ iff there are $a_1 < a_2 < \cdots < a_n \in (a, a+1)$ such that $\mathcal{M}, a_i \models F$ for $i \leq n$.
- $\mathcal{M}, a \models \overleftarrow{C}_n(F)$ iff there are $a_1 < a_2 < \cdots < a_n \in (a-1, a)$ such that $\mathcal{M}, a_i \models F$ for $i \leq n$.

Note that $C_1(P)$ (respectively, $\overleftarrow{C}_1(P)$) is equivalent to $\diamondsuit_{(-1,0)}(P)$ (respectively, $\diamondsuit_{=-1}(P)$).

In [19], we proved that all counting modalities are expressible in MTL over the expansion of the reals by two monadic predicate: integers and the even integers.

Let us illustrate the role of Int and show how to express all counting modalities $C_n(P)$ and $\overleftarrow{C}_n(P)$ (for $n \in Nat$) in MTL over $\mathbb{R}_{\mathbb{Z}}$. First, for every $k \in \mathbb{N}$, there is an LTL formula $Forward_k(P, Q)$ which expresses "from the current moment until the next occurrence of Q there are at least k points in P." Similarly, there is an LTL formula $Backward_k(P, Q)$ which expresses "between the current moment and the previous occurrence of Q (including the moment of this occurrence) there are at least k points in P." Finally, $C_n(P)$ - "P holds at least at n points within the next unit of time" - is equivalent to the conjunction of $\mathsf{Int} \rightarrow Forward_n(P, \mathsf{Int})$ and $\neg\mathsf{Int} \rightarrow \vee_{k=0}^{n}(Forward_k(P, \mathsf{Int}) \wedge \diamondsuit_{=1} Backward_{n-k}(P, \mathsf{Int}))$. The dual modality $\overleftarrow{C}_n(P)$ is expressed similarly.

3 Expressive Equivalence

Equivalence between temporal and first-order formulas with a single free variable is naturally defined as: F is equivalent to $\varphi(x)$ over a class $\mathcal{C}$ of structures iff for any $\mathcal{M} \in \mathcal{C}$ and $a \in \mathcal{M}$: $\mathcal{M}, a \models F \Leftrightarrow \mathcal{M}, a \models \varphi(x)$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be temporal logics. $\mathcal{L}$ is *expressively complete for* (or, at least as expressive as) $\mathcal{L}'$ over a class $\mathcal{C}$, if for every formula $F' \in \mathcal{L}'$ there is $F \in \mathcal{L}$ which is equivalent to F' over $\mathcal{C}$. In this case we write $\mathcal{L}' \preceq_{exp} \mathcal{L}$. Similarly, if $\mathcal{L}'$ is a first-order logic, $\mathcal{L}' \preceq_{exp} \mathcal{L}$ if for every formula $\varphi(x)$ in $\mathcal{L}$ with a single free variable, there is a formula $F \in \mathcal{L}$ equivalent to φ. $\mathcal{L}$ and $\mathcal{L}'$ are expressively equivalent (notation $\mathcal{L} =_{exp} \mathcal{L}'$) over $\mathcal{C}$ iff $\mathcal{L}' \preceq_{exp} \mathcal{L}$ and $\mathcal{L} \preceq_{exp} \mathcal{L}'$ over $\mathcal{C}$.

The fundamental result of Kamp [14,5] implies that a temporal logic with just two modalities Until and Since has the same expressive power as $FO[<]$ over the canonical linear time models $(\mathbb{N}, <)$, $(\mathbb{R}, <)$ and non-negative reals $(\mathbb{R}^{\geq 0}, <)$.

An influential consequence of Kamp's result is that *LTL* has emerged as the canonical temporal logic.

A technical notion that unifies the canonical linear time models is Dedekind completeness.

A linear order $(T, <)$ is *Dedekind complete* if every non-empty subset (of the domain) which has an upper bound has a least upper bound. The canonical linear time models $(\mathbb{N}, <)$, $(\mathbb{R}, <)$ and $(\mathbb{R}^{\geq 0}, <)$ are Dedekind complete, while the order of the rationals is not Dedekind complete.

Kamp's theorem states that *LTL* is expressively equivalent to $FO[<]$ over Dedekind complete orders.

Theorem 3.1 (Kamp [14]) *1. Given any LTL formula A there is an $FO[<]$ formula $\varphi_A(x)$ which is equivalent to A over all linear orders.*

2. Given any $FO[<]$ formula $\varphi(x)$ with one free variable, there is an LTL formula A_φ which is equivalent to φ over Dedekind complete orders.

Moreover, φ_A and A_φ are computable from A and φ.

The correspondence between predicate logics and temporal logics becomes considerably more complicated with the introduction of *metric* specifications.

All logics mentioned in Section 2.2 are less expressive than $FO[<, +1]$ over the reals. The translation from the formulas of these logics to equivalent formulas of $FO[<, +1]$ is straightforward.

Their expressive power can be summarized as follows: $QTL \prec_{exp} TLC$ [9,10], and $QTL \prec_{exp} MTL$ [1]. Moreover, since *TLC* is decidable, while *MTL* is undecidable, it follows that *TLC* cannot express $\Diamond_{=1} P$. In [10] we proved that *MTL* cannot express $C_2(P)$ - "P occurs twice in the next unit interval." Hence, the expressive power of *MTL* and *TLC* is incomparable.

Actually, the main result of [10] is much stronger. In particular, it implies that if $\mathcal{L}$ is a temporal logic with a finite set of modalities and $\mathcal{L} \preceq_{exp} FO[<, +1]$, then there is n such that a counting modality $C_n(P)$ is not expressible in $\mathcal{L}$.

As a consequence, in contrast to Kamp's theorem, no temporal logic with a finite set of modalities is expressively equivalent to $FO[<, +1]$ over the reals.

Our main result is that over the expansions of $(\mathbb{R}, <, +1)$ by a monadic predicate "the current moment is an integer" $FO[<, +1]$ is expressively equivalent to a finite base temporal logic *MTL*.

Theorem 3.2 (Main) *1. Given any MTL formula A there is an $FO[<, +1]$ formula $\varphi_A(x)$ which is equivalent to A over $\mathbb{R}_{\mathbb{Z}}$.*

2. Given any $FO[<, +1]$ formula $\varphi(x)$ with one free variable, there is a MTL formula A_φ which is equivalent to φ over $\mathbb{R}_{\mathbb{Z}}$.

Moreover, φ_A and A_φ are computable from A and φ.

Theorem 3.2 (1) is easily proved by the structural induction. The main technical contribution of our paper is the proof of Theorem 3.2 (2). The proof is constructive. An algorithm which for every $FO[<, +1]$ formula $\varphi(x)$ constructs a *MTL* formula which is equivalent to φ is easily extracted from our proof.

It is a routine exercise to adapt the proof of Theorem 3.2 to the non-negative reals, and to show that *MTL* and *FO*$[<,+1]$ are expressively equivalent over the non-negative reals expanded by a predicate interpreted as the set of natural numbers.

