The Complexity of Temporal Logic with Until and Since over Ordinals*

Stéphane Demri[1] and Alexander Rabinovich[2]

[1] LSV, ENS Cachan, CNRS, INRIA
demri@lsv.ens-cachan.fr
[2] Tel Aviv University, Ramat Aviv
rabinoa@post.tau.ac.il

Abstract. We consider the temporal logic with since and until modalities. This temporal logic is expressively equivalent over the class of ordinals to first-order logic thanks to Kamp's theorem. We show that it has a PSPACE-complete satisfiability problem over the class of ordinals. Among the consequences of our proof, we show that given the code of some countable ordinal α and a formula, we can decide in PSPACE whether the formula has a model over α. In order to show these results, we introduce a class of simple ordinal automata, as expressive as Büchi ordinal automata. The PSPACE upper bound for the satisfiability problem of the temporal logic is obtained through a reduction to the nonemptiness problem for the simple ordinal automata.

1 Introduction

The main models for time are $\langle \mathbb{N}, < \rangle$, the natural numbers as a model of *discrete time* and $\langle \mathbb{R}, < \rangle$, the real line as the model for *continuous time*. These two models are called the *canonical models of time*. A major result concerning linear-time temporal logics is Kamp's theorem [Kam68, GHR94] which says that LTL(U, S), the temporal logic having "*Until*" and "*Since*" as only modalities, is expressively complete for first-order monadic logic of order over the class of Dedekind complete linear orders. The canonical models of time are indeed Dedekind-complete. Another important class of Dedekind-complete orders is the class of ordinals, see e.g. an axiomatization of LTL(U, S) over ordinals in [Ven93].
In this paper the satisfiability problem for the temporal logic with until and since modalities over the class of ordinals is investigated. Our main results are the following: the satisfiability problem for LTL(U, S) over the class of ordinals is PSPACE-complete and a formula ϕ in LTL(U, S) has some α-model for some ordinal α iff it has an β-model for some $\beta < \omega^{|\phi|+2}$ where $|\phi|$ is the size of ϕ.

In order to prove these results we use an automata-based approach [VW94]. In Section 3, we introduce a new class of ordinal automata which we call simple ordinal automata. These automata are expressive equivalent to Büchi automata

* Partially supported by an invited professorship from ENS de Cachan and project AutoMathA (ESF).

N. Dershowitz and A. Voronkov (Eds.): LPAR 2007, LNAI 4790, pp. 531–545, 2007.

over ordinals [BS73]. However, the locations and the transition relations of these automata have additional structures as in [VW94, Roh97]. In particular, a location is a subset of a base set X. Herein, we provide a translation from formulae in LTL(U,S) into simple ordinal automata that allows to characterize the complexity of the satisfiability problem for LTL(U,S). However, the translation of the formula ϕ into the automaton $\mathcal{A}_\phi$ provides an automaton of exponential size in $|\phi|$ but the base of $\mathcal{A}_\phi$ has a cardinality linear in $|\phi|$.
Section 4 contains our main technical lemmas. We show there that every run in a simple ordinal automaton is equivalent to a short run. Consequently, we establish that a formula $\phi \in$ LTL(U,S) has an α-model iff it has a model of length $\mathrm{trunc}_{|\phi|+2}(\alpha)$ where $\mathrm{trunc}_{|\phi|+2}(\alpha)$ is a truncated part of α strictly less than $\omega^{|\phi|+2} \times 2$ (see the definition of truncation in Section 4).
In Section 5 we present two algorithms to solve the nonemptiness problem for simple ordinal automata. The first one runs in (simple) exponential time and does not take advantage of the short run property. The second algorithm runs in polynomial space and the short run property plays the main role in its design and its correctness proof.

In Section 6 we investigate several variants of the satisfiability problem and show that all of them are PSPACE-complete. Section 7 compares our results with related works. The satisfiability problem for LTL(U,S) over ω-models is PSPACE-complete [SC85]. Reynolds [Rey03, Rey] proved that the satisfiability problem for LTL(U,S) over the reals is PSPACE-complete. The proofs in [Rey03, Rey] are non trivial and difficult to grasp and it is therefore difficult to compare our proof technique with those of [Rey03, Rey] even though we believe cross-fertilization would be fruitful. We provide uniform proofs and we improve upper bounds for decision problems considered in [Cac06, DN07, Roh97]. We also compare our results and techniques with Rohde's thesis [Roh97]. Finally we show how our results entail most of the results from [DN07] and we solve some open problems stated there.

2 Temporal Logic with Until and Since

The formulae of LTL(U,S) are defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \mathsf{U} \phi_2 \mid \phi_1 \mathsf{S} \phi_2$$

where $p \in$ PROP for some set PROP of atomic propositions. Given a formula ϕ in LTL(U,S), we write $sub(\phi)$ to denote the set of subformulae of ϕ or their negation assuming that $\neg\neg\psi$ is identified with ψ. The size of ϕ is defined as the cardinality of $sub(\phi)$ and therefore implicitly we encode formulae as DAGs. This feature will be helpful for defining translations that increase polynomially the number of subformulae but for which the tree representation might suffer an exponential blow-up. We use the following abbreviations $\mathsf{G}\phi = \phi \wedge \neg(\top \mathsf{U} \neg\phi)$ and $\mathsf{F}\phi = \neg\mathsf{G}\neg\phi$ that do cause only a polynomial increase in size.

The satisfaction relation is inductively defined below where σ is an α-model of the form $\alpha \to \mathcal{P}(\mathrm{PROP})$ for some ordinal $\alpha > 0$ ($\beta < \alpha$):

- $\sigma, \beta \models p$ iff $p \in \sigma(\beta)$,
- $\sigma, \beta \models \neg\phi$ iff not $\sigma, \beta \models \phi$, $\sigma, \beta \models \phi_1 \wedge \phi_2$ iff $\sigma, \beta \models \phi_1$ and $\sigma, \beta \models \phi_2$,
- $\sigma, \beta \models \phi_1 \mathsf{U} \phi_2$ iff there is $\gamma \in (\beta, \alpha)$ such that $\sigma, \gamma \models \phi_2$ and for every $\gamma' \in (\beta, \gamma)$, we have $\sigma, \gamma' \models \phi_1$,
- $\sigma, \beta \models \phi_1 \mathsf{S} \phi_2$ iff there is $\gamma \in [0, \beta)$ such that $\sigma, \gamma \models \phi_2$ and for every $\gamma' \in (\gamma, \beta)$, we have $\sigma, \gamma' \models \phi_1$.

