An Unusual Temporal Logic

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Abstract. Kamp's theorem states that the temporal logic with modalities Until and Since has the same expressive power as the First-Order Monadic Logic of Order (FOMLO) over Real and Natural time flows. Kamp notes that there are expressions which deserve to be regarded as tense operators but are not representable within FOMLO. The words 'mostly' and 'usually' are examples of such expressions. We propose a formalization of 'usually' as a generalized Mostowski quantifier and prove an analog of Kamp's theorem.

1 Introduction

Temporal Logic (*TL*), introduced to Computer Science by Pnueli in [5], is a convenient framework for reasoning about "reactive" systems. This has made temporal logics a popular subject in the Computer Science community, enjoying extensive research in the past 40 years. In *TL* we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are expressed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1, \ldots, \varphi_k$ possibly on 'past' or 'future' points to a statement $M(\varphi_1, \ldots, \varphi_k)$ on the 'present' point t_0. The rule to determine the truth of a statement $M(\varphi_1, \ldots, \varphi_k)$ at t_0 is called a *truth table* of M. The choice of particular modalities with their truth tables yields different temporal logics. A temporal logic with modalities $M_1, \ldots, M_k$ is denoted by $TL(M_1, \ldots, M_k)$.

The simplest example is the one place modality $\Diamond P$ saying: "P holds some time in the future." Its truth table is formalized by $\varphi_\Diamond(x_0, X) := \exists x(x > x_0 \wedge P(x))$. This is a formula of the First-Order Monadic Logic of Order (*FOMLO*) - a fundamental formalism in Mathematical Logic where formulas are built using atomic propositions $P(x)$, atomic relations between elements $x_1 = x_2$, $x_1 < x_2$, Boolean connectives and first-order quantifiers $\exists x$ and $\forall x$. Two more natural modalities are the modalities Until ("*Until*") and Since ("*Since*"). $X\mathsf{Until}Y$ means that X will hold from now until a time in the future when Y will hold. $X\mathsf{Since}Y$ means that Y was true at some point of time in the past and since that point X was true until (not necessarily including) now.

The main *canonical*, linear time intended models are the non-negative integers $\omega := \langle \mathbb{N}, < \rangle$ for discrete time and the reals $\langle \mathbb{R}, < \rangle$ for continuous time.

Kamp's theorem [3] states that the temporal logic with modalities Until and Since and *FOMLO* have the same expressive power over the above two linear time canonical[1] models. After explaining his main theorem, Kamp writes:

> This still leaves open the question whether *all* English tense operators are representable in a language like *TL*. ... One easily verifies that indeed a very large number of expressions which are naturally classified as tense operators because of their function have first order definable tenses as their meanings. Yet there are expressions which deserve to be regarded as tense operators but which are nonetheless not representable within *TL*. The words 'mostly' and 'usually' are examples of such expressions. The impossibility of representing these particular expressions stems from the fact that their meanings involve a measure on time in an essential manner.

In this paper we suggest a formalization of "usually" over the standard discrete time $\omega := (\mathbb{N}, <)$ and prove a generalization of Kamp's theorem.

Here are three natural possibilities to formalize "P is unusual."

1. If P is finite.
2. If $\limsup_{n\to\infty} \frac{\text{the cardinality of } P\cap[0,n]}{n} = 0$.
3. If $\sum \frac{1}{p_i+1}$ finite, where $p_0 < p_1 < \cdots < p_i < \cdots$ is the enumeration of the elements of P.

$P \subseteq \mathbb{N}$ is usual if its complement is unusual. Note that "P is finite" is definable in *FOMLO* (over ω); however, formalizations (2)-(3) of "P is unusual" are not first-order definable.

A. Mostowski [4] initiated a study of so-called generalized quantifiers. Generalized quantifiers are now standard equipment in the toolboxes of both logicians and linguists.

The first-order logic with a (unary) generalized quantifier Q is obtained by extending the syntax of first-order logic by the rule if φ is a formula then $(Qx)\varphi$ is a formula. A (unary) generalized quantifier Q in a structure $\mathcal{M}$ is defined as a set $\mathcal{Q}$ of subsets of the domain of $\mathcal{M}$. The corresponding semantical clause for $(Qx)\varphi$ is $\mathcal{M} \models (Qx)\varphi(x, \bar{b})$ if $\{a \mid \mathcal{M} \models \varphi(a, \bar{b})\}$ is in $\mathcal{Q}$.

For a family of subsets $\mathcal{Q}$ of $\mathbb{N}$, we define a temporal modality $\langle Q\rangle$ as follows: $\langle Q\rangle\varphi$ holds iff the set of points where φ holds is in $\mathcal{Q}$.

Each of the above formalizations of unusual has the following properties:

1. If $P_1 \in \mathcal{Q}$ and $P_2 \subseteq P_1$ then $P_2 \in \mathcal{Q}$, i.e., if P_1 is unusual and $P_2 \subseteq P_1$, then so is P_2.
2. If $P_1, P_2 \in \mathcal{Q}$ then $P_1 \cup P_2 \in \mathcal{Q}$, i.e., if both P_i are unusual then their union is also unusual.
3. If $P_1 \in \mathcal{Q}$ and P_2 is finite then $P_1 \cup P_2 \in \mathcal{Q}$, i.e., if a finite set is added to an unusual event then the new set is still unusual.

[1] the technical notion which unifies $\langle \mathbb{N}, <\rangle$ and $\langle \mathbb{R}, <\rangle$ is Dedekind completeness.

