Random graphs

Homework assignment #2

Problem 1. Prove the following statements about long paths in sparse random graphs:

(a) Let m and n be integers and suppose that G is an n-vertex graph such that $e_G(A, B) > 0$ for every two disjoint sets A and B of m vertices of G. Prove that G contains a path of length $n - 2m$.

(b) Show that for every $\delta > 0$, there is a $K > 0$ such that a.a.s. $G_{n,K/n}$ contains a path of length at least $(1 - \delta)n$.

(c) Prove that if K is a sufficiently large constant, then a.a.s. $G \sim G_{Kn,K/n}$ has the following property. For every colouring $c\colon E(G) \to \{R, B\}$, there is a monochromatic path of length n, i.e., either $c^{-1}(R)$ or $c^{-1}(B)$ contains a path of length n.

Problem 2. Reconstruct Bollobás' original proof of the fact that a.a.s.

$$\chi\big(G_{n,p}\big) \leqslant (1 + o(1)) \cdot \frac{n}{2\log_{1/(1-p)} n}:$$

(a) Suppose that $q \in (0, 1)$ and $\varepsilon > 0$ are fixed constants, let $k = \lfloor (2 - \varepsilon) \log_{1/q} n \rfloor$, and let X denote the largest size of a collection of *pairwise edge-disjoint* copies of K_k in $G_{n,q}$. Show that $\mathbb{E}[X] \geqslant \Omega\big(n^2/(\log n)^C\big)$ for some absolute constant C.

(b) By considering the appropriate Doob martingale (the 'edge-exposure' martingale), prove that for some constant C and all sufficiently large n,

$$\Pr\left(\omega(G_{n,q}) \leqslant (2 - \varepsilon) \log_{1/q} n\right) = \exp\left(-\frac{n^2}{(\log n)^C}\right),$$

where $\omega(G)$ is the largest size of a clique in G.

Problem 3. Prove that for all integers $k \geqslant 3$ and $r \geqslant 2$, there exists a constant C such that if $p \geqslant Cn^{-2/k}$, then $G \sim G_{n,p}$ a.a.s. has the following property. For every $\varphi\colon V(G) \to \{1, \ldots, r\}$, there is an $i \in [r]$ such that the subgraph of G induced by $\varphi^{-1}(i)$ contains a copy of K_k.

Remark. It is also true that for every $k \geqslant 3$, there exists a constant c such that if $p \leqslant cn^{-2/k}$, then a.a.s. the vertex set of $G_{n,p}$ can be partitioned into two sets A and B such that neither $G[A]$ nor $G[B]$ contain a K_k. This is significantly harder to prove, but you might try to do it!