4 Proof Outline

The structure of our proof is similar to the proof of Kamp's theorem in [20]. We first recall the relevant notions and propositions from [6,20]. Then, we state their generalization to metric setting and prove expressive equivalence of *MTL* and *FO*$[<,+1]$ over $\mathbb{R}_{\mathbb{Z}}$.

Definition 4.1 (Decomposition and $\overrightarrow{\exists}\forall$-formulas) *Let Σ be a set of monadic predicate names.*

- *A decomposition formula (D-formula) over Σ is a formula $\chi(z_0,\dots,z_m)$ of the form:*

$$\begin{aligned}\exists x_n \dots \exists x_1 \exists x_0\, (x_n > x_{n-1} > \dots > x_1 > x_0) \wedge \\ \bigwedge_{i=0}^{m} z_i = x_{k_i} \wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) \wedge \bigwedge_{j=1}^{n} [(\forall y)_{>x_{j-1}}^{<x_j} \beta_j(y)] \\ \wedge\, (\forall y)_{>x_n} \beta_{n+1}(y) \wedge\, (\forall y)^{<x_0} \beta_0(y)\end{aligned} \qquad (1)$$

  where $\overline{z} = \{z_0,\dots,z_m\}$ and $\overline{x} = \{x_0,\dots,x_n\}$ are disjoint lists of variables, $0 \le k_i < k_j \le n$ for $i < j$ and all α_j, β_j are quantifier free formulas with one variable over Σ. Observe that $\chi(z_0,\dots,z_m)$ implies $\wedge_{i=0}^{m-1}(z_i < z_{i+1})$.
- *An $\overrightarrow{\exists}\forall$-formula over Σ is a conjunction of a D-formula as in (1) and $\bigwedge_{i=0}^{s}(u_i = z_{h(i)})$, where $u_0,\dots,u_s$ are variables and $h : \{0,\dots,s\} \to \{0,\dots,m\}$.*

The next definition plays a major role in the proof of Kamp's theorem [6,20].

Definition 4.2 *Let $\mathcal{M}$ be a structure with the signature including unary predicate names Σ, and $\mathcal{L}$ be a temporal logic. We denote by $\mathcal{L}[\Sigma]$ the set of unary predicate names $\Sigma \cup \{A \mid A \text{ is an } \mathcal{L}\text{-formula over } \Sigma\}$. The canonical $\mathcal{L}$-expansion of $\mathcal{M}$ is an expansion of $\mathcal{M}$ to a structure with unary predicate names $\mathcal{L}[\Sigma]$, where each predicate name $A \in \mathcal{L}[\Sigma]$ is interpreted as $\{a \in \mathcal{M} \mid \mathcal{M}, a \models A\}$.*

Note that if A is an $\mathcal{L}$-formula over $\mathcal{L}[\Sigma]$ predicates, then it is equivalent to an $\mathcal{L}$ -formula over Σ, and hence to an atomic formula in the canonical $\mathcal{L}$-expansions.

The $\overrightarrow{\exists}\forall$ formulas are defined as previously, but now they can use as atoms $\mathcal{L}$ definable predicates.

We say that first-order formulas in a signature which includes $\mathcal{L}[\Sigma]$ are equivalent over $\mathcal{M}$ (respectively, over a class $\mathcal{C}$ of structures) if they are equivalent in the canonical expansion of $\mathcal{M}$ (in the canonical expansion of every $\mathcal{M} \in \mathcal{C}$).

Propositions 4.4-4.5 were proved[1] in [20].

[1] For the sake of simplicity these propositions were stated for $\mathcal{L} := LTL$. However, their proofs are sound for any $\mathcal{L} \succeq_{exp} LTL$.

Proposition 4.3 (From $\overrightarrow{\exists}\forall$-formulas to temporal formulas) *Let $\mathcal{L}$ be a temporal logic such that $\mathcal{L} \succeq_{exp} LTL$. Then, every $\overrightarrow{\exists}\forall$-formula with one free variable is equivalent (over the canonical $\mathcal{L}$-expansions) to an $\mathcal{L}$ formula.*

Proposition 4.4 (From first-order formulas to $\overrightarrow{\exists}\forall$-formulas) *Let $\mathcal{L}$ be a temporal logic such that $\mathcal{L} \succeq_{exp} LTL$. Then every $FO[<]$ formula is equivalent (over the canonical $\mathcal{L}$ expansions of Dedekind complete orders) to a disjunction of $\overrightarrow{\exists}\forall$-formulas.*

Setting $\mathcal{L} := LTL$ in the next proposition we obtain Kamp's theorem.

Proposition 4.5 *Let $\mathcal{L} \succeq_{exp} LTL$ be a temporal logic. Then every $FO[<]$ formula with one free variable is equivalent (over the canonical $\mathcal{L}$-expansions) to an $\mathcal{L}$ formula.*

The structure of our proof is similar to that of Kamp's theorem. Recall that a substitution σ is a map from variables to terms. We use $\{t_0/z_0, \ldots, t_n/z_n\}$ for the substitution which maps z_i to t_i. For a formula ψ, the result of replacing free occurrences of z_i by t_i is denoted by $\psi\sigma$ (as usual, we have to avoid that the variables occurring in t_i are captured in $\psi\sigma$). Recall that the standard terms in $FO[<, +1]$ are variables or of the form $z + c$ or $\lfloor z \rfloor + c$, where $c \in \mathbb{Z}$. Every term of $FO[<, +1]$ is equivalent to a standard term. From now on we use the word "term" for "standard term."

Definition 4.6 *A simple (metric) formula is a formula of the form $\psi\sigma$, where ψ is an $\overrightarrow{\exists}\forall$-formula, and σ is a substitution.*

In a simple metric formula no bound variable is in the scope of function symbols $\lfloor\ \rfloor$ or $+c$ for $c \neq 0$, We will prove the next two Propositions which are adaptations of Propositions 4.4 and 4.3 to the metrical setting:

Proposition 4.7 (From simple formulas to *MTL* formulas) *Every simple metric formula with one free variable is equivalent (over the canonical MTL-expansions of $\mathbb{R}_{\mathbb{Z}}$) to an MTL formula.*

Proposition 4.8 (From first-order formulas to simple metric formulas) *Every $FO[<, +1]$ formula is equivalent (over the canonical MTL expansions of $\mathbb{R}_{\mathbb{Z}}$) to a disjunction of simple formulas.*

Propositions 4.8 and 4.7 immediately imply Theorem 3.2 (2) - our main result.

Proof. (of Theorem 3.2 (2).) Let $\varphi(x)$ be a $FO[<, +1]$ formula with one free variable. By Proposition 4.8, it is equivalent to a disjunction ψ_i of simple formulas. By Proposition 4.7, ψ_i is equivalent to a MTL formula A_i. Therefore, φ is equivalent to a MTL formula $\vee A_i$. □

Our proofs are organized as follows. The next section presents simple Lemmas. Proposition 4.7 is proved in Section 6 and Proposition 4.8 is proved in Section 7. The proofs of Propositions 4.7 and 4.8 often reuse Propositions 4.4 and 4.3.