Our main theorem states that for every family $\mathcal{Q}$ of subsets of $\mathbb{N}$ with properties (1)-(3) the temporal logic with modalities Until and Since and $\langle Q\rangle$ is expressively equivalent over $\omega := (\mathbb{N}, <)$ to the extension of *FOMLO* by the generalized quantifier Q. Moreover, our meaning preserving translations between these logics are computable and independent of $\mathcal{Q}$.

The rest of the paper is organized as follows. In Section 2 we recall the definitions of the monadic logic, the temporal logics and state Kamp's theorem. In Section 3 we provide a formalization of unusual as a Mostowski generalized quantifier and state our main result. In Section 4 we prove the main theorem. This proof is based on our simple proof of Kamp's theorem [6]. The proof of one proposition is postponed to Section 5. Section 6 states further results and open questions.

2 Kamp's Theorem

In this section we recall the definitions of the first-order monadic logic of order, the temporal logics and state Kamp's theorem.

Fix a set Σ of *atoms*. We use $P, R, S \dots$ to denote members of Σ. The syntax and semantics of both logics are defined below with respect to such Σ.

First-Order Monadic Logic of Order In the context of *FOMLO*, the atoms of Σ are referred to (and used) as *unary predicate symbols*. Formulas are built using these symbols, plus two binary relation symbols: $<$ and $=$, and a set of first-order variables (denoted: $x, y, z, \dots$). Formulas are defined by the grammar:

$$\varphi ::= \ x < y \ | \ x = y \ | \ P(x) \ | \ \neg\varphi_1 \ | \ \varphi_1 \vee \varphi_2 \ | \ \varphi_1 \wedge \varphi_2 \ | \ \exists x \varphi_1 \ | \ \forall x \varphi_1$$

where $P \in \Sigma$. We will also use the standard abbreviated notation for ***bounded quantifiers***, e.g., $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$, and $(\forall x)^{<z}(\dots)$ denotes $\forall x((x < z) \rightarrow (\dots))$, and $((\forall x)^{<z_2}_{>z_1}(\dots)$ denotes $\forall x((z_1 < x < z_2) \rightarrow (\dots))$, etc.

Semantics. Formulas are interpreted over *labeled linear orders* which are called *chains*. A ***Σ-chain*** is a triplet $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{T}$ is a set - the *domain* of the chain, $<$ is a linear order relation on $\mathcal{T}$, and $\mathcal{I} : \Sigma \rightarrow \mathcal{P}(\mathcal{T})$ is the *interpretation* of Σ (where $\mathcal{P}$ is the powerset notation). We use the standard notation $\mathcal{M}, t_1, t_2, \dots, t_n \models \varphi(x_1, x_2, \dots, x_n)$ to indicate that the formula φ with free variables among $x_1, \dots, x_n$ is satisfiable in $\mathcal{M}$ when x_i are interpreted as elements t_i of $\mathcal{M}$. For atomic $P(x)$ this is defined by: $\mathcal{M}, t \models P(x)$ iff $t \in \mathcal{I}(P)$; the semantics of $<, =, \neg, \wedge, \vee, \exists$ and $\forall$ is defined in a standard way.

Temporal Logics In the context of temporal logics the atoms of Σ are used as *atomic propositions* (also called *propositional atoms*). Formulas are built using these atoms, and a set (finite or infinite) B of ***modality names***, where a non-negative integer ***arity*** is associated with each $\mathsf{M} \in B$. The syntax of *TL* with the ***basis*** B over the signature Σ, denoted by ***TL(B)***, is defined by the grammar:

$$F ::= \ P \ | \ \neg F_1 \ | \ F_1 \vee F_2 \ | \ F_1 \wedge F_2 \ | \ \mathsf{M}(F_1, F_2, \dots, F_n)$$

where $P \in \Sigma$ and $\mathsf{M} \in B$ is an n-place modality. As usual, ***True*** denotes $P \vee \neg P$ and ***False*** denotes $P \wedge \neg P$.

Semantics. The semantics defines when a temporal formula holds at a *time-point* (or *moment* or element of the domain) in a chain.

The semantics of each n-place modality $\mathsf{M} \in B$ is defined by a 'rule' specifying how the set of moments where $\mathsf{M}(F_1, \ldots, F_n)$ holds (in a given structure) is determined by the n sets of moments where each of the formulas F_i holds. Such a 'rule' for M is formally specified (over time flow $(\mathcal{T}, <)$), by an operator $\mathcal{O}_{\mathsf{M}} : (\mathcal{P}(\mathcal{T}))^n \longrightarrow \mathcal{P}(\mathcal{T})$, which assigns to each n tuples of subsets of $\mathcal{T}$ a subset of $\mathcal{T}$.

The semantics of $TL(B)$ formulas is then defined inductively: Given a structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ and a moment $t \in \mathcal{M}$ (read $t \in \mathcal{M}$ as $t \in \mathcal{T}$), define when a formula F ***holds*** in $\mathcal{M}$ at t - notation: $\mathcal{M}, t \models F$ - as follows:

- $\mathcal{M}, t \models P$ iff $t \in \mathcal{I}(P)$ for any propositional atom P.
- $\mathcal{M}, t \models F \vee G$ iff $\mathcal{M}, t \models F$ or $\mathcal{M}, t \models G$; similarly ("pointwise") for $\wedge$, $\neg$.
- $\mathcal{M}, t \models \mathsf{M}(F_1, \ldots, F_n)$ iff $t \in \mathcal{O}_{\mathsf{M}}(T_1, \ldots, T_n)$ where $\mathsf{M} \in B$ is an n-place modality, $F_1, \ldots, F_n$ are formulas and $T_i := \{s \in \mathcal{T} : \mathcal{M}, s \models F_i\}$.

