BPS polynomials and Welschinger invariants [P. Bousseau]

Joint with H. Argüz 2506.02770 [Talk on 16/04/2026]

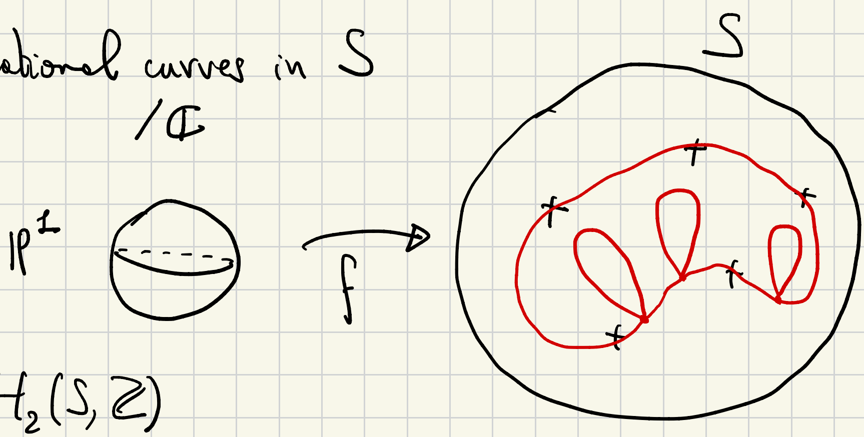
$$\text{Counting curves on surfaces} \begin{cases} \nearrow \mathbb{C} \\ \rightarrow \mathbb{R} \\ \searrow \text{Tropical/Refined} \end{cases}$$

S : rational smooth projective surface / $\mathbb{C}$

$S = \mathbb{P}^2$ blowup at n points in general position
$(\mathbb{P}^1 \times \mathbb{P}^1)$

$n \leq 3$: Toric surface $\qquad$ $n \leq 8$: del Pezzo surfaces

Rational curves in S / $\mathbb{C}$



$\beta \in H_2(S, \mathbb{Z})$

$f_*[\mathbb{P}^1] = \beta$ $\qquad$ Fix m points in S

$$m = \beta \cdot c_1(S) - 1$$

$n = 0$ $\quad \mathbb{P}^2$ $\quad$ degree d curves through $m = 3d - 1$ points

Rational $\qquad\qquad$ in general position

$$GW^S_{0,\beta} = \begin{cases} \text{Naive Count} & n \leq 8 \\ \text{Gromov-Witten count} & n > 8 \end{cases}$$

$\in \mathbb{Z}$

$\mathbb{P}^2$ Kontsevich recursion formula $\quad$ WDVV

Göttsche - Pandharipande ~ 98

Ex: . $S = \mathbb{P}^2$, $d = 3$, $m = 8$, $GW^S_{0,\beta} = 12$ $\leftarrow$

. $S = \mathbb{P}^2$ blown up in $n = 6$ points

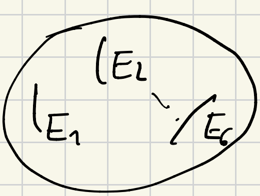
$$\beta = 6H - \sum_{i=1}^{6} 2E_i \qquad m = 5 \qquad GW^S_{0,\beta} = 3240 \leftarrow$$

$/\mathbb{R}$ $\quad$ $S = \mathbb{P}^2$ blown-up at n <u>real</u> <u>points</u>

standard Real structure

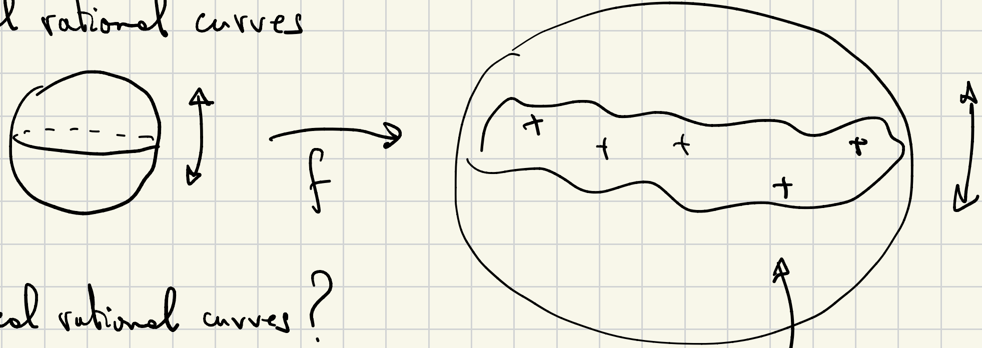
Real Locus $= \mathbb{RP}^2$

$\mathbb{P}^2$



Real rational curves

$\mathbb{RP}^1$



Real rational curves?