5 Notations and Observations

Notations. As usual, $\diamond_{=2}P$ abbreviates $\diamond_{=1}\diamond_{=1}P$, and $\diamond_{=c}$ for $c \in Z$ is defined similarly. We denote by FreeVar(φ) the set of free variables of φ.

Let $\sigma := \{t_0/z_0, \dots, t_i/z_n\}$ be a substitution. We use dom(σ) for $\{z_0, \dots, z_n\}$ and Term(σ) for $\{\sigma(z) \mid z \in \mathrm{dom}(\sigma)\}$. Recall that the terms of $FO[<, +1]$ are of the form z, $\lfloor z \rfloor + c$ and $z+c$, where z is a variable; and in a simple metric formula no bound variable is in the scope of function symbols $\lfloor\ \rfloor$ or $+c$ for $c \neq 0$,

For a quantifier free formula φ we denote by Term(φ) the set of terms that appear in φ. For a simple formula $\varphi := \psi\sigma$ we use Term(φ) for $\{\sigma(z) \mid z \in \mathrm{FreeVar}(\psi)\}$. For a Boolean combination φ of simple and quantifier free formulas φ_i we denote by Term(φ) the union of Term(φ_i).

In this section we state simple lemmas which will be used in the proofs of Propositions 4.7 and 4.8. All these lemmas easily follow from the definitions.

Lemma 5.1 *Every atomic $FO[<, +1]$ formula is equivalent to a disjunction of simple formulas.*

Let T be a set of terms. An *order constraint Ord* over T is a conjunction of formulas of the form $t = t'$ and $t' < t$ for $t, t' \in T$. An order constraint *Ord* is *linear* if for every $t_1, t_2 \in$ Term(*Ord*): either *Ord* implies $t_1 < t_2$, or *Ord* implies $t_2 < t_1$, or *Ord* implies $t_1 = t_2$.

Let φ be a simple formula $\psi\sigma$, where $\psi := \bigwedge_{i=0}^{s}(u_i = z_{h(i)}) \wedge \chi(z_0, \dots, z_m)$ is an $\overrightarrow{\exists}\forall$-formula as in Definition 4.1. We denote by Ord_φ, the (linear) order constraint generated by φ over Term(φ), which is defined as $\bigwedge_{i=0}^{s} \sigma(u_i) = \sigma(z_{h(i)}) \wedge \bigwedge_{j=0}^{m-1} \sigma(z_j) < \sigma(z_{j+1})$.

Lemma 5.2 *1. Let Ord be an order constraint. Then Ord is equivalent to a disjunction of simple formulas, and ¬Ord is equivalent to a disjunction of simple formulas.*
2. If φ is a simple formula, then $\neg\varphi$ is equivalent to a disjunction of simple formulas.
3. A Boolean combination of simple formulas is equivalent to a disjunction of simple formulas.

Proof. (1) is immediate.

(2) Let $\varphi := \psi\sigma$, where ψ is an $\overrightarrow{\exists}\forall$-formula. Then $\neg\varphi$ is equivalent to $\neg Ord_\varphi \vee (\neg\psi)\sigma$. Since, $\neg\psi$ is an $FO[<]$ formula, by Proposition 4.4, it is equivalent to a disjunction $\vee\psi_i$ of $\overrightarrow{\exists}\forall$-formulas. Therefore, $(\neg\psi)\sigma$ is equivalent to a disjunction of simple formulas, and $\neg\varphi$ is equivalent to a disjunction of simple formulas.

(3) immediately by (2). □

Lemma 5.3 *1. If Ord is an order constraint, then Ord is equivalent to a disjunction of linear order constraints Ord_i such that Term(Ord_i) = Term(Ord) for every i.*

2. *If φ is a simple formula and Ord is an order constraint, then $\varphi \wedge Ord$ is equivalent to a disjunction of simple formulas φ_i such that $Term(\varphi_i) = Term(\varphi) \cup Term(Ord)$ for every i.*

Lemma 5.4 *Let $\chi(z_0, \ldots, z_m)$ be a D-formula as in (1) (hence, χ implies $\bigwedge_{i=0}^{m-1}(z_i < z_{i+1})$).*

1. *χ is equivalent to a conjunction $\wedge_{i=0}^{m-1}\chi_i(z_i, z_{i+1})$ of D-formulas with two variables.*
2. *More generally, if $0 = l_0 < l_1 < l_2 < \cdots < l_s = m$, then χ is equivalent to a conjunction $\wedge_{i=0}^{s-1}\chi_i(z_{l_i}, \ldots, z_{l_{i+1}})$ of D-formulas with free variables as displayed.*
3. *Let z be a fresh variable. Then $\chi \wedge z < z_0$ is equivalent to a D-formula χ' with $FreeVar(\chi') = \{z, z_0, \ldots, z_m\}$. Similarly, for $\varphi \wedge z_m < z$.*
4. *A conjunction of D-formulas with the same set of free variables is equivalent to a disjunction of (other) D-formulas with the same set of free variables.*

Lemma 5.5 (Shifting monadic predicates by a constant) *Let $\varphi(z_0, \ldots, z_n)$ be an $FO[<, +1]$ formula, $c \in \mathbb{Z}$, and let φ^c be obtained from φ when every monadic predicate P in φ is replaced by (a monadic predicate definable by) $\Diamond_{=c}P$.*

1. *Then $\mathcal{M}, a_0, \ldots, a_n \models \varphi^c$ iff $\mathcal{M}, a_0 + c, \ldots, a_n + c \models \varphi$.*
2. *If φ is a D (respectively, $\overrightarrow{\exists}\forall$ or simple) formula, then φ^c is a D (respectively, $\overrightarrow{\exists}\forall$ or simple) formula.*
3. *If ψ is an $\overrightarrow{\exists}\forall$-formula, then $\psi\sigma$ is equivalent to $\psi^c\sigma^{-c}$, where $\sigma^{-c}(z) := \sigma(z) - c$ for every $z \in FreeVar(\psi)$.*

6 From Simple Formulas to *MTL* Formulas - Proof of Proposition 4.7

In this section we prove Proposition 4.7 which states that every simple metric formula with one free variable is equivalent (over the canonical *MTL*-expansions of $\mathbb{R}_{\mathbb{Z}}$) to an *MTL* formula. Proposition 4.7 immediately follows from Claims 1 and 2 below.

Claim 1. A simple formula with one free variable z is equivalent to a disjunction of formulas of one of the following forms:

(A) $z = \lfloor z \rfloor \wedge \chi(z_0, z_1)\sigma_0$, where χ is a D-formula as in (1) and $\sigma_0 := \{\lfloor z \rfloor + c/z_0, \lfloor z \rfloor + c + 1/z_1\}$.