Truth tables: Practically, most standard modalities studied in the literature can be specified in *FOMLO*: A *FOMLO* formula $\varphi(x, P_1, \ldots, P_n)$ (with a single free first-order variable x and with n predicate symbols P_i) is called an n***-place first-order truth table***. Such a truth table φ ***defines*** an n-ary modality M whose semantics is given by an operator $\mathcal{O}_{\mathsf{M}}$ such that for any time flow $(\mathcal{T}, <)$, for any $T_1, \ldots, T_n \subseteq \mathcal{T}$ and for any structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{I}(P_i) = T_i$:

$$\mathcal{O}_M(T_1, \ldots, T_n) = \{t \in \mathcal{T} : \mathcal{M}, t \models \varphi(x, P_1, \ldots, P_n)\}$$

Example 2.1. *Below are truth-table definitions for the well known "**Eventually**", the (binary) **strict**-*Until *and **strict**-*Since *of [3,1].*

- $\Diamond$ *("**Eventually**") is defined by:* $\varphi_{\Diamond}(x, P) := (\exists x')_{>x} P(x')$
- Until *is defined by :* $\varphi_{\mathsf{Until}}(x, P_1, P_2) := (\exists x')_{>x}(P_2(x') \wedge (\forall y)^{<x'}_{>x} P_1(y))$
- Since *is defined by:* $\varphi_{\mathsf{Since}}(x, P_1, P_2) := (\exists x')^{<x}(P_2(x') \wedge (\forall y)^{<x}_{>x'} P_1(y))$

Example 2.2 (Modality $\langle Q \rangle$)**.** *Let $\mathcal{Q}$ be a family of subsets of the domain $\mathcal{T}$ of a structure $\mathcal{M}$. We can define a unary modality $\langle Q \rangle$ by the operator*

- $\mathcal{O}(T_1) := \begin{cases} \mathcal{T} \text{ if } T_1 \in \mathcal{Q} \\ \emptyset \quad \text{otherwise.} \end{cases}$

In the next section we will formalize "usually" by special families of subsets of $\mathbb{N}$. It is clear that there are $\mathcal{Q}$ such that the corresponding modality $\langle Q \rangle$ has no first-order truth table.

Kamp's Theorem Equivalence between temporal and monadic formulas is naturally defined: F is equivalent to $\varphi(x)$ over a class $\mathcal{C}$ of structures iff for any $\mathcal{M} \in \mathcal{C}$ and $t \in \mathcal{M}$: $\mathcal{M}, t \models F \Leftrightarrow \mathcal{M}, t \models \varphi(x)$. If $\mathcal{C}$ is the class of all chains, we will say that F is equivalent to φ.

A linear order $(T,<)$ is *Dedekind complete* if every non-empty subset (of the domain) which has an upper bound has a least upper bound. The canonical linear time models $\omega := (\mathbb{N},<)$ and $(\mathbb{R},<)$ are Dedekind complete, while the order of the rationals is not Dedekind complete. A chain is Dedekind complete if its underlying linear order is Dedekind complete.

The fundamental theorem of Kamp's states that $TL(\mathsf{Until},\mathsf{Since})$ is expressively equivalent to *FOMLO* over Dedekind complete chains.

Theorem 2.3 (Kamp [3])**.** *1. Given any $TL(\mathsf{Until},\mathsf{Since})$ formula A there is a FOMLO formula $\varphi_A(x)$ which is equivalent to A over all chains.*

2. Given any FOMLO formula $\varphi(x)$ with one free variable, there is a $TL(\mathsf{Until},\mathsf{Since})$ formula which is equivalent to φ over Dedekind complete chains.

3 An Unusual Quantifier and Modality

3.1 Generalized Quantifier

The syntax of the first-order logic with a unary generalized quantifier Q (notation $FO[Q]$) is obtained by extending the usual first-order syntax by the new quantifier.

The formulas of $FO[Q]$ are built by the usual formation rules and the following new (variable-binding) formation rule:

– if x is a variable and φ is a formula of $FO[Q]$, then so is $(Qx)\varphi$, and Qx binds all free occurrences of x in φ.

The semantics of $FO[Q]$ is provided by enriching the domain of first-order structures with a set $\mathcal{Q}$ of subsets of its domain and extending the usual definition of satisfaction by a clause for $(Qx)\varphi$:

$$\mathcal{M}, b_1,\dots,b_n \models (Qx)\varphi(x,y_1,\dots,y_n) \text{ if } \{a \mid \mathcal{M},a,b_1,\dots,b_n \models \varphi(x,y_1,\dots,y_n)\} \text{ is in } \mathcal{Q}.$$

FOMLO$[Q]$ denotes the extension of *FOMLO* by a generalized quantifier Q.

For a generalized quantifier Q we also introduce **modality** $\langle Q\rangle$, defined by $\mathcal{M},t \models \langle Q\rangle\varphi$ iff $\{a \mid \mathcal{M},a \models \varphi\} \in \mathcal{Q}$. Note that if $\mathcal{M},t \models \langle Q\rangle\varphi$, then $\mathcal{M},t' \models \langle Q\rangle\varphi$ for every t'.