The satisfiability problem for LTL(U, S) consists in determining, given a formula ϕ, whether there is a model σ such that $\sigma, 0 \models \phi$.

We recall that well orders are particular cases of Dedekind complete linear orders. Indeed, a chain is Dedekind complete iff every non-empty bounded subset has a least upper bound. Kamp's theorem applies herein.

Theorem 1. *(I) [Kam68]* LTL(U, S) *over the class of ordinals is as expressive as the first-order logic over the class of structures* $\langle \alpha, < \rangle$ *where* α *is an ordinal. (II) [BS73] The satisfiability problem for* LTL(U, S) *over the class of ordinals is decidable.*

Hence, LTL(U, S) is a fundamental logic to be studied. Moreover, another key result is the PSPACE-completeness of LTL(U, S) restricted to ω-models [SC85].

We recall below a definability result that will be used in Sections 6 and 7. Ordinals strictly below ω^ω can be defined in LTL(U, S) with the truth constant $\top$ (no propositional variable).

Lemma 2. *Given an ordinal* $0 < \alpha = \omega^{k_1} a_{k_1} + \cdots \omega^{k_m} a_{k_m} < \omega^\omega$ *with* $k_1 > \ldots > k_m \geq 0$, $a_{k_1}, \ldots a_{k_m} > 0$, *there is a formula* def_α *in* LTL(U, S) *of linear size in* $\Sigma_i(k_i \times a_{k_i})$ *such that for any model* σ, *we have* $\sigma, 0 \models \mathrm{def}_\alpha$ *iff* σ *is of length* α.

3 Translation from Formulae to Simple Ordinal Automata

In Section 3.1, we introduce a new class of ordinal automata which we call simple ordinal automata. These automata are expressive equivalent to Büchi automata over ordinals [BS73]. However, the locations and the transition relations of these automata have additional structures. In Section 3.2, we provide a translation from LTL(U, S) into simple ordinal automata which assigns to every formula in LTL(U, S) an automaton that recognizes exactly its models. We borrow the automata-based approach for temporal logics from [VW94, KVW00].

3.1 Simple Ordinal Automata

Definition 3. *A simple ordinal automaton* $\mathcal{A}$ *is a structure* $\langle X, Q, \delta_{next}, \delta_{lim} \rangle$ *such that*

- X *is a finite set (the basis of* $\mathcal{A}$*),* $Q \subseteq \mathcal{P}(X)$ *(the set of locations),*
- $\delta_{next} \subseteq Q \times Q$ *is the next-step transition relation,*
- $\delta_{lim} \subseteq \mathcal{P}(X) \times Q$ *is the limit transition relation.*

$\mathcal{A}$ can be viewed as a finite directed graph whose set of nodes is structured. Given a simple ordinal automaton $\mathcal{A}$, an α-path (or simply a path) is a map $r : \alpha \to Q$ for some $\alpha > 0$ such that

- for every $\beta + 1 < \alpha$, $\langle r(\beta), r(\beta + 1)\rangle \in \delta_{next}$,
- for every limit ordinal $\beta < \alpha$, $\langle \text{always}(r, \beta), r(\beta)\rangle \in \delta_{lim}$ where

$$\text{always}(r, \beta) \stackrel{\text{def}}{=} \{a \in X : \exists\ \gamma < \beta \text{ such that } a \in \cap_{\gamma' \in (\gamma, \beta)}\ r(\gamma')\}.$$

The set always(r, β) contains exactly the elements of the basis that belong to every location from some $\gamma < \beta$ until β. We sometimes write always(r) instead of always(r, α) when α is a limit ordinal or always(r) instead of always$(r, \alpha - 1)$ when α is a successor ordinal and $\alpha - 1$ is a limit ordinal.

Given an α-path r, for $\beta, \beta' < \alpha$ we write

- $r_{\geq \beta}$ to denote the restriction of r to positions greater or equal to β,
- $r_{\leq \beta}$ to denote the restriction of r to positions less or equal to β,
- $r_{[\beta, \beta')}$ to denote the restriction of r to positions in $[\beta, \beta')$ (half-open interval).

A simple ordinal automaton with acceptance conditions is a structure of the form $\langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim}\rangle$ where

- $I \subseteq Q$ is the set of initial locations,
- $F \subseteq Q$ is the set of final locations for accepting runs whose length is some successor ordinal,
- $\mathcal{F} \subseteq \mathcal{P}(X)$ encodes the accepting condition for runs whose length is some limit ordinal.

Given a simple ordinal automaton with acceptance conditions, an accepting run is a path $r : \alpha \to Q$ such that $r(0) \in I$ and

- if α is a successor ordinal, then $r(\alpha - 1) \in F$,
- otherwise $\{a \in X : \exists\ \gamma < \alpha \text{ such that } a \in \cap_{\gamma' \in (\gamma, \alpha)}\ r(\gamma')\} \in \mathcal{F}$.

The nonemptiness problem for simple ordinal automata consists in checking whether $\mathcal{A}$ has an accepting run.

Our current definition for simple ordinal automata does not make them language acceptors since they have no alphabet. It is possible to add in the definition a finite alphabet Σ and to define the next-step transition relation as a subset of $Q \times \Sigma \times Q$. If we do so, our model of automata can recognize the same languages as the usual ordinal automata with Muller acceptance conditions in the limit transitions. The proof is not very difficult. Additionally, the current definition can be viewed as the case either when the alphabet is a singleton or when the read letter is encoded in the locations through the dedicated elements of the basis. This second reading will be in fact used implicitly in the sequel.

We also write $\mathcal{A}$ to denote either a simple ordinal automaton or its extension with acceptance conditions.