(B) $\lfloor z \rfloor < z < \lfloor z \rfloor + 1 \wedge \chi(z_0, z_1, z_2)\sigma_0$, where χ is a D-formula as in (1) and $\sigma_0 := \{\lfloor z \rfloor + c/z_0, z + c/z_1, \lfloor z \rfloor + c + 1/z_2\}$.

Claim 2. Any formula of the form (A) or (B) is equivalent to an *MTL* formula.

Proof of Claim 2. The only non-trivial metric constraint in formulas of these forms is that the distance between two integer points $\lfloor z \rfloor + c$ and $\lfloor z \rfloor + c + 1$ is one. This can be easily formalized in $FO[<]$ using the monadic predicate Int. Below are formal details.

We will translate formulas of the form (B) to equivalent *MTL* formulas (the translation of formulas of the form (A) is simpler).

If φ is of the form (B), then it is equivalent to the conjunction of $\neg\mathsf{Int}(z)$ and $\big(\exists z_0 z_2(\mathsf{Int}(z_0) \wedge \mathsf{Int}(z_2) \wedge z_0 < z_1 < z_2 \wedge (\forall u)^{<z_2}_{>z_0}\neg\mathsf{Int}(y) \wedge \chi)\big)\sigma$, where $\sigma := \{z+c/z_1\}$. Since $\exists z_0 z_2(\mathsf{Int}(z_0) \wedge \mathsf{Int}(z_2) \wedge z_0 < z_1 < z_2 \wedge (\forall u)^{<z_2}_{>z_0}\neg\mathsf{Int}(y) \wedge \chi)$ is an $FO[<]$ formula, it is equivalent to an *MTL* formula A, by Proposition 4.5. Therefore, φ is equivalent to an *MTL* formula $\neg\mathsf{Int} \wedge \diamond_{=c} A$. □

Proof of Claim 1. We assume that a least term of φ w.r.t. Ord_φ is of the form $\lfloor z \rfloor + c$ (otherwise, by Lemma 5.3 we can rewrite φ as a disjunction of simple formulas with this property). There is $N \in \mathbb{N}$ such that Ord_φ implies that all terms in $\mathrm{Term}(\varphi)$ are less than $\lfloor z \rfloor + c + N$.

Let $T := \{\lfloor z \rfloor + c + j \mid j = 0, \ldots, N\} \cup \{z + c + j \mid j = 0, \ldots, N-1\}$. Note that $\mathrm{Term}(\varphi) \subsetneq T$.

Let Ord_i (for $i < K \in \mathbb{N}$) be all satisfiable linear orders on T (there are finitely many such orders). Then φ is equivalent to $\vee_i(\varphi \wedge Ord_i)$. Hence, (by Lemma 5.3), φ is equivalent to a disjunction of simple formulas φ_i with $\mathrm{Term}(\varphi_i) = T$.

Since, $z + c, \lfloor z \rfloor + c \in T$, it follows that either $Ord_{\varphi_i} \rightarrow z = \lfloor z \rfloor$ or $Ord_{\varphi_i} \rightarrow z > \lfloor z \rfloor$. If Ord_{φ_i} implies $z = \lfloor z \rfloor$, we show that φ_i is equivalent to a disjunction of formulas of the form (A); if Ord_{φ_i} implies $z > \lfloor z \rfloor$, we show that φ_i is equivalent to a disjunction of formulas of the form (B).

We will show the second assertion (the first one is simpler). Assume $Ord_{\varphi_i} \rightarrow z > \lfloor z \rfloor$, then there is $\chi(z_0, \ldots, z_{2N})$ as in (1) such that φ_i is equivalent to $\lfloor z \rfloor < z \wedge \chi\sigma$ where $\sigma(z_{2j}) = \lfloor z \rfloor + c + j$ and $\sigma(z_{2j-1}) = z + c + j$ for $j = 0, \ldots, N$.

By Lemma 5.4(2), χ is equivalent to $\wedge_{j=0}^{N-1}\chi_j(z_{2j}, z_{2j+1}, z_{2j+2})$ where χ_j are D formulas with $\mathrm{FreeVar}(\chi_j) = \{z_{2j}, z_{2j+1}, z_{2j+2}\}$.

Replace in χ_j each monadic predicate P by a predicate definable by $\diamond_{=c+j}P$, and rename its free variables $z_{2j}, z_{2j+1}, z_{2j+2}$ to z_0, z_1, z_2; the result is a D-formula $\psi_j(z_0, z_1, z_2)$. Then by Lemma 5.5, we obtain that $\chi\sigma$ is equivalent to $(\wedge_{j=0}^{N-1}\psi_j(z_0, z_1, z_2))\sigma_0$, where $\sigma_0 := \{\lfloor z \rfloor + c/z_0, z + c/z_1, \lfloor z \rfloor + c + 1/z_2\}$.

Finally, $\wedge_{j=0}^{N-1}\psi_j(z_0, z_1, z_2)$ is equivalent, by Lemma 5.4(4), to a disjunction of D formulas with free variables z_0, z_1, z_2. Therefore, φ_i is equivalent to a disjunction of formulas of the form (B). □

7 From First-order Formulas to Simple Formulas - Proof of Proposition 4.8

In this section we prove Proposition 4.8 which states that every $FO[<, +1]$ formula is equivalent (over the canonical *MTL* expansions of $\mathbb{R}_\mathbb{Z}$) to a disjunction of simple formulas.

The main technical result of this section is:

Proposition 7.1 *If φ is a simple formula, then $\exists z\varphi$ is equivalent to a Boolean combination of simple formulas.*

Proposition 4.8 follows (by a straightforward structural induction) from Proposition 7.1 and Lemmas 5.1 and 5.2(3).

In [17], we proved that for every $N \in \mathbb{N}$, every $FO[<,+1]$ sentence (no free variable) is equivalent to an *MTL* formula over the class of real intervals of length $< N$. The following locality properties of formulas with a single free variable play a key role in our proof of Proposition 4.7: if $\varphi(z)$ is a simple formula with one free variable, then there is $N \in \mathbb{N}$ such that Ord_φ implies that the distance between t_1 and t_2 is $< N$ for every $t_1, t_2 \in \text{Term}(\varphi)$. This locality property fails for formulas with several free variables. Yet, for every formula φ we can decompose Ord_φ into local components, as stated in Lemma 7.3.

Definition 7.2 *A linear order constraint Ord is local if there is $N \in \mathbb{N}$ such that Ord implies that for every $t_1, t_2 \in Term(Ord)$, the distance between t_1 and t_2 is less than N (i.e., $Ord \rightarrow (t_2 < t_1 + N \wedge t_1 < t_2 + N)$).*

A linear constraint can be decomposed into local constraints and a linear order between them.