3.2 Usual and Unusual over $\mathbb{N}$

Let us start with some intuitive requirements on unusual sets. If P never happens (respectively, always holds) then P is unusual (respectively, is not unusual). If P_1 is unusual and $P_2 \subseteq P_1$, then P_2 is unusual. If both P_1 and P_2 are unusual, then their union is also unusual. It is also natural to require that a finite subset of an infinite set is unusual. These lead to the following definition.

Given a set X, an **unusual** family on X is a set $\mathcal{Q}$ consisting of subsets of X such that

1. $\emptyset \in \mathcal{Q}$ and $X \notin \mathcal{Q}$.
2. If A and B are subsets of X, A is a subset of B, and B is an element of $\mathcal{Q}$, then A is also an element of $\mathcal{Q}$.

3. If A and B are elements of $\mathcal{Q}$, then so is the union of A and B.
4. If A is finite then A is in $\mathcal{Q}$.

In model theory, an ideal $\mathcal{Q}$ on a set X is a family of subsets of X which satisfies (1)-(3). A filter is a dual notion to an ideal. Hence, a family $\mathcal{Q}$ of subsets of X is a filter, if it is a non-empty proper subset of $\mathcal{P}(X)$ and it is closed under superset and finite intersection. In model theory ideals (respectively, filters) are considered as families of small (respectively, big) subsets of X.

Several collections of "small" subsets of $\mathbb{N}$ are presented below:

1. $\mathcal{Q}_1 := \{P \mid P \text{ is finite}\}$.
2. $\mathcal{Q}_2 := \{P \mid \limsup_{n\to\infty} \frac{\text{the cardinality of } P\cap[0,n]}{n} = 0\}$.
3. Van der Waerden ideal is the family $\{P \mid P$ does not contain an arithmetic progression of arbitrary length$\}$.

Let $p_0 < p_1 < \cdots < p_i < \cdots$ be the enumeration of the elements of P.

4. $\mathcal{Q}_4 := \{P \mid \sum \frac{1}{p_i+1} \text{ is finite }\}$.
5. P is 1-sparse if for every n there is N such that $[m, m+n]$ contains at most one element from P for every $m > N$.
6. P is 1-thin if $\lim_{n\to\infty} \frac{p_n}{p_{n+1}} = 0$.
7. P is almost 1-thin if $\limsup_{n\to\infty} \frac{p_n}{p_{n+1}} < 1$.

Note that 1-sparse (respectively, 1-thin, or almost 1-thin) sets are not closed under union. Hence, these families of sets are not ideals.

A set is sparse (respectively, thin or almost thin) if it is finite or a finite union of 1-sparse (respectively, 1-thin, or almost 1-thin) sets. The family of sparse (respectively, thin or almost thin) is an ideal.

The families defined in examples (1)-(4), as well as the families of sparse, thin and almost thin sets are unusual. The family $\{P \mid$ the set of even elements of P is finite$\}$ is also unusual.

A generalized **quantifier** Q is **unusual** if its corresponding family of subsets of $\mathbb{N}$ is unusual. Dually, we say that a family $\mathcal{Q}$ of sets is **usual** if $\{\mathbb{N} \setminus P \mid P \in \mathcal{Q}\}$ is unusual. The corresponding quantifier and modality are *usual.* The next Lemma states some immediate equivalences:

Lemma 3.1. *If Q is an unusual quantifier. Then:*

1. $(Qx)(\varphi_1 \vee \varphi_2)$ is equivalent to $((Qx)\varphi_1) \wedge (Qx)\varphi_2$.
2. $(Qx)(\varphi \wedge x < z)$ is equivalent to True.
3. If x does not occur free in φ, then $(Qx)(\varphi \wedge \psi)$ is equivalent to $\neg\varphi \vee (Qx)\psi$,
4. Assume that x does not occur free in φ. Then $(Qx)\varphi$ is equivalent to $\neg\varphi$.

3.3 Expressive Equivalence

Theorem 3.2 (Main)**.** *Let $\mathcal{Q}$ be an unusual family of subsets of $\mathbb{N}$. Let Q and $\langle Q\rangle$ be the corresponding generalized quantifier and modality. Then*

1. *Given any $TL(\mathsf{Until}, \mathsf{Since}, \langle Q\rangle)$ formula A there is a $FOMLO[Q]$ formula $\varphi_A(x)$ which is equivalent (over ω-chains) to A.*
2. *Given any $FOMLO[Q]$ formula $\varphi(x)$ with one free variable, there is a $TL(\mathsf{Until}, \mathsf{Since}, \langle Q\rangle)$ formula A_φ which is equivalent (over ω-chains) to φ.*

Moreover, φ_A and A_φ are computable from φ and A and independent of Q.

The meaning preserving translation from $TL(\mathsf{Until}, \mathsf{Since}, \langle Q\rangle)$ to $FOMLO[Q]$ is easily obtained by structural induction. The main technical contribution of our paper is a proof of Theorem 3.2 (2). The proof is constructive. An algorithm which for every $FOMLO[Q]$ formula $\varphi(x)$ constructs a $TL(\mathsf{Until}, \mathsf{Since}, \langle Q\rangle)$ formula which is equivalent to φ is easily extracted from our proof.

4 Proof of the Main Theorem

First, we introduce $\overrightarrow{\exists}\forall$ formulas which are instances of the Decomposition formulas of [2,6].