3.2 Translation from LTL(U, S) Formulae to Simple Ordinal Automata

As usual, a set Y is a maximally Boolean consistent subset of $sub(\phi)$ when the following conditions are satisfied: for every $\psi \in sub(\phi)$, $\neg\psi \in Y$ iff $\psi \notin Y$ and for every $\psi_1 \wedge \psi_2 \in sub(\phi)$, $\psi_1 \wedge \psi_2 \in Y$ iff $\psi_1, \psi_2 \in Y$. Given a formula ϕ, the simple ordinal automaton $\mathcal{A}_\phi = \langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim}\rangle$ is defined as follows:

- $X = sub(\phi)$.
- Q is the set of maximally Boolean consistent subsets of $sub(\phi)$.
- I is the set of locations that contain ϕ and no since formulae.
- F is the set of locations with no elements of the form $\psi_1 \mathsf{U} \psi_2$.
- $\mathcal{F}$ is the set of sets Y such that $\{\psi_1, \neg\psi_2, \psi_1 \mathsf{U} \psi_2\} \not\subseteq Y$, for every $\psi_1 \mathsf{U} \psi_2 \in X$.
- For all $q, q' \in Q$, $\langle q, q'\rangle \in \delta_{next}$ iff the conditions below are satisfied:
  (next$_\mathsf{U}$) for $\psi_1 \mathsf{U} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{U} \psi_2 \in q$ iff either $\psi_2 \in q'$ or $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q'$,
  (next$_\mathsf{S}$) for $\psi_1 \mathsf{S} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{S} \psi_2 \in q'$ iff either $\psi_2 \in q$ or $\psi_1, \psi_1 \mathsf{S} \psi_2 \in q$.
- For all $Y \subseteq X$ and $q \in Q$, $\langle Y, q\rangle \in \delta_{lim}$ iff the conditions below are satisfied:
  (lim$_\mathsf{U}$1) if $\psi_1, \neg\psi_2, \psi_1 \mathsf{U} \psi_2 \in Y$, then either $\psi_2 \in q$ or $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q$,
  (lim$_\mathsf{U}$2) if $\psi_1, \psi_1 \mathsf{U} \psi_2 \in q$ and $\psi_1 \in Y$, then $\psi_1 \mathsf{U} \psi_2 \in Y$,
  (lim$_\mathsf{U}$3) if $\psi_1 \in Y$, $\psi_2 \in q$ and $\psi_1 \mathsf{U} \psi_2$ is in the basis X, then $\psi_1 \mathsf{U} \psi_2 \in Y$,
  (lim$_\mathsf{S}$) for every $\psi_1 \mathsf{S} \psi_2 \in sub(\phi)$, $\psi_1 \mathsf{S} \psi_2 \in q$ iff ($\psi_1 \in Y$ and $\psi_1 \mathsf{S} \psi_2 \in Y$).

Even though the conditions above can easily be shown correct, at this stage it might sound mysterious how they have been made up. For some of them, their justification comes with the proof of Lemma 4.

Let σ be an α-model and ϕ be a formula in LTL(U, S). The Hintikka sequence for σ and ϕ is an α-sequence $H^{\sigma,\phi}$ defined as follows: for every $\beta < \alpha$,

$$H^{\sigma,\phi}(\beta) \stackrel{\text{def}}{=} \{\psi \in sub(\phi) \ : \ \sigma, \beta \models \psi\}.$$

Now we can state the correctness lemma.

Lemma 4

(I) *If $\sigma, 0 \models \phi$, then $H^{\sigma,\phi}$ is an accepting run of $\mathcal{A}_\phi$.*
(II) *If r is an accepting run of $\mathcal{A}_\phi$, then there is a model σ such that $\sigma, 0 \models \phi$ and r is the Hintikka sequence for σ and ϕ.*
(III) *ϕ is satisfiable iff $\mathcal{A}_\phi$ has an accepting run.*

4 Short Run Properties

Let $\mathcal{A}$ be a simple ordinal automaton and Y be a subset of its basis. Y is said to be present in $\mathcal{A}$ iff there is a limit transition of the form $\langle Y, q\rangle$ in $\mathcal{A}$. Given a set Y present in $\mathcal{A}$, its weight, noted weight(Y), is the maximal l such that $Y_1 \subset Y_2 \subset \cdots \subset Y_l$ is a sequence of present subsets in $\mathcal{A}$ and $Y_1 = Y$. Obviously, weight(Y) $\leq |X| + 1$.

Given a path $r : \alpha \to Q$ in $\mathcal{A}$ with $\alpha \geq \omega+1$, its weight, noted weight(r), is the maximal value in the set $\{\text{weight}(\text{always}(r, \beta)) : \beta < \alpha,\ \beta \text{ is a limit ordinal}\}$.

By convention, if a path is of length strictly less than $\omega + 1$, then its weight is zero (no limit transition is fired). Furthermore, we write exists(r) to denote the set $\bigcup_{\beta<\alpha} r(\beta)$ and all(r) to denote the set $\bigcap_{\beta<\alpha} r(\beta)$. For example, all($r$) corresponds to the set of elements from the basis that are present in all locations of the run r. Let r, r' be two paths of respective length $\alpha+1$ and $\alpha'+1$, we say that they are congruent (noted $r \sim r'$) iff the conditions below are meet: $r(0) = r'(0)$, $r(\alpha) = r'(\alpha')$, exists(r) = exists(r') and all(r) = all(r'). We can easily adapt the congruence relation to runs r and r' of length some limit ordinal by requiring that always(r) = always(r') instead of the condition on final locations for runs of length some successor ordinal.

Let r_1 be a path of length $\alpha + 1$ and r_2 be a path of length β such that $r_1(\alpha) = r_2(0)$. The concatenation $r_1 \cdot r_2$ is the path r of length $\alpha + \beta$ such that for $\gamma \in [0, \alpha]$, $r(\gamma) = r_1(\gamma)$ and for $\gamma \in [0, \beta)$, $r(\alpha+\gamma) = r_2(\gamma)$. For every ordinal α, the concatenation of α-sequences of paths is defined similarly. The relation $\sim$ can be viewed as a congruence for the concatenation operation on paths.

Lemma 5

(I) *Let $r \cdot r_0 \cdot r'$, r_1 be two paths such that $r_0 \sim r_1$. Then, $r \cdot r_1 \cdot r'$ is a path that is congruent to $r \cdot r_0 \cdot r'$.*

(II) *Let $r_0^0, r_0^1, r_0^2, \ldots$ and $r_1^0, r_1^1, r_1^2, \ldots$ be two ω-sequences of pairwise consecutive paths such that for $i \geq 0$, $r_0^i \sim r_1^i$ and their length is a successor ordinal. If $r \cdot r_0^0 \cdot r_0^1 \cdot r_0^2 \cdot \ldots \cdot r'$ is a path, then it is congruent to $r \cdot (r_1^0 \cdot r_1^1 \cdot r_1^2 \cdot \ldots) \cdot r'$.*

The proof of the above lemma is by an easy verification.