Lemma 7.3 *Let Ord be a satisfiable linear constraint. Then, there are $Ord_0, \dots, Ord_k$ such that:*

1. *Ord_i are local constraints.*
2. *$Term(Ord) = \cup_i Term(Ord_i)$.*
3. *$FreeVar(Ord_i) \cap FreeVar(Ord_j) = \emptyset$ for $i \neq j$.*
4. *Let t_i^{leat} be a leat and $t_i^{greatest}$ be a greatest term in Ord_i. Then, Ord is equivalent to $\bigwedge_{i=0}^{k} Ord_i \wedge \bigwedge_{i=0}^{k-1} (t_i^{greatest} < t_{i+1}^{least})$.*

Terminology. Ord_i and Term(Ord_i), as above, are called *local components* of *Ord*. Whenever *Ord* is clear from the context and $z \in \text{FreeVar}(Ord_i)$, then Ord_i is also called the local component of z.

Proof. Define an equivalence relation $\sim$ on Term(*Ord*) as: $t_1 \sim t_2$ if there is $N \in \mathbb{N}$ such that *Ord* implies that the distance between t_1 and t_2 is less than N. It is easy to see that $\sim$ is a convex equivalence relation, i.e., if $t_1 \sim t_2$, and $Ord \rightarrow (t_1 < t < t_2)$, then $t_1 \sim t \sim t_2$. Let T_i (for $i = 0, \dots, k$) be the equivalence classes of $\sim$. It is clear that $\text{FreeVar}(T_i) \cap \text{FreeVar}(T_j) = \emptyset$ for $i \neq j$. Let Ord_i be the order *induced* by *Ord* on T_i, i.e., for $t_1, t_2 \in T_i$: (1) $t_1 < t_2 \in Ord_i$ iff $Ord \rightarrow t_1 < t_2$ and (2) $t_1 = t_2 \in Ord_i$ iff $Ord \rightarrow t_1 = t_2$.

It is easy to see that Ord_i are local orders which satisfy the conclusion of Lemma 7.3. □

Let us proceed with a proof of Proposition 7.1. Given a simple formula φ with $z \in \text{FreeVar}(\varphi)$. We show that $\exists z \varphi$ is equivalent to a Boolean combination of simple formulas, according to the following cases:

Case 1. Ord_φ has only one local component.

Case 2. The local component of z is the last or the first local component of Ord_φ.

Case 3. There are local components before and after the local component of z.

Let Ord_i be the local component of z. For each of the above cases we will consider two subcases: (A) z is the only variable in FreeVar(Ord_i), and (B) There are other variables in FreeVar(Ord_i).

The road map of the proof is as follows:

Subcase 1.A immediately follows from Proposition 4.7. The proof of subcase 1.B is very similar to the proof of Proposition 4.7; however, due to additional variables, the notations are heavier.

Subcases 2.A and 3.A easily follow from Proposition 4.7.

Subcases 2.B and 3.B are reducible to Case 1, using standard logical equivalences.

Though the proof is lengthy, it is simple.

7.1 Case 1

We consider two subcases:

Subcase A. $\text{FreeVar}(\varphi) = \{z\}$. In this subcase, by Proposition 4.7, there is an *MTL* formula A equivalent to φ. Hence, $\exists z\varphi$ is equivalent (in the canonical *MTL* expansion) to an $\overrightarrow{\exists}\forall$-sentence $\exists x A(x)$. It is also equivalent to an *MTL* formula $B := \diamond A \vee A \vee \overleftarrow{\diamond} A$, where $\diamond A$ (respectively, $\overleftarrow{\diamond} A$) abbreviates True$\mathsf{Until}A$ (respectively, True$\mathsf{Since}A$).

Subcase B. There is $u \in \text{FreeVar}(\varphi)$ such that u is not z.

The proof for subcase B is similar to the proof of Proposition 4.7 (see Section 6).

First, we can assume that Ord_φ is satisfiable (otherwise, the formula is equivalent to False). Let $T := \text{Term}(\varphi)$. We can assume that $\lfloor u \rfloor + c$ is a least term and $\lfloor u \rfloor + c + N$ is a greatest term in T, and if $t \in T$ and $Ord_\varphi \to (\lfloor u \rfloor + c \leq t + d < \lfloor u \rfloor + c + N)$, then $t + d \in T$ (otherwise, use Lemma 5.3 to rewrite φ as a disjunction of formulas with these properties).

Next, we eliminate all terms of the form $\lfloor v \rfloor + d$ for each variable v which is different from u. Indeed, if such term t occurs in T, then $Ord_\varphi \to (\lfloor v \rfloor + d = \lfloor u \rfloor + c + i)$ for some $i < N$. Hence, we can replace t by $\lfloor u \rfloor + c + i$.

Therefore, we can assume that the set of terms $T := \text{Term}(\varphi)$ of our formula φ has the following properties:

1. $\lfloor u \rfloor + c$ is the least and $\lfloor u \rfloor + c + N$ is the greatest element of T.
2. Let $V := \text{FreeVar}(\varphi)$. Then, there are $c_v \in \mathbb{Z}$ for $v \in V$ such that $T = \{\lfloor u \rfloor + c + i \mid i \leq N\} \cup \{v + c_v + i \mid i = 0, \ldots, N-1\}$.

Define an equivalence relation $\approx$ on V as $v \approx v'$ if Ord_φ implies that v and v' have the same fractional part, i.e., if $Ord_\varphi \to (v + c_v = v' + c_{v'})$ for constants $c_v, c_{v'}$ defined in (2). Assume that $\approx$ has l equivalence classes $V_0, \ldots, V_l$. Define $v(i)$ to be a variable in V_i. Furthermore, we can assume that Ord_φ implies that $v(i) + c_{v(i)} < v(j) + c_{v(j)}$ for $i < j$.

Now, φ is equivalent to the conjunction of $E := \bigwedge_{i=1}^{l}(\bigwedge_{v \in V_i \setminus v(i)} v(i) + c_{v(i)} = v + c_v)$ and $\chi(z_0, \ldots, z_{N \times (l+1)})\sigma$, where χ is a D formula and $\sigma(z_{j \times (l+1)}) :=$

$\lfloor u \rfloor + c + j$ for $j = 0, \dots, N$ and $\sigma(z_{i+j\times(l+1)}) := v(i) + c_{v(i)} + j$ for $i = 1, \dots, l$ and $j = 0, \dots, N-1$.

If the first conjunct E has an occurrence of z, i.e., $z + d = v + d'$ occurs there, then we can replace all occurrences of z in φ by $v + d' - d$. The resulting formula φ' does not have free occurrences of z and is equivalent to φ. Therefore, $\exists z\varphi$ is equivalent to φ'.

If E has no occurrence of z, then $\exists z\varphi$ is equivalent to $E \wedge \exists z(\chi\sigma)$.

Therefore, it remains to prove that $\exists z(\chi\sigma)$ is equivalent to a Boolean combination of simple formulas.

Our strategy is similar to the proof of Proposition 4.7 (see Section 6).