Definition 4.1 ($\overrightarrow{\exists}\forall$-formulas)**.** *Let Σ be a set of monadic predicate names. An $\overrightarrow{\exists}\forall$-formula over Σ is a formula of the form:*

$$\begin{aligned}
\psi(z_0, \dots, z_m) := {} & \exists x_n \dots \exists x_1 \exists x_0 \\
& \left(\bigwedge_{k=0}^{m} z_k = x_{i_k} \right) \wedge (x_n > x_{n-1} > \dots > x_1 > x_0) && \text{“\textit{ordering of } } x_i \text{ \textit{and} } z_j\text{”} \\
\wedge & \bigwedge_{j=0}^{n} \alpha_j(x_j) && \text{“\textit{Each} } \alpha_j \text{ \textit{holds at} } x_j\text{”} \\
\wedge & \bigwedge_{j=1}^{n} [(\forall y)^{<x_j}_{>x_{j-1}} \beta_j(y)] && \text{“\textit{Each} } \beta_j \text{ \textit{holds along} } (x_{j-1}, x_j)\text{”} \\
\wedge & (\forall y)_{>x_n} \beta_{n+1}(y) && \text{“}\beta_{n+1} \text{ \textit{holds everywhere after} } x_n\text{”} \\
\wedge & (\forall y)^{<x_0} \beta_0(y) && \text{“}\beta_0 \text{ \textit{holds everywhere before} } x_0\text{”}
\end{aligned}$$

with a prefix of $n+1$ existential quantifiers and with all α_j, β_j quantifier free formulas with one variable over Σ. (ψ has $m+1$ free variables $z_0, \dots, z_m$ and $m+1 \le n+1$ existential quantifiers are dummy and are introduced just in order to simplify notations.)

Definition 4.2 ($\vee\overrightarrow{\exists}\forall$-formulas)**.** *A formula is a $\vee\overrightarrow{\exists}\forall$ formula if it is equivalent to a disjunction of $\overrightarrow{\exists}\forall$-formulas.*

The set of $\vee\overrightarrow{\exists}\forall$ formulas is closed under disjunction, conjunction and existential quantification. The set of $\vee\overrightarrow{\exists}\forall$ formulas is not closed under negation. However, the negation of a $\vee\overrightarrow{\exists}\forall$ formula is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula in the expansion of chains by all $TL(\mathsf{Until}, \mathsf{Since})$ definable predicates (see Proposition 4.7).

The next definition plays a major role in the proof of Kamp's theorem [2,6].

Definition 4.3. *Let $\mathcal{M}$ be a Σ chain and $\mathcal{L}$ be a temporal logic. We denote by $\mathcal{L}[\Sigma]$ the set of unary predicate names $\Sigma \cup \{A \mid A$ is an $\mathcal{L}$-formula over $\Sigma\}$. The canonical $\mathcal{L}$-expansion of $\mathcal{M}$ is an expansion of $\mathcal{M}$ to an $\mathcal{L}[\Sigma]$-chain, where each predicate name $A \in \mathcal{L}[\Sigma]$ is interpreted as $\{a \in \mathcal{M} \mid \mathcal{M}, a \models A\}$[2]. We say that first-order formulas in the signature $\mathcal{L}[\Sigma] \cup \{<\}$ are equivalent over $\mathcal{M}$ (respectively, over a class of Σ-chains $\mathcal{C}$) if they are equivalent in the canonical expansion of $\mathcal{M}$ (in the canonical expansion of every $\mathcal{M} \in \mathcal{C}$).*

Note that if A is a $\mathcal{L}$ formula over $\mathcal{L}[\Sigma]$ predicates, then it is equivalent to a $\mathcal{L}$ formula over Σ, and hence to an atomic formula in the canonical $\mathcal{L}$-expansions.

The $\overrightarrow{\exists}\forall$ and $\vee\overrightarrow{\exists}\forall$ formulas are defined as previously, but now they can use as atoms $\mathcal{L}$ definable predicates.

The next Proposition was proved in [6].

Proposition 4.4. *Let $\mathcal{L}$ be a temporal logic which contains modalities* Until *and* Since*. Every FOMLO formula is equivalent (over the canonical $\mathcal{L}$ expansions of ω-chains) to a disjunction of $\overrightarrow{\exists}\forall$-formulas.*

The $\vee\overrightarrow{\exists}\forall$ formulas with one free variable can be easily translated to temporal formulas.

Proposition 4.5 (From $\vee\overrightarrow{\exists}\forall$-formulas to $\mathcal{L}$ formulas)**.** *If $\mathcal{L}$ contains modalities* Until *and* Since*, then every $\vee\overrightarrow{\exists}\forall$ formula with one free variable is equivalent (over the canonical $\mathcal{L}$-expansions) to an $\mathcal{L}$ formula.*

The proof of the next proposition is postponed to Sect. 5.