Lemma 6. *Let $r : \alpha \to Q$ be a path in $\mathcal{A}$. Then, there is a path $r' : \alpha' \to Q$ such that $\alpha' < \omega^{\mathrm{weight}(r)+1}$ and $r \sim r'$.*

Lemma 6 states a crucial property for most of complexity results established in the sequel. Indeed, for usual ordinal automata, it is not possible to get this polynomial bound as an exponent of ω for the length of the short paths. Actually, the exponent is linear in the cardinal of its basis and can be logarithmic in the number of locations for large automata. By combination of Lemma 4 and Lemma 6, we obtain the following interesting result.

Corollary 7. *If ϕ is satisfiable, then ϕ has an α-model with $\alpha < \omega^{|\phi|+2}$.*

This can be still be refined a little more by observing that for each $\omega^i \times a$ occurring in the Cantor normal form of the length of a small model of ϕ (strictly less than $\omega^{|\phi|+2}$), a is bounded by $2^{|\phi|-1}$ since the cardinal of the set of locations of $\mathcal{A}_\phi$ is bounded by $2^{|\phi|-1}$.

For $n \in \mathbb{N}$, let trunc_n be the function that assigns to every ordinal $\alpha > 0$ an ordinal in $(0, \omega^n 2)$ as follows. α can be written in the form $\alpha = \omega^n \gamma + \beta$ with $\beta \in [0, \omega^n)$. Then $\mathrm{trunc}_n(\alpha) = \omega^n \times min(\gamma, 1) + \beta$.

Lemma 8. *Let $\mathcal{A}$ be a simple ordinal automaton.*

(I) *If r is a path of length $\omega^{\mathrm{weight}(r)+1} \times \alpha$ for some countable ordinal $\alpha > 0$, then there is a path r' of length $\omega^{\mathrm{weight}(r)+1}$ such that $r \sim r'$.*

(II) *If a path r has length $\omega^{\mathrm{weight}(r)+1}$, then for every ordinal $\alpha > 0$, there is a path r' of length $\omega^{\mathrm{weight}(r)+1} \times \alpha$ such that $r \sim r'$.*

(III) *If r is a path of length α and $\beta \approx_{|X|+1} \alpha$, then there is a path r' of length β such that $r \sim r'$.*

Only in (I), α is supposed to be countable. Because of the translation from formulae to automata, we can also establish a pumping lemma at the level of formulae.

Lemma 9

(I) *Let $\mathcal{A}$ be a simple ordinal automaton with acceptance conditions and α, β be ordinals such that $\alpha \approx_{|X|+1} \beta$. Then, $\mathcal{A}$ has an accepting run of length α iff $\mathcal{A}$ has an accepting run of length β.*

(II) *Let ϕ be a formula in $\mathrm{LTL}(\mathsf{U}, \mathsf{S})$ and α, β be ordinals such that $\alpha \approx_{|\phi|+2} \beta$. Then ϕ has an α-model iff ϕ has a β-model.*

Proof. (I) Direct consequence of Lemma 6 and Lemma 8 since accepting runs can be viewed as paths.

(II) By Lemma 4, ϕ has an α-model iff $\mathcal{A}_\phi$ has an accepting run r of length α. Since $|\phi| + 1$ bounds the weight of any path in $\mathcal{A}_\phi$ and by (I), we get that $\mathcal{A}_\phi$ has an accepting run r of length α iff $\mathcal{A}_\phi$ has an accepting run r of length β. Equivalently, ϕ has a β-model. □

5 Checking Nonemptiness of Simple Ordinal Automata

In this section, we provide algorithms to check whether a simple ordinal automata admits accepting runs. The first one is in EXPTIME. Our optimal algorithm runs in polynomial space in the size of the basis.

Let $\mathcal{A}$ be a simple ordinal automaton $\langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim}\rangle$. We provide below an algorithm to check given $q, q' \in Q$ and $n \in \mathbb{N}$ whether there is path $r : \alpha + 1 \to Q$ such that $r(0) = q$, $r(\alpha) = q'$ and $\alpha < \omega^n$. Given an $(\alpha+1)$-path we write $\mathrm{abs}(r)$ to denote the quadruple $\langle r(0), \mathrm{exists}(r), \mathrm{all}(r), r(\alpha)\rangle$. We define a family of relations containing the quadruples of the form $\mathrm{abs}(r)$. Each relation R_i below is therefore a subset of $R_i \subseteq Q \times \mathcal{P}(X)^2 \times Q$.

- $R_0 = \{\langle q, q \cup q', q \cap q', q'\rangle : \langle q, q'\rangle \in \delta_{next}\}$,
- For $i \in \mathbb{N}$,

$$R'_i = \{\langle q_0, \bigcup_{i=0}^{m} E_i, \bigcap_{i=0}^{m} A_i, q_{m+1}\rangle :$$

$$\exists \langle q_0, E_0, A_0, q_1\rangle R_i \langle q_1, E_1, A_1, q_2\rangle R_i \cdots R_i \langle q_m, E_m, A_m, q_{m+1}\rangle\}$$

- For $i \in \mathbb{N}$, R_{i+1} is defined from R'_i as follows: $\langle q, E, A, q'\rangle \in R_{i+1}$ iff
  - either $\langle q, E, A, q'\rangle \in R'_i$
  - or there exist a limit transition $\langle Y, q'\rangle \in \delta_{lim}$ and a path

$$\langle q_0, E_0, A_0, q_1\rangle R_i \langle q_1, E_1, A_1, q_2\rangle R_i \cdots R_i \langle q_m, E_m, A_m, q_{m+1}\rangle$$

such that
(a) $q_0 = q_{m+1}$, (b) $\bigcap_{i=0}^{m} A_i = Y$, (c) $\langle q, E', A', q_0\rangle \in R'_i$ for some E', A',
(d) $E = (E' \cup q') \cup \bigcup_{i=0}^{m} E_i$, $A = (A' \cap q') \cap \bigcap_{i=0}^{m} A_i$.

Because $R_i \subseteq R_{i+1}$ for all i, for some $N \leq 2^{4\times|X|} + 1$, $R_{N+1} = R_N$. The bound $2^{4\times|X|} + 1$ takes simply into account that $Q \subseteq \mathcal{P}(X)$.

Lemma 10. *(I) If $\langle q, E, A, q'\rangle \in R_n$, then there is an $(\alpha + 1)$-path such that $\mathrm{abs}(r) = \langle q, E, A, q'\rangle$ and $\alpha < \omega^n$. (II) Conversely, let $r : \alpha + 1 \to Q$ be a path such that $\alpha < \omega^n$. Then $\mathrm{abs}(r) \in R'_n$.*

We provide below a first complexity result.

