Weighted Twisted Stable Maps

(mostly joint w/ Martin Olsson)

§1 A "classical" ~~story~~: GW theory & weighted stable maps

no twisted yet

Recall Gromov-Witten theory:

$\cancel{X}$ sm ~~proj~~ ~~var~~ DM stack $\Rightarrow$ $\boxed{\text{Gromov-Witten}}$ $\Rightarrow \langle \ldots \rangle^{X}_{g,n,d} : H^*(\cancel{X})^{\otimes n} \to \mathbb{Q}$

$I_n(\mathcal{X})$

depends on $\overline{\mathcal{M}}_{g,n}(\cancel{X}, d)$ moduli of stable maps

no weighted, yet

twisted (Abramovich-Vistoli)

$ev_i : \overline{\mathcal{M}}_{g,n}(\cancel{X}, d) \longrightarrow I_n(\cancel{X})$

$[\overline{\mathcal{M}}_{g,n}(\cancel{X}, d)]^{vir} \in A_*(\overline{\mathcal{M}}(\cancel{X}))$ virtual class

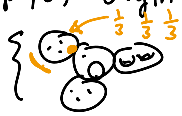
$\langle \ldots \rangle = \text{push to point}\left([\overline{\mathcal{M}}(\cancel{X})]^{vir} \cap (ev_1^*\gamma_1 \ldots ev_n^*\gamma_n)\right)$

Problem: $\overline{\mathcal{M}}_{g,n}(X,d)$ has complicated geometry.

Idea: Find a geometrically simpler compactification of an open subset of $\overline{\mathcal{M}}_{g,n}(X,d)$ w/ ev_i & $[\,\cdot\,]^{vir}$

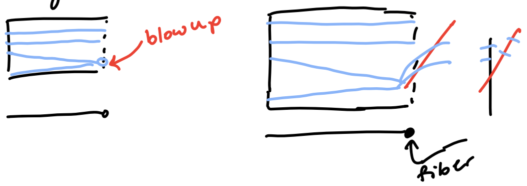
Stage 1 Hassett's weighted stable curves (the case when $X = pt$)

$\overline{\mathcal{M}}_{g,n}(pt, 0) = \overline{\mathcal{M}}_{g,n}$ = stable curves

$= \{$  $\mid$ distint marked points, each component has $2g-2+\#\text{nodes}+\#\text{mks} > 0 \}$

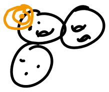
compactification of $\mathcal{M}_{g,n} = \{$  $\}$

Ex: $\mathcal{M}_{g,n}$ is not compact:



$\overline{\mathcal{M}}_{g,\underline{a}}(pt, 0) = \overline{\mathcal{M}}_{g,\underline{a}}$ = weighted stable curves

$\underline{a} \in (0,1]^n$

$= \{$  $\mid$ if $\{s_i\}_{i \in I}$ coincide then $\sum_{i \in I} a_i \le 1$, each component has $2g-2+\#\text{nodes}+\sum a_i > 0 \}$

Ex $\overline{\mathcal{M}}_{0,5} = Bl_{3\,pts}(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow \overline{\mathcal{M}}_{0,(\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\,1)} = \mathbb{P}^2$

Stage 2 Alexeev-Guy / Bayer-Manin's weighted stable maps

$\overline{\mathcal{M}}_{g,n}(X,d)$ = ~~stable maps~~ weighted stable maps $= \left\{ \begin{array}{c} s_i \curvearrowright C \xrightarrow{f} X \\ \downarrow \\ S \end{array} \;\middle|\; \begin{array}{l} \text{~~disjoint~~ sections } \sum_{i \in I} a_i \le 1 \\ \text{each compon is weighted stable} \\ \text{OR } f \text{ has pos degree} \end{array} \right\}$

$n \leftarrow \underline{a}$

weighted prestable curve

there are morphisms $\overline{\mathcal{M}}_{g,n}(X,d) \xrightarrow{Q} \overline{\mathcal{M}}_{g,\underline{a}}(X,d)$

Def

$$\begin{tikzcd}
\overline{\mathcal{M}}(X) \arrow[r, "ev_i"] \arrow[d, "Q"'] & X \\
\overline{\mathcal{M}}_{\underline{a}}(X) \arrow[ur, "ev_i"'] &
\end{tikzcd}$$

Fails! For weighted twisted

$$[\overline{\mathcal{M}}_{\underline{a}}(X)]^{vir} = Q_*[\overline{\mathcal{M}}(X)]^{vir}$$

Cor GW invariants = weighted GW invariants

NOTE: $\overline{\mathcal{M}}_{g,\underline{a}}(X,d)$ is still not that simple...