m points are all real

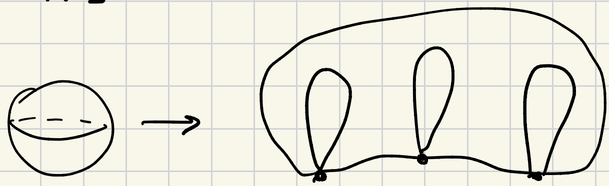
Not deformation invariant

$\mathbb{P}^2$ $d=3$ $m=8$ 8, 10 or 12 real curves

Welschinger: # real rational curves

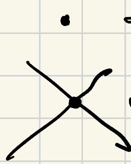
with sign

$(-1)^{\#\text{ isolated nodes}}$



Real node $xy=0$

$\left.\begin{array}{l} x^2+y^2=0 \\ x^2-y^2=0 \end{array}\right]$



$W_\beta^s \in \mathbb{Z}$ independent of choices!

Ex: . $\mathbb{P}^2$ $d=3$ $m=8$ $W_\beta^s = 8$

. $\mathbb{P}^2$ blown up in 6 pts, β, $W_\beta^s = 1000$

Real version of WDVV equation Solomon

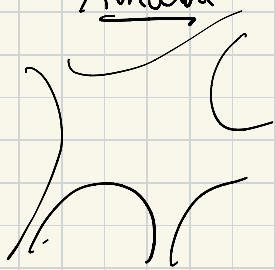
Horev-Solomon

If $n \leq 3$: S toric surface $\rightarrow$ Tropical description of $GW_{0,\beta}^S$ and W_β^S.

$(\mathbb{C}^*)^2 \curvearrowright S$

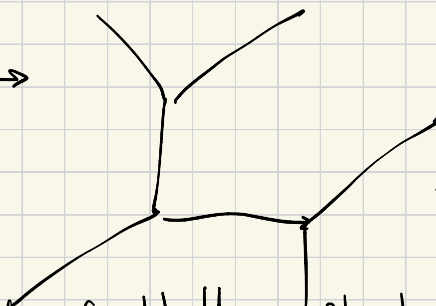
$$\begin{array}{ccc} & & (z_1, z_2) \in (\mathbb{C}^*)^2 \\ \downarrow & & \downarrow \\ \mathbb{R}^2 & & (\log|z_1|, \log|z_2|) \end{array}$$

Amoeba



Limit:

Amoeba $\longrightarrow$



Piecewise linear graph in $\mathbb{R}^2$

Tropical curve.

Tropical correspondence thm (Mikhalkin, Shustin, ...)

$$\begin{cases} GW_{0,\beta}^S = \displaystyle\sum_{\substack{h : \Gamma \to \mathbb{R}^2 \\ \uparrow \\ \text{3-valent}}} \prod_{\substack{v \\ \text{vertices} \\ \text{of } \Gamma}} m_v^{\mathbb{C}} \\ W_\beta^S = \displaystyle\sum_{h : \Gamma \to \mathbb{R}^2} \prod_{v \text{ vertices of } \Gamma} m_v^{\mathbb{R}} \end{cases}$$

Block-Göttsche:

$$\underbrace{BG_\beta^S(q)}_{\substack{\text{Tropical}\\ \text{definition.}}} := \sum_{\substack{h:\Gamma\to\mathbb{R}^2\\ \cdots}} \prod_v [m_{C,v}]_q \quad \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

$$[m_{C,v}]_q = \frac{q^{\frac{m_{C,v}}{2}} - q^{-\frac{m_{C,v}}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad q\text{-integer}$$

$(n \leq 3)$

$$\xrightarrow{q\to 1} GW_{0,\beta}^S$$

$$\xrightarrow[q\to -1]{} W_\beta^S$$

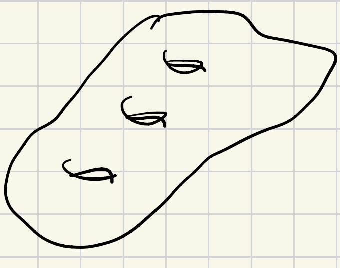
Ex: . $\mathbb{P}^2$ $d=3$ $n=8$ $BG = q^{-1} + 10 + q$

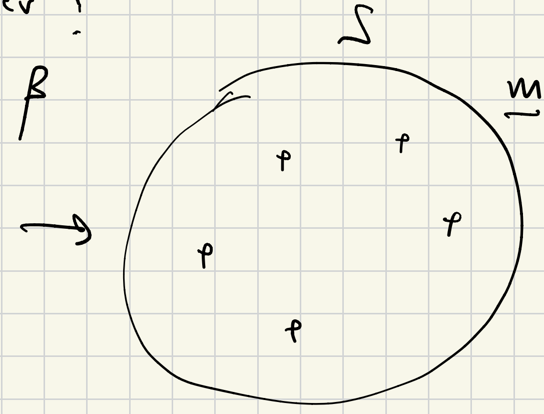
. $\mathbb{P}^2$ blownup in 6pts $n=6>3$: no def of BG.