We are going to prove:

Claim 1. $\chi\sigma$ is equivalent to a disjunction of formulas of the form:

(C) $\psi(z_0, \dots z_{l+1})\sigma_0$, where
1. ψ is a D-formula, and Ord_ψ implies $z_0 < z_1 < \dots < z_{l+1}$.
2. $\sigma_0(z_0) = \lfloor u \rfloor + c$, $\sigma_0(z_{l+1}) = \lfloor u \rfloor + c + 1$ and $\sigma_0(z_i) = v(i) + c_{v(i)}$ for $i = 1, \dots, l$.
3. All variables $v(i)$ are different from each other.

Claim 2. If ψ and σ are as in (C), then $\exists z(\psi\sigma)$ is equivalent to a simple formula.

Claims 1 and 2 imply that $\exists z(\chi\sigma)$ is equivalent to a disjunction of simple formulas.

The proof of Claim 2 is easy. Indeed, $\exists x\big(\alpha\{x + c/x_i\}\big)$ is equivalent to $\exists x_i\alpha$, whenever x is not free in α. Since z is $v(i)$ for some i, we obtain by the above equivalence that $\exists z(\psi\sigma_0)$ is equivalent to $(\exists z_i\psi)\sigma_0$. Observe that $\exists z_i\psi$ is an $\vec{\exists}\forall$-formula. Therefore, $\exists z(\psi\sigma_0)$ is equivalent to a simple formula $(\exists z_i\psi)\sigma_0$.

The proof of Claim 1 is similar to the proof of Claim 1 in Section 6.

Namely, we can rewrite $\chi(z_0, \dots, z_{N\times(l+1)})$ as a conjunction of D-formulas $\chi_0(z_0, \dots, z_{l+1}), \dots, \chi_i(z_{i\times(l+1)}, \dots, z_{(i+1)\times(l+1)}), \dots, \chi_{N-1}(z_{(N-1)\times(l+1)}, \dots, z_{N\times(l+1)})$, with free variables as displayed. Replace in χ_i each monadic predicate P by a predicate definable by $\Diamond_{=c+i}P$, and rename its free variables $z_{i\times(l+1)}, \dots, z_{(i+1)\times(l+1)}$ to $z_0, \dots z_l$; the result is a D formula $\psi_i(z_0, z_1, \dots, z_{l+1})$.

By Lemma 5.5, $\chi\sigma$ is equivalent to $(\bigwedge_i \psi_i)\sigma_0$. Finally, since ψ_i are D-formulas and $\mathrm{FreeVar}(\psi_i) = \{z_0, \dots, z_{l+1}\}$, we obtain, by Lemma 5.4, that $\bigwedge_i \psi_i$ is equivalent to a disjunction of D-formulas, and $\chi\sigma$ is equivalent to a disjunction of formulas of the form (C).

This completes the proof of Claim 2.

7.2 Case 2

Let $Ord := Ord_\varphi$ and assume that Ord is decomposed as in Lemma 7.3, and $z \in \mathrm{FreeVar}(Ord_k)$ (the case when z in the first local component is dual).

Let $Ord_{<k}$ be the order induced by Ord on $\cup_{i=0}^{k-1} T_i$ and Ord' be the order induced by Ord on $t_{k-1}^{greatest} \cup T_k$. Then, φ is equivalent to $\varphi_1 \wedge \varphi_2$, where φ_1 and φ_2 are simple formulas such that $\mathrm{Term}(\varphi_1) = \cup_{i=0}^{k-1} T_i$, $\mathrm{Term}(\varphi_2) = \{t_{k-1}^{greatest}\} \cup T_k$. and $Ord_{\varphi_2} = Ord'$. Since z is not free in φ_1, we obtain that $\exists z\varphi$ is equivalent

to $\varphi_1 \wedge \exists z\varphi_2$. So, it remains to prove that $\exists z\varphi_2$ is equivalent to a Boolean combination of simple formulas.

φ_2 has two local components and the first one contains only one term.

We have reduced Case 2 to a slightly simpler version:

Ord_φ has two local components: $T_0 = \{t\}$ and T_1 such that $z \in \text{FreeVar}(T_1)$.

We consider two subcases:

Subcase A. z is the only free variable T_1.

Subcase B. There is $u \in \text{FreeVar}(\varphi)$ such that u is not z.

Subcase A. Let t_1 be a least term in T_1. We can assume that it is of the form $\lfloor z \rfloor + c$. Indeed, otherwise t_1 is $z + c$. From the following equivalence

$$(t < z + c \wedge z + c \neq \lfloor z \rfloor + c) \leftrightarrow (t < \lfloor z \rfloor + c \vee (\lfloor z \rfloor + c < t \wedge t < z + c))$$

we obtain that φ is equivalent to a disjunction of $(\varphi \wedge t < \lfloor z \rfloor + c)$ and of $\varphi \wedge z > t > \lfloor z \rfloor + c$. The second disjunct is equivalent to a formula with one local component. Hence, $\exists z(\varphi \wedge t > \lfloor z \rfloor + c)$ is equivalent to a Boolean combination of simple formulas by case 1. The first disjunct has the desirable property that the minimal term of T_1 is $\lfloor z \rfloor + c$.

Next, φ is equivalent to $\psi(z_0, z_1, \ldots, z_m)\sigma$ where ψ is an $\overrightarrow{\exists}\forall$-formula, $\sigma(z_0) = t$, $\sigma(z_1) = \lfloor z \rfloor + c$ and $\sigma(z_i) \in T_1$ for $i > 1$.

Hence, φ is equivalent to $\psi_1(z_0, z_1)\sigma \wedge \psi_2(z_1, \ldots z_m)\sigma$ for $\overrightarrow{\exists}\forall$-formulas ψ_1 and ψ_2.

Since $\psi_2\sigma$ contains only one free variable, it is equivalent to an *MTL* formula A, by Proposition 4.7.

Hence, $\exists z\varphi$ is equivalent to $(\exists z_1\theta(z_0, z_1))\{t/z_0\}$, where $\theta(z_0, z_1)$ expresses the following:

1. $\psi_1(z_0, z_1) \wedge \mathsf{Int}(z_1)$ and
2. A holds somewhere in the interval $[z_1 - c, z_1 - c + 1)$, i.e., $\Diamond_{=-c}(A \vee \Diamond_{(0,1)} A)(z_1)$.

Therefore, $\theta(z_0, z_1)$ is equivalent to an $FO[<]$ formula (in the canonical *MTL*-expansions). Hence, $\exists z_1\theta(z_0, z_1)$ is equivalent to a disjunction of $\overrightarrow{\exists}\forall$-formulas and $\exists z\varphi$ is equivalent to a disjunction of simple formulas.

Subcase B. By standard logical equivalences this subcase is reducible to Case 1 considered in Section 7.1. Below are the details. T_1 is a local component (w.r.t. Ord_φ). Therefore, there is $N \in \mathbb{N}$ such that $\lfloor u \rfloor - N$ is less than all elements in T_1.

φ is equivalent to a disjunction of $\varphi_1 := t \geq \lfloor u \rfloor - N \wedge \varphi$ and $\varphi_2 := t < \lfloor u \rfloor - N \wedge \varphi$.