Stage 3 (Maulik) – Ciocane-Fontanine – Kim's quasimap theory & big I-functions

source curve can also bubble from basepoints of f:

$\mathbb{P}^1_{\mathbb{A}^1 \smallsetminus \{0\}} \xrightarrow{[ax^2 : xy : y^2]} \mathbb{P}^2$ ⟵ line bundle w/ 3 sections, ~~not all zero~~

Example: $Qmap_{0,\delta}([\mathbb{A}^N/\mathbb{G}_m] \times \mathbb{P}^1, (d,1))$

$= \mathbb{P}^{(d+1)(N+1)-1} \times \underbrace{\mathbb{P}^1 \times \ldots \mathbb{P}^1}_{n-3}$

Results: (CF-Kim) can use this to compute genus-0! GW invariants of GIT quotients

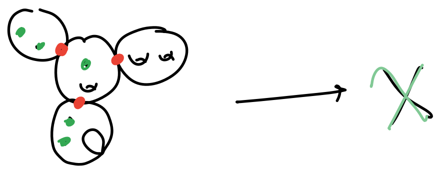
§2 Reasons to extend to orbifold setting

If you allow X to be a proper DM stack $\mathcal{X}$, then:

- This, physically, amounts to X have mild singularities
- Costello: $g=0$ invariants of X^n/S_n related to $g=n-1$ invariants of X

What changes when you do GW with $\mathcal{X}$ a DM stack?

What are <u>twisted</u> stable maps?



$\mathcal{C} = (C, \{s_i\}_{i=1}^n, M_S \to M_S' \; \{r_i\}_{i=1}^n)$

integers ≥ 1

↑ simple Inclusion of log strs

Challenge: Find geometrically simpler compactification(s) of $\mathcal{M}_{g,n}(\mathcal{X},d) = \{$  $\to \mathcal{X}\}$

w/ ev_i's, $[\;]^{vir}$

and relate new weighted twisted GW invariants to old.

§ 3 Theorems making progress toward this challenge

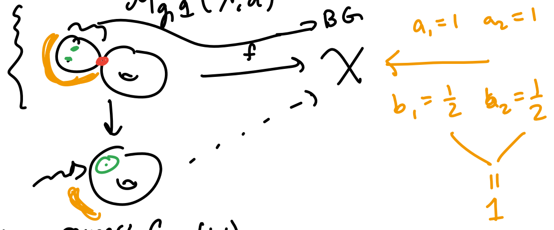
Thm A (Olsson-W) There are algebraic stacks of weighted stable generalized log twisted curves and contraction morphisms $\mathcal{M}^{glt}_{g,\underline{a}} \to \mathcal{M}^{glt}_{g,\underline{b}}$ $\underline{b} \le \underline{a}$

Thm B (Olsson-W) There are proper algebraic stacks of weighted twisted stable maps to $\mathcal{X}$ and contraction morphisms, for any $\underline{b} \le \underline{a}$

$$\overline{\mathcal{M}}^{cont}_{g,\underline{a}}(\mathcal{X},d) \longrightarrow \overline{\mathcal{M}}_{g,\underline{b}}(\mathcal{X},d)$$

$$\cap\!|$$

$$\overline{\mathcal{M}}_{g,\underline{a}}(\mathcal{X},d)$$



Thm-in-progress C (W)

$\overline{\mathcal{M}}_{g,\underline{a}}(\mathcal{X},d)$ have evaluation maps & virtual classes.

⇒ Can define GW.

Conjecture D (…, based on a theorem of Bayer-Cadman $\mathcal{X} = \mathbb{C}^N/\mu_r$)

If $\underline{b} \le \underline{a}$

$$[\overline{\mathcal{M}}^{cont}_{g,\underline{a}}(\mathcal{X},d)]^{vir} = Q^*[\overline{\mathcal{M}}_{g,\underline{b}}(\mathcal{X},d)]^{vir} \cdot \left(\begin{array}{l}\text{explicit}\\ \text{formula in}\\ \text{terms of } \psi \text{ \&}\\ \text{divisor classes}\end{array}\right)$$