Proposition 4.6. *(Closure under unusual quantifier) Let Q be an unusual quantifier on $\mathbb{N}$ and $\mathcal{L}$ be a temporal logic which contains modalities* Until*,* Since *and $\langle Q\rangle$. If ψ is an $\overrightarrow{\exists}\forall$-formula, then $(Qx)\psi$ is equivalent (over the canonical $\mathcal{L}$ expansions of ω-chains) to a disjunction of $\overrightarrow{\exists}\forall$-formulas.*

As a consequence we obtain:

Proposition 4.7. *Let Q be an unusual quantifier on $\mathbb{N}$ and $\mathcal{L}$ be a temporal logic which contains modalities* Until*,* Since *and $\langle Q\rangle$. Every FOMLO$[Q]$ is equivalent (over the canonical $\mathcal{L}$ expansions of ω-chains) to a disjunction of $\overrightarrow{\exists}\forall$-formulas.*

Now, we are ready to prove the unusual version of Kamp's Theorem:

Theorem 4.8. *Let Q be an unusual quantifier on $\mathbb{N}$. For every FOMLO$[Q]$ formula $\varphi(x)$ with a single free variable, there is a TL(*Until*,* Since*, $\langle Q\rangle$) formula which is equivalent (on ω-chains) to φ.*

Proof. By Proposition 4.7, $\varphi(x)$ is equivalent to a disjunction of $\overrightarrow{\exists}\forall$ formulas. By Proposition 4.5, an $\overrightarrow{\exists}\forall$ formula is equivalent to a *TL*(Until, Since, $\langle Q\rangle$) formula. Hence, $\varphi(x)$ is equivalent to a *TL*(Until, Since, $\langle Q\rangle$) formula. □

This completes the proof of our main theorem, except for Proposition 4.6 which is proved in the next section.

[2] We often use "$a \in \mathcal{M}$" instead of "a is an element of the domain of $\mathcal{M}$"

5 Proof of Proposition 4.6

In this section we say that "formulas are equivalent in a chain $\mathcal{M}$" instead of "formulas are equivalent in the canonical $\mathcal{L}$-expansion of $\mathcal{M}$." We also say that "formulas are equivalent" instead of "formulas are equivalent in the canonical $\mathcal{L}$-expansions of chains over ω."

If ψ has at most one free variable then, by Proposition 4.5, ψ is equivalent to a $TL(\mathsf{Until}, \mathsf{Since}, \langle Q \rangle)$ formula A. Hence, $(Qx)\psi$ is equivalent to a temporal logic formula $\langle Q \rangle A$.

Let $\psi(z_0, \ldots, z_m)$ be an $\overrightarrow{\exists}\forall$-formula as in Definition 4.1 with $m \geq 1$. W.l.o.g. assume that $\psi \rightarrow \wedge_{i=0}^{m-1} z_i < z_{i+1}$.

If x is not free in ψ then, by Lemma 3.1, $(Qx)\psi$ is equivalent to a $\neg\psi$ and hence to a $\vee\overrightarrow{\exists}\forall$ formula by Proposition 4.4.

If $x \in \{z_0, \ldots, z_{m-1}\}$, then there are at most finitely many x which satisfy ψ, therefore $(Qx)\psi$ is equivalent to True.

If x is z_m then ψ is equivalent to the conjunction of an $\overrightarrow{\exists}\forall$-formula $\psi_1(z_0, \ldots, z_{m-1})$ and an $\overrightarrow{\exists}\forall$-formula $\psi_2(z_{m-1}, z_m)$ with two free variables z_{m-1} and z_m. By Lemma 3.1, $(Qz_m)\psi$ is equivalent to $\neg\psi_1 \vee (Qz_m)\psi_2(z_{m-1}, z_m)$. By Proposition 4.4, it is sufficient to show that $(Qz_m)\psi_2(z_{m-1}, z_m)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.

It is easy to show that any $\overrightarrow{\exists}\forall$ formula with the free variables z_0, z_1 is equivalent to a formula of the following form:

$$\exists x_0 \ldots \exists x_n [(z_0 = x_0 < \cdots < x_n = z_1) \wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) \wedge \bigwedge_{j=1}^{n} (\forall y)_{>x_{j-1}}^{<x_j} \beta_j(y)] \quad (1)$$

where α_i, β_i are quantifier free.

Therefore, to complete our proof it is sufficient to prove the following lemma:

Lemma 5.1. *Let $\psi(z_0, z_1)$ be a formula as in* (1). *Then $(Qz_1)\psi$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.*

In the rest of this section we prove Lemma 5.1. Our proof is organized as follows. In Lemma 5.3 we prove an instance of Lemma 5.1 where all β_i are equivalent to True. Then we derive a more general instance (Corollary 5.5) where $\beta_1(x)$ holds for all $x > z_0$. Finally, in Lemma 5.6(2) we prove the full version of Lemma 5.1. First, we introduce some helpful notations.

Notation 5.2. *We use the abbreviated notation $[\alpha_0, \beta_1 \ldots, \alpha_{n-1}, \beta_n \alpha_n](z_0, z_1)$ for the $\overrightarrow{\exists}\forall$-formula as in (1).*

In this notation Lemma 5.1 can be rephrased as $(Qz_1)[\alpha_0, \beta_1 \ldots, \alpha_{n-1}, \beta_n \alpha_n](z_0, z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.

We start with the instance of Lemma 5.1 where all β_i are True.

Lemma 5.3. $(Qz_1)\exists x_0 \exists x_1 \ldots \exists x_n \, (z_0 = x_0 < x_1 < \cdots < x_n = z_1) \wedge \bigwedge_{i=0}^{n} P_i(x_i)$ *is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.*

Proof. This formula is equivalent to the disjunction of $(Qz_1)P_n(z_1)$ and $\neg\exists x_0\exists x_1\ldots\exists x_{n-1}\,(z_0 = x_0 < x_1 < \cdots < x_{n-1})\wedge \bigwedge_{i=0}^{n-1} P_i(x_i)$. The first disjunct is equivalent to $\langle Q\rangle P_n$. The second disjunct is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula by Proposition 4.4. Hence, this formula is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula. □

The next Lemma does not deal with generalized quantifiers.