First question: geometric interpretation of $BG_\beta^S(q)$?

$GW_{0,\beta}^S$ Extra parameter?

Genus g curves β





$[\overline{M}_g(S,\beta,\dots)]^{vir}$ g-dim virtual class

stable maps

$\mathbb{E}$ Hodge bundle $\overset{\mathrm{Rk}\, g}{}$ $H^0(C,\omega_C)$ dim g

$$\mathbb{E} \downarrow \overline{M}_g(S,\beta,\dots)$$

$(f: C \longrightarrow S)$ $\lambda_g := c_g(\mathbb{E})$

$$\left[GW^S_{g,\beta} := \int_{[\overline{M}_g(S,\beta,\dots)]^{vir}} (-1)^g \underline{\underline{\lambda_g}} \in \mathbb{Q} \right]$$

<u>Thm</u> (B, 2017) $\underline{n \leq 3}$

$$BG^S_\beta(q) = \frac{\sum_g GW^S_{g,\beta}\, u^{2g+m-1}}{\left(2\sin\left(\frac{u}{2}\right)\right)^{m-1}}$$

$$q = e^{iu} = \sum_{n\geq 0} \frac{(iu)^n}{n!}$$

$q \to 1$

$u \mapsto 0$

$\rightsquigarrow GW^S_{0,\beta}$

$q \to -1$???

$u = \pi$???

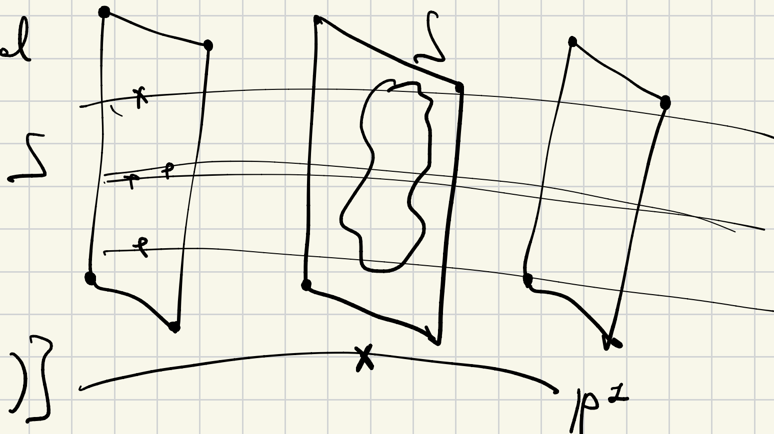
Thm: (AB) $\forall n$ $\exists$ $\mathrm{BPS}_\beta(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$

s.t. $$\mathrm{BPS}_\beta(q) = \frac{\sum_g GW^S_{g,\beta}\, u^{2g+m-1}}{\left(2\sin\left(\frac{u}{2}\right)\right)^{m-1}}$$

$q = e^{iu}$

$GW^S_{g,\beta}$ = genus g GW invt of a 3-fold $S \times \mathbb{P}^1$

dim 3 : expected / virtual dim is independent of g!



$$\Rightarrow \underbrace{[\overline{M}_g(S\times\mathbb{P}^1, \beta, \dots)]}_{\text{dim } 0}$$

$$e(H^1(C, f^* \underbrace{\underbrace{N_{S|S\times\mathbb{P}^1}}_{\mathcal{O}_S}}_{\mathcal{O}_C}))$$

All g GW of 3-folds

Integrality / BPS conjectures or results

[Zinger, Doan-Walpuski]

$H^1(C, \mathcal{O}_C)$

$= H^0(C, \omega_C)^\vee$

Rem: $n \leq 3$ $\quad$ BPS $=$ BG.

. $\mathbb{P}^2$ blown up in $\underset{n}{\overset{\shortparallel}{6}}$ pts, β $\rightsquigarrow$ $GW_{0,\beta}^S = 3240$

$$\left[\begin{aligned} BPS_\beta^S = q^{-4} + 13q^{-3} + 100q^{-2} + 547q^{-1} + 1918 \\ q^4 + 13q^3 + 100q^2 + 547q \end{aligned}\right]$$

$$BPS_\beta^S(q=1) = GW_{0,\beta}^S$$

$q=-1$

1000

[Conj: $\forall n$ $\quad \underbrace{BPS_\beta^S(q=-1)} = \underline{W_\beta^S}$

[Thm (AB) Conj is True for $\underline{n \leq 6}$.

(likely: $n = 7, 8$?) $\quad \underline{n > 8}$

E. Brugallé [IKS .....]