Lemma 11. *The nonemptiness problem for simple ordinal automata with acceptance conditions can be checked in exponential time in $|X|$.*

Proof. Let $\mathcal{A}$ be of the form $\langle X, Q, I, F, \mathcal{F}, \delta_{next}, \delta_{lim}\rangle$. $\mathcal{A}$ has an accepting run iff either (A) there are $q_0 \in I$, $q_f \in F$ and $E, A \subseteq X$ such that $\langle q_0, E, A, q_f\rangle \in R'_n$ for some n or (B) there are $q_0 \in I$, and a run r from q_0 such that always$(r) \in \mathcal{F}$. (A) deals with accepting runs of length some successor ordinal, whereas (B) deals with accepting runs of length some limit ordinal.

By Lemma 6 and Lemma 10(II), in order to check (A), it is sufficient to test for $\langle q_0, E, A, q_f\rangle \in I \times \mathcal{P}(X)^2 \times F$ whether $\langle q_0, E, A, q_f\rangle \in R'_{|X|+2} \subseteq R_{|X|+3}$. Since $|Q|$ is in $\mathcal{O}(2^{|X|})$, computing $R_{|X|+3}$ takes $|X|+3$ steps that requires polynomial time in $|\mathcal{A}|$ and exponential time in $|X|$, we obtain the desired result. Observe that we can take advantage of the fact that computing the transitive closure of a relation and the maximal strongly connected components can be done in polynomial time in the size of the relations.

By Ramsey theorem, (B) is equivalent to the following condition: there are $q \in Q$, $E, E', A \subseteq X$, $A' \in \mathcal{F}$ and runs r_1 and r_2 such that $\mathrm{abs}(r_1) = \langle q_0, E, A, q\rangle$ and $\mathrm{abs}(r_2) = \langle q, E', A', q\rangle$.

Hence. in order to check these, it is enough to check whether there are $q_0 \in I$, $q \in Q$ and $E, A \subseteq X$ such that $\langle q_0, E, A, q\rangle \in R'_{|X|+2}$, $\langle q, E', A', q\rangle \in R'_{|X|+2}$ and $A' \in \mathcal{F}$. This can be done in exponential time as for (A). □

The proof of Lemma 11 mentions Lemma 6 but the exponential time upper bound can be obtained by observing that an exponential number of steps, such as $2^{4\times|X|} + 1$ would provide the same bound in the worst case. As a corollary of Lemma 11, satisfiability for LTL(U, S) is in EXPTIME. Moreover, this can be improved as shown in the proof of Theorem 13 presented in Section 6.

We improve below the bound in Lemma 11.

Theorem 12. *The nonemptiness problem for simple ordinal automata can be checked in polynomial space in $|X|$.*

Proof. Following the proof of Lemma 11, $\mathcal{A}$ has an accepting run iff (A) there are $q_0 \in I$, $q_f \in F$ and $E, A \subseteq X$ such that $\langle q_0, E, A, q_f\rangle \in R_{|X|+3}$ or (B) there are $q_0 \in I$, $q \in Q$ and $E', A' \subseteq X$ such that $\langle q_0, E', A', q\rangle \in R_{|X|+3}$, $\langle q, E', A', q\rangle \in$

$R_{|X|+3}$ and $A' \in \mathcal{F}$. X denotes the basis of $\mathcal{A}$. In order to check (A), the non-deterministic algorithm guesses $q_0 \in I$, $q_f \in F$ and $E, A \subseteq X$ (encoded in polynomial space in $\mathcal{O}(|X|)$ and test whether PATH$(\mathcal{A}, \langle q_0, E, A, q_f\rangle, |X|+3)$ returns true. Condition (B) admits a similar treatment. The non-deterministic algorithm PATH defined below works in polynomial space in $|X|$ assuming that the last argument is polynomial in $|X|$ which is the case with $|X|+3$. Figure 1 contains the definition of the function PATH (some details are omitted).

In (2.), guessing on-the-fly a long sequence means that only two consecutive quadruples are kept in memory at any time. We need a counter to guarantee that $m < 2^{4\times|X|+1}$ and it requires only space in $\mathcal{O}(|X|)$. Moreover, in order to check $E = \bigcup_j E_j$ and $A = \bigcap_j A_j$ we need two auxiliary variables that bookkeep the E_j and A_j so far respectively. Similar techniques are used in (3.) to guarantee that this non-deterministic algorithm requires only polynomial space in $\mathcal{O}(|X|+N)$ (we only need more variables and steps). It is straightforward to show that

PATH$(\mathcal{A}, \langle q, E, A, q'\rangle, N)$

- If $N = 0$ then (if (either $E \neq q \cup q'$ or $A \neq q \cap q'$ or $\langle q, q'\rangle \notin \delta_{next}$) then `abort` else return $\top$);
- If $N > 0$ then go non-deterministically to 1.,2. or 3.
  - **(1.)** Return PATH$(\mathcal{A}, \langle q, E, A, q'\rangle, N-1)$
  - **(2.)** Guess on-the-fly a sequence

$$\langle q_0, E_0, A_0, q_1\rangle, \langle q_1, E_1, A_1, q_2\rangle, \ldots, \langle q_m, E_m, A_m, q_{m+1}\rangle$$

    such that
    - $m < 2^{4\times|X|+1} + 1$,
    - for $0 \leq i \leq m$, PATH$(\mathcal{A}, \langle q_i, E_i, A_i, q_{i+1}\rangle, N-1)$ returns $\top$,
    - $E = \bigcup_j E_j$ and $A = \bigcap_j A_j$
    - $q = q_0$, $q' = q_{m+1}$;
  - **(3.)** We guess here two long sequences:
    - **(3.1)** Guess on-the-fly a sequence

$$\langle q_0, E_0, A_0, q_1\rangle, \langle q_1, E_1, A_1, q_2\rangle, \ldots, \langle q_m, E_m, A_m, q_{m+1}\rangle$$

      such that
      - $m < 2^{4\times|X|+1} + 1$,
      - for $0 \leq i \leq m$, PATH$(\mathcal{A}, \langle q_i, E_i, A_i, q_{i+1}\rangle, N-1)$ returns $\top$,
      - $E' = \bigcup_j E_j$ and $A' = \bigcap_j A_j$;
      - $q_0 = q$;
    - **(3.2)** Guess a limit transition $\langle Y, q'\rangle \in \delta_{lim}$ and on-the-fly a sequence $\langle q'_0, E'_0, A'_0, q'_1\rangle, \langle q'_1, E'_1, A'_1, q'_2\rangle, \ldots, \langle q'_m, E'_{m'}, A'_{m'}, q'_{m'+1}\rangle$ such that
      - $m' < 2^{4\times|X|+1}$,
      - for $0 \leq i \leq m'$, PATH$(\mathcal{A}, \langle q'_i, E'_i, A'_i, q'_{i+1}\rangle, N-1)$ returns $\top$,
      - $E = (E' \cup q'_{m'+1}) \cup \bigcup_j E'_j$,
      - $A = (A' \cap q'_{m'+1}) \cap \bigcap_j A'_j$, $Y = \bigcap_j A'_j$, $q'_0 = q_{m+1}$;
- Return $\top$.