Lemma 5.4. *$((\forall y)_{>z_0}\beta_1)\wedge[\alpha_0,\beta_1,\alpha_1,\beta_2,\ldots,\alpha_{n-1},\beta_n,\alpha_n](z_0,z_1)$ is equivalent to $((\forall y)_{>z_0}\beta_1)\wedge\exists x_0\exists x_1\ldots\exists x_n\,(z_0 = x_0 < x_1 < \cdots < x_n = z_1)\wedge \bigwedge_{i=0}^{n}\alpha'_i(x_i)$, where α'_i are atoms.*

As a consequence we obtain:

Corollary 5.5. *Let $\psi(z_0,z_1)$ be $((\forall y)_{>z_0}\beta_1(y))\wedge[\alpha_0,\beta_1,\alpha_1,\beta_2,\ldots,\alpha_{n-1},\beta_n,\alpha_n](z_0,z_1)$. Then $(Qz_1)\psi$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.*

Proof. Immediately by Lemmas 3.1(3), 5.3, and 5.4. □

Now we are ready to prove Lemma 5.1, i.e., $(Qz_1)[\alpha_0,\beta_1\ldots,\beta_{n-1},\alpha_{n-1},\beta_n,\alpha_n](z_0,z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.

Lemma 5.6. *1. Let $\psi(z_0,z_1)$ be $((\exists y)_{>z_0}\neg\beta_1(y))\wedge[\alpha_0,\beta_1,\alpha_1,\beta_2,\ldots,\alpha_{n-1},\beta_n,\alpha_n](z_0,z_1)$. Then $(Qz_1)\psi$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.*

2. $(Qz_1)[\alpha_0,\beta_1,\alpha_1,\beta_2,\ldots,\alpha_{n-1},\beta_n,\alpha_n](z_0,z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula..

Proof. We prove (1) and (2) simultaneously by induction on n. Observe that A is equivalent to $(((\exists y)_{>z_0}\neg\beta_1(y))\wedge A)\vee(((\forall y)_{>z_0}\beta_1(y))\wedge A)$. Hence, if (1) holds for n, then by Corollary 5.5, Lemma 3.1(1) and the closure of $\vee\overrightarrow{\exists}\forall$ formulas under conjunction we obtain that (2) holds for n. Therefore, for the inductive step it is sufficient to prove that if (1) and (2) hold for n then (2) holds for $n+1$.

Note that $(\exists y)_{>z_0}\neg\beta_1(y)$ implies that there is at most one z such that $[\alpha_0,\beta_1,\alpha_1](z_0,z)$ and $\neg(\exists y)_{>z}[\alpha_0,\beta_1,\alpha_1](z_0,y)$.

If there is no such z, then $(Qz_1)\psi$ is equivalent to True.

So, we assume that there is a unique such z. It is definable by the formula

$$def(z_0,z) := [\alpha_0,\beta_1,\alpha_1](z_0,z)\wedge\neg(\exists y)_{>z}[\alpha_0,\beta_1,\alpha_1](z_0,y). \qquad (2)$$

It is sufficient to show that $(\exists z)_{>z_0}\,def(z)\wedge(Qz_1)[\alpha_0,\beta_1,\alpha_1,\ldots,\beta_{n+1},\alpha_{n+1}](z_0,z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula ψ'. Then $(Qz_1)\psi$ is equivalent to $(\forall y)_{>z_0}\beta_1(y)\vee(\neg\exists z\,def)\vee(\exists z\,def\wedge\psi')$, and by Proposition 4.4, to a $\vee\overrightarrow{\exists}\forall$ formula.

We prove this by induction on n. The *basis* is trivial.

Inductive step $n\mapsto n+1$. Define:

$$\begin{aligned}
A_i^-(z_0,z) &:= [\alpha_0,\beta_1,\ldots,\beta_i,\alpha_i](z_0,z) && i=1,\ldots,n\\
A_i^+(z,z_1) &:= [\alpha_i,\beta_{i+1},\ldots\beta_{n+1}\alpha_{n+1}](z,z_1) && i=1,\ldots,n\\
A_i(z_0,z,z_1) &:= A_i^-(z_0,z)\wedge A_i^+(z,z_1) && i=1,\ldots,n\\
B_i^-(z_0,z) &:= [\alpha_0\beta_1,\ldots,\beta_{i-1},\alpha_{i-1},\beta_i,\beta_i](z_0,z) && i=1,\ldots,n+1\\
B_i^+(z,z_1) &:= [\beta_i,\beta_i,\alpha_i\beta_{i+1}\alpha_{i+1},\ldots,\beta_{n+1},\alpha_{n+1}](z,z_1) && i=1,\ldots,n+1\\
B_i(z_0,z,z_1) &:= B_i^-(z_0,z)\wedge B_i^+(z,z_1) && i=1,\ldots,n+1
\end{aligned}$$

If the interval (z_0, z_1) is non-empty, these definitions imply

$$[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1) \Leftrightarrow (\forall z)^{<z_1}_{>z_0} \big(\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i \big)$$

$$[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1) \Leftrightarrow (\exists z)^{<z_1}_{>z_0} \big(\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i \big)$$