Hence, $\exists z\varphi$ is equivalent to $(\exists z\varphi_1) \vee (\exists z\varphi_2)$.

We are going to show that both disjuncts are equivalent to a Boolean combination of simple formulas; hence, so is $\exists z\varphi$.

Indeed, Ord_{φ_1} has only one local component (since $t < t'$ for every $t' \in T_1$ and $u \in \text{FreeVar}(T_1)$). Therefore, by Case 1, $\exists z\varphi_1$ is equivalent to a Boolean combination of simple formulas.

φ_2 is equivalent to $\psi(z_0, z_1, \ldots, z_m)\sigma$ where (1) ψ is an $\overrightarrow{\exists}\forall$-formula such that $Ord_\psi \rightarrow z_0 < z_1 \wedge \bigwedge_{i=2}^{m}(z_1 < z_i)$ and (2) $\sigma(z_0) = t$, $\sigma(z_1) = \lfloor u \rfloor - N$ and $\sigma(z_i) \in T_1$ for $i > 1$.

Therefore, ψ is equivalent to $\psi_1(z_0, z_1) \wedge \psi_2(z_1, \ldots z_m)$, where ψ_i are $\overrightarrow{\exists}\forall$-formulas. Now, $\psi\sigma$ is equivalent to $\psi_1\sigma \wedge \psi_2\sigma$ and (a) $z \notin \mathrm{FreeVar}(\psi_1\sigma)$ and (b) $\psi_2\sigma$ has only one local component. Hence, $\exists z\varphi_2$ is equivalent to a conjunction of simple formulas $\psi_1\sigma$ and of $\exists z(\psi_2\sigma)$ which is equivalent, by case 1, to a Boolean combination of simple formulas.

This completes the proof of subcase B of case 2.

7.3 Case 3

First, similarly to Case 2, we can reduce this case to a version with three local components, where the minimal and the maximal components have one term.

Next, let the local components of Ord_φ be T_0, T_1 and T_2, where $T_0 = \{t_0\}$ and $T_2 = \{t_2\}$ and $z \in \mathrm{FreeVar}(T_1)$. Consider two subcases:

Subcase A. z is the only free variable in T_1.

Subcase B. There is $u \in \mathrm{FreeVar}(\varphi)$ such that u is not z.

In subcase A, we can further assume that the least term of T_1 is $\lfloor z \rfloor + c$ and the greatest is $\lfloor z \rfloor + d$ for some $c, d \in \mathbb{N}$.

Then, φ is equivalent to $\psi(z_0, z_1, \ldots z_m, z_{m+1})\sigma$, where ψ is an $\overrightarrow{\exists}\forall$-formula and

- $Ord_\psi \rightarrow (z_0 < z_1 < z_m < z_{m+1} \wedge \bigwedge_{i=2}^{m-1}(z_1 < z_i < z_m)$ and
- $\sigma(z_0) = t_0$, $\sigma(z_{m+1}) = t_2$, and $\sigma(z_1) = \lfloor z \rfloor + c$, $\sigma(z_m) = \lfloor z \rfloor + d$ are integers, and $\sigma(z_i) \in T_1$ for $i = 2, \ldots, m-1$.

Hence, ψ is equivalent to a conjunction $\psi_1(z_0, z_1) \wedge \psi_2(z_1, \ldots z_m) \wedge \psi_3(z_m, z_{m+1})$ of $\overrightarrow{\exists}\forall$ formulas.

The only free variable in $\psi_2\sigma$ is z, and therefore, by Proposition 4.7, $\psi_2\sigma$ is equivalent to $A(z)$, where A is an atomic predicate (in the canonical *MTL*-expansion).

Therefore, $\exists z\varphi$ is equivalent to $(\exists z_1 z_m \theta(z_0, z_1, z_m, z_{m+1}))\{t_0/z_0, t_2/z_{m+1}\}$, where $\theta(z_0, z_1, z_2, z_3)$ expresses the following:

1. $\psi_1(z_0, z_1) \wedge \mathsf{Int}(z_1) \wedge \mathsf{Int}(z_m) \wedge \psi_3(z_m, z_{m+1})$ and
2. $\mathsf{Int}(z_1) \wedge \mathsf{Int}(z_m) \wedge z_m = z_1 + d - c$ and
3. A holds somewhere in the interval $[z_1 - c, z_1 - c + 1)$, i.e., $\Diamond_{=-c}(A \vee \Diamond_{(0,1)} A)(z_1)$.

The second item states: "there are $d - c - 1$ integer points in (z_1, z_2) and $\mathsf{Int}(z_1) \wedge \mathsf{Int}(z_2)$," and it is expressible by a $FO[<]$ formula over $\mathbb{R}_{\mathbb{Z}}$. Therefore, $\theta(z_0, z_1, z_m, z_{m+1})$ is equivalent to a $FO[<]$.

Hence, $\exists z_1 z_m \theta$ is equivalent to a disjunction of $\overrightarrow{\exists}\forall$-formulas and $\exists z\varphi$ is equivalent to a disjunction of simple formulas.

By standard logical equivalences, subcase B is reducible to case 1 or case 2. We skip the details.

8 Conclusion and Related Works

A major result concerning temporal logics is Kamp's theorem [14,5] which implies that the temporal logic with two modalities "P Until Q" and "P Since Q" is expressively equivalent to First-Order Monadic Logic of Order ($FO[<]$) over the standard linear time intended models - the Naturals $\langle \mathbb{N}, < \rangle$ for discrete time and the Reals $\langle \mathbb{R}, < \rangle$ for continuous time.

$FO[<]$ is a fundamental formalism; however, $FO[<]$ cannot express over the reals properties like "P holds exactly after one unit of time." Such local metric properties are easily expressible in $FO[<, +1]$ - the extension of $FO[<]$ by $+1$ function. In contrast to the Kamp theorem, no temporal logic with a finite set of modalities is expressively equivalent over the reals to $FO[<, +1]$ [10].

Actually, in [10] a much stronger result is proved. Recall that counting modalities $C_n(P)$ - "P will hold at least at n points within the next unit of time" are defined by $FO[<, +1]$ formulas. In [10], we proved that no temporal logic with a finite *or infinite* family of modalities which are defined by $FO[<, +1]$ formulas with bounded quantifier depth can express over $\mathbb{R}$ all the modalities $C_n(P)$.

$FO[<, +1]$ lacks expressive power to specify the natural global metric property "the current moment is an integer."

Surprisingly, our main result states that $FO[<, +1]$ has the same expressive power as the temporal logic MTL (with only six modalities) over the expansion of the reals by a monadic predicate "x is an integer." We could use alternative notations. Let $FO[<, +1, Int]$ be the expansion of the monadic first-order logic by a unary function symbol $+1$ and a unary relation symbol Int interpreted over the reals as the plus one function and as the set of integers. Let $MTL[Int]$ be obtained from MTL by adding modality Int defined by $\mathcal{M}, a \models Int$ iff a is an integer. Our main result - Theorem 3.2 - can be rephrased as $FO[<, +1, Int]$ is expressively equivalent over $\mathbb{R}$ to $MTL[Int]$. (Technically, it is slightly more convenient in our proof to treat Int as a monadic predicate and not as a modality.)