Fig. 1. Algorithm PATH

PATH($\mathcal{A}, \langle q, E, A, q' \rangle, N$) has a computation that returns $\top$ (all the guesses were correct) iff $\langle q, E, A, q' \rangle \in R_N$. Finally Savitch's Theorem allows to conclude that nonemptiness can be checked in deterministic polynomial space in $|X|$. □

6 Complexity of Satisfiability Problems

We establish new complexity results for problems related to LTL(U, S) satisfiability thanks to the intermediate results we have established so far.

6.1 Complexity of LTL(U, S)

Here is the main result of the paper.

Theorem 13. *The satisfiability problem for* LTL(U, S) *over the class of ordinals is* PSPACE*-complete.*

Proof. By Lemma 4, given a formula ϕ in LTL(U, S), there is an automaton $\mathcal{A}_\phi$ whose accepting runs correspond exactly to models of ϕ. In order to check nonemptiness of $\mathcal{A}_\phi$, we do not build it explicitly (as usual) but we run the algorithm from the proof of Theorem 12 and we compute the locations, and transition relations of $\mathcal{A}_\phi$ on demand. Hence, we obtain a polynomial space non-deterministic algorithm since the basis of $\mathcal{A}_\phi$ has a cardinality in $\mathcal{O}(|\phi|)$ and checking whether a subset of X is a location of $\mathcal{A}_\phi$ or $\langle q, q' \rangle \in \delta_{next}$ or $\langle Y, q \rangle \in \delta_{lim}$ can be done in polynomial space in $\mathcal{O}(|\phi|)$. Again by Savitch's Theorem, we get that the satisfiability problem for LTL(U, S) is in PSPACE. The PSPACE lower bound can be easily shown inherited from LTL. □

Thanks to Kamp's theorem, we get the following corollary.

Corollary 14. *Let* LTL($\mathsf{U}, \mathsf{S}, \mathsf{O}_1, \ldots, \mathsf{O}_k$) *be an extension of* LTL(U, S) *with k first-order definable temporal operators. Then the satisfiability problem for the logic* LTL($\mathsf{U}, \mathsf{S}, \mathsf{O}_1, \ldots, \mathsf{O}_k$) *over the class of ordinals is in* PSPACE*.*

Indeed, every formula $\mathsf{O}_i(p_1, \ldots, p_{n_i})$ encoded as a DAG can be translated into an equivalent formula in LTL(U, S) encoded as a DAG over the propositional variables $p_1, \ldots, p_{n_i}$. Since $\mathsf{O}_1, \ldots, \mathsf{O}_k$ and their definition in LTL(U, S) are constant of LTL($\mathsf{U}, \mathsf{S}, \mathsf{O}_1, \ldots, \mathsf{O}_k$) we obtain a translation in polynomial-time (with our definition for the size of formulae).

6.2 A Family of Satisfiability Problems

The satisfiability problem for LTL(U, S) asks for the existence of a model for a given formula. A natural variant of this problem consists in fixing the length of the models in advance as for LTL. The satisfiability problem for LTL(U, S) over α-models, noted SAT(α, LTL(U, S)), is defined as follows: given a formula ϕ in LTL(U, S), is ϕ satisfiable over an α-model? In this subsection we prove that SAT(α, LTL(U, S)) is in PSPACE for every countable ordinal α. First we consider the case of ordinals strictly less than ω^ω. Recall that for every $\alpha < \omega^\omega$ there is a formula def_α in LTL(U, S) such that for every β-model σ, we have $\sigma, 0 \models \mathrm{def}_\alpha$ iff $\beta = \alpha$.

Corollary 15. *For every $\alpha < \omega^\omega$, the problem* $\mathrm{SAT}(\alpha, \mathrm{LTL}(\mathsf{U}, \mathsf{S}))$ *is in* PSPACE.

Proof. ϕ has a α-model iff $\psi = \phi \wedge \mathrm{def}_\alpha$ is satisfiable over the class of ordinals. Thanks to Lemma 2 and Theorem 13, we obtain the PSPACE upper bound. □

Now we consider the case of a countable ordinal $\alpha \geq \omega^\omega$. Let α' be the unique ordinal strictly less than ω^ω such that $\alpha = \omega^\omega \times \gamma + \alpha'$ for some ordinal γ. Note that for every k, $\mathrm{trunc}_k(\alpha) = \mathrm{trunc}_k(\omega^k + \alpha') < \omega^\omega$. By Lemma 9, ϕ has an α-model iff ϕ has a $\alpha_{|\phi|}$-model with $\alpha_{|\phi|} = \mathrm{trunc}_{|\phi|+2}(\alpha) = \mathrm{trunc}_{|\phi|+2}(\omega^{|\phi|+2} + \alpha')$. Hence, ϕ has an α-model iff $\phi \wedge \mathrm{def}_{\alpha_{|\phi|}}$ is satisfiable (over the class of countable ordinals). Since the size of $\mathrm{def}_{\alpha_{|\phi|}}$ is polynomial in the size of ϕ, we derive from Theorem 13 the following result.

Corollary 16. *For every countable $\alpha \geq \omega^\omega$, the problem* $\mathrm{SAT}(\alpha, \mathrm{LTL}(\mathsf{U}, \mathsf{S}))$ *is in* PSPACE.

Corollaries 15, 16 and the arguments similar to the arguments in the proof of Corollary 14 imply the result below.

Theorem 17. *For every finite set $\{\mathsf{O}_1, \ldots, \mathsf{O}_k\}$ of first-order definable temporal operators and every countable ordinal α, the satisfiability problem for the logic* $\mathrm{LTL}(\mathsf{O}_1, \ldots, \mathsf{O}_k)$ *restricted to α-models is in* PSPACE.