Hence, for every $\varphi(z_0, z)$:

$$((\exists z)^{<z_1}_{>z_0} \varphi(z_0, z)) \wedge [\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$$

is equivalent to $(\exists z)^{<z_1}_{>z_0} \big(\varphi(z_0, z) \wedge (\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i)\big)$. In particular,

$$\begin{array}{c} (\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge [\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1) \\ \text{is equivalent to} \\ (\exists z)^{<z_1}_{>z_0} \big(\mathit{def}(z_0, z) \wedge (\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i)\big), \end{array} \tag{3}$$

where *def* was defined in equation (2). To proceed we use the following simple properties of the unusual quantifier:

Lemma 5.7. *Assume that z_1 does not occur free in φ, and $\exists! z \varphi$. Then*

1. $(Qz_1)(\exists z)^{<z_1}(\varphi \wedge C)$ *is equivalent to* $(\exists z)(\varphi \wedge (Qz_1)C)$
2. $(Qz_1)\exists z(\varphi \wedge \bigvee C_i)$ *is equivalent to* $\bigwedge \exists z(\varphi \wedge (Qz_1)C_i)$

Now $(\exists z)_{>z_0} \mathit{def}(z_0, z) \wedge (Qz_1)[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$ is equivalent (by Lemma 5.7(1)) to $(Qz_1)(\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge [\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$ is equivalent, by (3), to $(Qz_1)\big((\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge (\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i)\big)$ is equivalent (by Lemma 5.7(2)) to

$$\big(\bigwedge_{i=1}^{n} (\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge (Qz_1)A_i \big) \wedge \big(\bigwedge_{i=1}^{n+1} (\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge (Qz_1)B_i \big) \tag{4}$$

We are going to show that $(Qz_1)A_i$ $(i = 1, \ldots, n)$ and $(Qz_1)B_i$ $(i = 2, \ldots, n+1)$, and $(\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge (Qz_1)B_1$ are equivalent to $\vee \overrightarrow{\exists} \forall$ formulas and therefore, by Proposition 4.4, we obtain that (4) is equivalent to a $\vee \overrightarrow{\exists} \forall$ formula.

Recall that $A_i := A_i^-(z_0, z) \wedge A_i^+(z, z_1)$ and $B_i := B_i^-(z_0, z) \wedge B_i^+(z, z_1)$. By Lemma 3.1(3), we obtain that $(Qz_1)A_i$ is equivalent to $\neg A_i^- \vee (Qz_1)A_i^+$. By the inductive assumption $(Qz_1)A_i^+$ is equivalent to a $\vee \overrightarrow{\exists} \forall$ formula for $i = 1, \ldots, n$. Hence, by Proposition 4.4, $(Qz_1)A_i$ is equivalent to a $\vee \overrightarrow{\exists} \forall$ formula. Similar arguments show that $(Qz_1)B_i$ is equivalent to a $\vee \overrightarrow{\exists} \forall$ formula for $i = 2, \ldots, n+1$.

Finally, $\mathit{def}(z_0, z)$ implies that there is no $x > z$ such that $\alpha_1(x)$ and β_i holds on $[z, x)$. Therefore, B_1^+ is equivalent to False and $(Qz_1)B_1^+$ is equivalent to True. Hence, $(\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z) \wedge (Qz_1)B_1$ is equivalent to a $\vee \overrightarrow{\exists} \forall$ formula $(\exists z)^{<z_1}_{>z_0} \mathit{def}(z_0, z)$.

This completes our proof of Lemma 5.1 and of Proposition 4.6. □

6 Further Results and Open Questions

We provided a natural interpretation of usual/unusual over $\mathbb{N}$ and proved an analog of Kamp's theorem. We can consider several unusual quantifiers $\mathcal{Q}_1, \dots \mathcal{Q}_k$ and prove that $FOMLO[Q_1, \dots, Q_k]$ and $TL(\mathsf{Until}, \mathsf{Since}, \langle Q_1 \rangle, \dots, \langle Q_k \rangle)$ have the same expressive power over ω. Our result can be easily extended to the time domain of integers; however, in this case we have to require that if $\mathcal{Q}$ is a family of unusual sets over integers and $P \in \mathcal{Q}$, then neither $(-\infty, k]$ nor $[k, \infty)$ is a subset of P. It is open how to formalize "usually/unusually" over the reals.

Standard notions of "fairness" are based on the ideal of finite sets. For example, strong fairness is formalized as: if P_1 occurs infinitely often, then P_2 occurs infinitely often. It is natural to base fairness on an unusual modality $\langle Q \rangle$, and define Q-fairness as $Fair_Q(P_1, P_2) := \langle Q \rangle P_2 \rightarrow \langle Q \rangle P_1$. More general notions of "fairness" can be introduced by using several unusual quantifiers; e.g., $Fair_{Q_1,Q_2}(P_1, P_2) := \langle Q_2 \rangle P_2 \rightarrow \langle Q_1 \rangle P_1$.

Unfortunately, in our extension a phrase like "It is unusual that the weather is sunny when it rains" is not expressible, and further extensions are needed to express such a binary unusual modality.

We can show that under each of the seven interpretations of unusual described in Section 3.2, the problem whether a $TL(\mathsf{Until}, \mathsf{Since}, \langle Q \rangle)$ formula is satisfiable is PSPACE-complete. Moreover, the interpretations (2)-(7) of unusual give the same set of satisfiable $TL(\mathsf{Until}, \mathsf{Since}, \langle Q \rangle)$ formulas.

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