Our proof uses some techniques from [17], where we proved a result that can be viewed as an extension of Kamp's theorem to metric logics over bounded real time domains: for every $N \in \mathbb{N}$, $FO[<, +1]$ and MTL are expressively equivalent over the class of real intervals of length $< N$. Note that for every MTL formula A there is $FO[<, +1]$ formula ψ_A which is equivalent to A over all real time intervals[2]. For every $FO[<, +1]$ formula ψ with one free variable, we constructed in [17] an MTL formula A_ψ^N which is equivalent to ψ over the real intervals of length $< N$; MTL formulas $A_\psi^{N_1}$ and $A_\psi^{N_2}$ are different for $N_1 \neq N_2$. It can be proved that there is no uniform (independent from N) translation from $FO[<, +1]$ to an equivalent (over $[0, N)$ interval) MTL formula. Finally, note that for every $N \in \mathbb{N}$, there is a $FO[<, +1]$ formula $int_N(t)$ which defines the set of integers in the interval $[0, N)$. Indeed, let $\alpha_0(t)$ be $\forall t'(t \leq t)$ and $\alpha_{i+1}(t) := \exists t'(\alpha_i(t') \land t = t' + 1)$ for $i < N - 1$. The unique element which satisfies $\alpha_i(t)$ in $[0, N)$ is i; hence, $int_N(t)$ can be defined as $\vee_{i=0}^{N-1} \alpha_i(t)$. Therefore, the expansion

[2] Formally, $FO[<, +1]$ over bounded intervals uses a binary relation "x at distance one from y" instead of $+1$ function.

of interval $[0, N)$ by a monadic predicate "x is an integer" does not increase the expressive power of $FO[<, +1]$.

Our results were obtained in 2012, independently of the result of Paul Hunter [12] which states that the temporal logic $MTLC$ which in addition to MTL modalities has all counting modalities is expressively equivalent to $FO[<, +1]$ (without the need for the additional unary predicate for the integers). Though $MTLC$ has infinitely many modalities, Hunter's result implies the main result of this paper, since one can express the counting modalities in MTL, using the monadic predicate for the integers, as shown in the last paragraph of Section 2. On the other hand, Hunter's result can be proved by a minor modification of our proof. In particular, Propositions 4.7 and 4.8 hold when MTL is replaced by $MTLC$ and $\mathbb{R}_{\mathbb{Z}}$ is replaced by $\mathbb{R}$.

The proof techniques of this paper and of [12] - though possessing common elements - are quite different.

In [13], the logic $FO[<, +\mathbb{Q}]$ was introduced. This logic adds to $FO[<, +1]$ an infinite family of unary function symbols: $+q$ for each rational q. Every fragment of $FO[<, +\mathbb{Q}]$ which uses only finitely many $+q$ functions is strictly less expressive than $FO[<, +\mathbb{Q}]$. Therefore, no temporal logic with finitely many modalities is expressively equivalent to $FO[<, +\mathbb{Q}]$. The main result of [13] states that $FO[<, +\mathbb{Q}]$ is expressively equivalent to $MTL_{\mathbb{Q}}$, where $MTL_{\mathbb{Q}}$ is a temporal logic obtained from MTL by adding modality $\diamond_{=q}$, for every rational q, and modalities $\diamond_{(0,q)}$ and $\diamond_{(-q,0)}$ for every positive rational q. Recall that a counting modality $C_2(P)$ - "P will hold at least twice within the next unit of time" is definable by an $FO[<, +1]$ formula $\psi(z_0) := \exists x_1 \exists x_2 (z_0 < x_1 < x_2 < z_0 + 1) \land P(x_1) \land P(x_2)$, and $C_2(P)$ is not expressible in MTL over the reals [10]. Let us illustrate how $C_2(P)$ was expressed in $MTL_{\mathbb{Q}}$ using fractional constants [13]. The idea is to consider three cases according to whether P is true twice in the interval $(z_0, z_0 + \frac{1}{2}]$, twice in the interval $[z_0 + \frac{1}{2}, z_0 + 1)$ or once in $(z_0, z_0 + \frac{1}{2})$ and $(z_0 + \frac{1}{2}), z_0 + 1)$. The last case is equivalent to an $MTL_{\mathbb{Q}}$ formula $\diamond_{(0,\frac{1}{2})} P \land \diamond_{=1}(\diamond_{(-\frac{1}{2},0)} P)$; an $MTL_{\mathbb{Q}}$ formula $\diamond_{(0,\frac{1}{2})}(P \land \diamond_{(0,\frac{1}{2})} P)$ holds in the first case and implies $C_2(P)$; an $MTL_{\mathbb{Q}}$ formula $\diamond_{=1}(\diamond_{(-\frac{1}{2},0)}(P \land \diamond_{(-\frac{1}{2},0)} P)$ holds in the second case and also implies $C_2(P)$. Therefore, $C_2(P)$ is equivalent to a disjunction of these three formulas.

Note that every predicate logic is expressively equivalent to a modal logic with an infinite set of modalities. For every predicate formula $\psi(t)$ with one free first-order variable, one can consider the modality with a truth table defined by ψ. The modal logic with all these modalities and the predicate logic are expressively equivalent. Hence, if a predicate logic is expressively equivalent to no temporal logic with a finite set of modalities, one can try to find an equivalent temporal logic with an infinitely many modalities which are "natural" or "simple" in some sense.

Table **??** lists predicate logics and corresponding expressively equivalent temporal logics and summarizes our comparison. Note, thought both $MTLC$ and $MTL_{\mathbb{Q}}$ use infinitely many modalities, all modalities in $MTL_{\mathbb{Q}}$ are defined by $FO[<, +\mathbb{Q}]$ formulas of quantifier depth at most two, while the $MTLC$ modalities cannot be defined in a fragment of $FO[<, +1]$ of bounded quantifier depth.

Predicate logic	Temporal Logic	Models	Cardinality of the set of modalities	Reference
$FO[<]$	LTL	All Dedekind complete linear orders	finite	[14]
$FO[<,+1]$	MTL	intervals $[0,N)$	finite	[17]
$FO[<,+1]$	MTL	$\mathbb{R}_{\mathbb{Z}}$	finite	this paper
$FO[<,+1,Int]$	$MTL[Int]$	$\mathbb{R}$	finite	this paper
$FO[<,+1]$	$MTLC$	$\mathbb{R}$	infinite	[12]
$FO[<,+\mathbb{Q}]$	$MTL_{\mathbb{Q}}$	$\mathbb{R}$	infinite	[13]

Table 1: Predicate logics and corresponding expressively equivalent temporal logics

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