Observe that (1) if α is finite, then $\mathrm{SAT}(\alpha, \mathrm{LTL}(\mathsf{O}_1, \ldots, \mathsf{O}_k))$ is NP-complete whereas (2) if α is infinite, then PSPACE-hardness for $\mathrm{SAT}(\alpha, \mathrm{LTL}(\mathsf{U}, \mathsf{S}))$ follows from the PSPACE-completeness of $\mathrm{SAT}(\omega, \mathrm{LTL}(\mathsf{U}, \mathsf{S}))$.

6.3 Uniform Satisfiability

Büchi (see, e.g., [BS73]) has shown that there is a *finite* amount of data concerning any countable ordinal that determines its monadic theory.

Definition 18 (Code of an ordinal). *Let α be a countable ordinal and let m be in $[1, \omega]$.*

1. *Write $\alpha = \omega^m \alpha' + \zeta$ with $\zeta < \omega^m$ (this can be done in a unique way), and let*
$$p_m(\alpha) := \begin{cases} -2 \text{ if } \alpha' = 0 \\ -1 \text{ if } 0 < \alpha' < \omega_1 \end{cases}.$$
2. *If $\zeta \neq 0$, write $\zeta = \sum_{i \leq n} \omega^{n-i} \cdot a_{n-i}$ where $a_i \in \omega$ for $i \leq n$ and $a_n \neq 0$ (this can be done in a unique way), and let $t_m(\alpha) := (a_n, \ldots, a_0)$. If $\zeta = 0$, let $t(\alpha) = -3$.*
3. *The m-*code *of α is the pair $(p_m(\alpha), t_m(\alpha))$.*

The following is implicit in [BS73].

Theorem 19 (Code Theorem). *There is an algorithm that, given a monadic second-order sentence ϕ and the ω-*code *of a countable ordinal α, determines whether $\langle \alpha, < \rangle \models \phi$.*

Lemma 9 can be rephrased as "the $(|\phi| + 2)$-code of an ordinal α determines whether ϕ has a model of length α".

Let $C = (b, a_n, \ldots a_0)$ be an m-code. Its size is defined as $n+a_0+a_1+\cdots+a_n$. It is clear that for $m_1 < m_2$ the m_2-code of an ordinal determines its m_1-code and there is a linear time algorithm, that given m_2-code of an ordinal and $m_1 < m_2$ computes the m_1-code of the ordinal.

The arguments used in the proof of Corollary 16 show the following theorem.

Theorem 20 (Uniform Satisfiability)

(I) *There is a polynomial-space algorithm that, given an* LTL(U, S) *formula* ϕ *and the* ω-code *of a countable ordinal* α*, determines whether* ϕ *has an* α*-model.*

(II) *There is a polynomial-space algorithm that, given an* LTL(U, S) *formula* ϕ *and the* $(|\phi|+2)$*-code of a countable ordinal* α*, determines whether* ϕ *has an* α*-model.*

7 Related Work

In this section, we compare our results with those from the literature. Because of lack of place, we omit to cite works in which models of length higher than ω are considered for formal verification of computer systems, see e.g. [GW94].

7.1 Comparison with Rohde's Thesis

In [Roh97], it is shown that an uniform satisfiability problem for temporal logic with until (and without since) can be solved in exponential-time. The inputs of this problem are a formula in LTL(U) and the representation of an ordinal. The satisfiability problem is also shown in EXPTIME. In order to obtain this upper bound, formulae are shown equivalent to alternating automata and a reduction from alternating automata into a specific subclass of non-deterministic automata is given. Finally, a procedure for testing nonemptiness is provided. Here are the similarities between [Roh97] and our results.

1. We follow an automata-based approach and the class of non-deterministic automata in [Roh97] and ours have a structured set of locations and limit transitions use elements that are true from some position.
2. Existence of α-paths in the automata depends on some truncation of α.
3. The logical decision problems can be solved in exponential-time.

However, our work improves considerably some results from [Roh97].

1. Our temporal logic includes the until and since operators (instead of until only) and it is therefore as expressive as first-order logic.
2. We establish a tight PSPACE upper bound (instead of EXPTIME) thanks to the introduction of a class of simple ordinal automata.
3. Our proofs are much shorter and transparent (instead of the lengthy developments from [Roh97]).

Consequently, the developments from [Roh97] and ours follow the same approach with different definitions for automata, different intermediate lemmas and distinct final complexity bounds. On the other hand, the structure of the whole proof to obtain the main complexity bounds is similar.

7.2 Comparison with Reynolds' Results

Even though the results for linear-time temporal logics from [Rey03, Rey] involve distinct models, our automata-based approach has similarities with these works that uses a different proof method, namely mosaics. Indeed, equivalence classes of the relation $\sim$ between runs of length a successor ordinal roughly correspond to mosaics from [Rey03]. We recall the main results below.

Theorem 21. *(I) The satisfiability problem for the temporal logic with until and since over the reals is* PSPACE*-complete. (II) The satisfiability problem for* LTL(U) *over the class of all linear orders is* PSPACE*-complete.*

The proofs in [Rey03, Rey] are much more involved than our proofs since the orders are more complex than the class of ordinals. Unfortunately, we do not understand these proofs fully and find it difficult to compare to our proof.

7.3 Quantitative Temporal Operators

In this section, we show that the main results from [DN07] are subsumed by the current paper. We also solve an open problem from [Cac06, DN07]. For every fixed countable ordinal $\alpha \leq \omega$, let us introduce the logic LTL($\mathcal{O}_\alpha$) where the set of temporal operators $\mathcal{O}_\alpha$ is defined as follows: $\{\mathsf{X}^\beta : \beta < \omega^\alpha\} \cup \{\mathsf{U}^\beta : \beta \leq \omega^\alpha\}$. The models of LTL($\mathcal{O}_\alpha$) as those of LTL($\mathsf{U}, \mathsf{S}$) and the formulae of LTL($\mathcal{O}_\alpha$) are precisely defined by: $\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathsf{X}^\beta\phi \mid \phi_1 \mathsf{U}^{\beta'} \phi_2$. The size of a formula ϕ is the number of subformulae occurring in ϕ plus the sum of all the natural numbers occurring in ϕ either as a coefficient or as an exponent of ω. The satisfaction relation is inductively defined below where σ is a model for LTL($\mathcal{O}_\alpha$) (we omit the obvious clauses):

- $\sigma, \beta \models \mathsf{X}^{\beta'}\phi$ iff $\beta + \beta'$ is a position of σ and $\sigma, \beta + \beta' \models \phi$,
- $\sigma, \beta \models \phi_1 \mathsf{U}^{\beta'} \phi_2$ iff there is $0 < \gamma < \beta'$ such that $\beta + \gamma$ is a position of σ, $\sigma, \beta + \gamma \models \phi_2$ and for every $0 < \gamma' < \gamma$, we have $\sigma, \beta + \gamma' \models \phi_1$.

The satisfiability problem for LTL($\mathcal{O}_\alpha$) consists in determining, given a formula ϕ, whether there is a model σ such that $\sigma, 0 \models \phi$. The main results of [Cac06, DN07] are the following: for every $k \geq 1$, the satisfiability problem for LTL($\mathcal{O}_k$) restricted to models of length ω^k is PSPACE-complete and LTL($\mathcal{O}_\omega$) restricted to models of length ω^ω is decidable.

Observe that LTL($\mathcal{O}_k$) cannot express the temporal operator U over the class of countable ordinals but it can do it on models of length ω^k. Hence, each logic LTL($\mathcal{O}_k$) is less expressive than LTL(U, S).

Moreover, it is easy to show that for every $\alpha \leq \omega$, the logic LTL($\mathcal{O}_\alpha$) is expressive equivalent (over the class of countable ordinals) to its sublogic over the following set $\mathcal{O}'_\alpha$ of temporal operators:

$$\mathcal{O}'_\alpha = \{\mathsf{X}^{\omega^i} : \omega^i < \omega^\alpha, i \in \mathbb{N}\} \cup \{\mathsf{U}^{\omega^\beta} : \omega^\beta \leq \omega^\alpha, \beta \leq \omega\}.$$

This set is finite when α is finite. Moreover, there is a linear time (and logarithmic space) meaning preserving translation from LTL($\mathcal{O}_\alpha$) into LTL($\mathcal{O}'_\alpha$).

We obtain alternative proofs for known results and we get new results.

Theorem 22. *For every $k \geq 1$,*

(I) *the satisfiability problem for* LTL($\mathcal{O}_k$) *over ω^k-models is in* PSPACE*,*
(II) *the satisfiability problem for* LTL($\mathcal{O}'_k$) *restricted to ω^k-models is* PSPACE*-complete,*
(III) *for every countable infinite ordinal α, the satisfiability problem for* LTL($\mathcal{O}'_k$) *restricted to α-models is* PSPACE*-complete.*

(III) is an instance of Theorem 17. (II) is an instance of (III). (I) can be shown by observing that there is logarithmic space meaning preserving translation from LTL($\mathcal{O}_k$) to LTL($\mathcal{O}'_k$). (I) is the main result of [DN07] with the unary encoding of natural numbers occurring in ordinal expressions.

Finally, the corollary below improves the non-elementary bounds obtained in [Cac06, DN07] for LTL($\mathcal{O}_\omega$) by reducing this temporal logic to the monadic second-order logic, and then to Büchi ordinal automata.

Corollary 23. *Satisfiability for* LTL($\mathcal{O}_\omega$) *over the class of ω^ω-models is* PSPACE*-complete.*

8 Conclusion

In the paper, we have shown that the linear-time temporal logic with until and since over the class of ordinals, namely LTL(U, S) has a PSPACE-complete satisfiability problem. Thanks to Kamp's theorem [Kam68], we know that LTL(U, S) is a fundamental temporal logic since it is as expressive as first-order logic over the class of ordinals. In order to establish this tight complexity characterization, we have introduced the class of simple ordinal automata. This class of automata is more structured than usual ordinal automata and the sets of locations have some structural properties, typically it is a subset of the powerset of some set (herein called the basis). As a consequence, we are also able to improve some results from [Roh97, DN07]. For instance the uniform satisfiability problem is PSPACE-complete and we obtain alternative proofs for results in [DN07].

Extensions of our results include that the satisfiability problem for the language LTL($\mathsf{O}_1, \ldots, \mathsf{O}_k$) where the O_is form a finite set of MSO definable temporal operators is in PSPACE by adapting the developments from [VW94] and showing that our simple ordinal automata augmented with alphabet has the expressive power of standard ordinal automata. Furthermore, our results can be

extended to scattered linear orderings, see e.g. [BC07]. Indeed, one should add right limit transitions, using the terminology from [BC07] and adapt the developments herein.

Acknowledgments. We would like to thank the anonymous referees for helpful suggestions and remarks.

References

[BC07] Bruyère, V., Carton, O.: Automata on linear orderings. Journal of Computer and System Sciences 73, 1–24 (2007)

[BS73] Büchi, J.R., Siefkes, D.: *The monadic second order theory of all countable ordinals*. Lecture Notes in Mathematics, vol. 328. Springer, Heidelberg (1973)

[Cac06] Cachat, T.: Controller synthesis and ordinal automata. In: Graf, S., Zhang, W. (eds.) ATVA 2006. LNCS, vol. 4218, pp. 215–228. Springer, Heidelberg (2006)

[DN07] Demri, S., Nowak, D.: Reasoning about transfinite sequences. International Journal of Foundations of Computer Science 18(1), 87–112 (2007)

[GHR94] Gabbay, D., Hodkinson, I., Reynolds, M.: Temporal Logic - Mathematical Foundations and Computational Aspects, vol. 1. OUP, Oxford (1994)

[GW94] Godefroid, P., Wolper, P.: A partial approach to model checking. I & C 110(2), 305–326 (1994)

[Kam68] Kamp, J.: Tense Logic and the theory of linear order. PhD thesis, UCLA, USA (1968)

[KVW00] Kupferman, O., Vardi, M.Y., Wolper, P.: An automata-theoretic approach to branching-time model checking. J. ACM 47(2), 312–360 (2000)

[Rey] Reynolds, M.: The complexity of the temporal logic over the reals. Under submission

[Rey03] Reynolds, M.: The complexity of the temporal logic with until over general linear time. Journal of Computer and System Sciences 66(2), 393–426 (2003)

[Roh97] Rohde, S.: Alternating Automata and The Temporal Logic of Ordinals. PhD thesis, University of Illinois (1997)

[SC85] Sistla, A., Clarke, E.: The complexity of propositional linear temporal logic. J. ACM 32(3), 733–749 (1985)

[Ven93] Venema, Y.: Completeness via completeness: Since and until. In: de Rijke, M. (ed.) Diamonds and Defaults, pp. 279–286. Kluwer Academic Publishers, Dordrecht (1993)

[VW94] Vardi, M., Wolper, P.: Reasoning about infinite computations. I & C 115, 1–37 (1994)