A variational inequality for discontinuous solutions of degenerate parabolic equations

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Abstract

The Beltrami framework for image processing and analysis introduces a non-linear parabolic problem, called in this context the Beltrami flow. We study in the framework for functions of bounded variation, the well-posedness of the Beltrami flow in the one-dimensional case. We prove existence and uniqueness of the weak solution using lower semi-continuity results for convex functions of measures. The solution is defined via a variational inequality, following Temam's technique for the evolution problem associated with the minimal surface equation.

Keywords: Degenerate equation, Beltrami framework, lower semi-continuity, convex functionals, weak solution.

1 Introduction

Non-linear PDEs are used extensively in recent years for different tasks in image processing. In many cases the mathematical properties of these equations are not rigorously treated. We study in this work the well-posedness of a non-linear parabolic problem, called in this context the *Beltrami flow*, that emerges in the Beltrami framework [18] for image denoising. The Beltrami flow is known (see [17], [18]) as a powerful edge preserving technique for denoising of signals and images. This flow originates from the minimization of the area of the two-dimensional image manifold embedded in $\mathbb{R}^3$ for gray-scale images, and in $\mathbb{R}^5$, for color images. A short review of the Beltrami flow is presented in Section 3 below.

Let Ω be a bounded set in $\mathbb{R}^n$ and its boundary $\partial\Omega$ of class C^1. We are interested in establishing the well-posedness for the following Neumann problem:

$$(P_1) \qquad \begin{aligned} & u_t = \operatorname{div}(g(Du)), \quad (x,t) \in \Omega \times (0,T) && (1.1) \\ & u(x,0) = u_0(x), \quad x \in \Omega && (1.2) \\ & \frac{\partial u}{\partial \nu}|_{S_T} = 0. && (1.3) \end{aligned}$$

where $u_0 \in BV(\Omega) \bigcap L^\infty(\Omega)$, $g(\xi) = \nabla_\xi G(\xi)$, and G is a convex function with linear growth as $||\xi|| \to \infty$.

We study here mainly the one-dimensional case. The Beltrami flow in the one-dimensional case is a particular case of equation (1.1). Few results for the n-dimensional case will be discussed though as well.

This problem was much considered in the recent years and various results are known. We present in detail the most relevant results in the next subsection. Equation (1.1) is a particular case of a more general class of degenerate equations studied in Andreu-Caselles-Mazon [3]. In that work, the existence and uniqueness of the weak solution for this problem were proved via the concepts and techniques of the entropy solution.

In the present work we propose a generalization of Temam's definition of the weak solution via a variational inequality [14]. For the particular class of equations characterized by equation (1.1), the variational inequalities approach enables to give a shorter and simpler proof of the well-posedness for the problem (P_1) in the BV space.

The structure of the paper is as follows: In Section 2 we review previous relevant works. In Section 3 we shortly describe the Beltrami framework and flow. We remind some known facts about functions with bounded variations in section 4. We motivate the definition of the weak solution in Section 5 and in Section 6 we give our main result. Section 7 provides some comments on flows in higher dimensions. We conclude in Section 8.

2 Previous related works

For many PDE-based models in image processing the only known result of existence and uniqueness is under the condition that the initial data are of Lipschitz type (see [11]). This assumption is, in general, inappropriate for images or signals. This is due to the fact that images contain in general edges, i.e. discontinuities. The proper space for images should be therefore the BV space which allows discontinuities. It is necessary thus to study *weak* solutions in the more realistic BV space to the suggested PDE-based models.

We describe, in the rest of this section, the main known results of well-posedness for problems which are related to ours.

In [8] the existence and uniqueness of the following problem is considered:

$$u_t = (\phi(u)b(u_x))_x, \quad (x,t) \in \mathbb{R} \times (0,T) \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2.2}$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and strictly positive, and $b : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly increasing and odd function that is approaching a constant value at infinity. The initial data is a strictly increasing bounded function. However, this approach cannot be used for generalizations to higher dimensions.

Rosenau [16] studied equations of type

$$u_t = \frac{\partial}{\partial x}\Big(\frac{u_x}{(1+u_x^2)^{1/2}}\Big)$$

in the context of thermodynamical theory of phase transition. He showed that free-energy functionals have a unique infinite-gradient limit which assures a finite energy.

Barenblatt [7] considered various flows of relevance in image processing and studied them in the limit of very large gradients. He arrived at the modified equation

$$u_t = \frac{u_{xx}}{(u_x^2)^{1+\alpha}}, \alpha \in \mathbb{R}_+.$$

Through the analysis of intermediate-asymptotics solutions for the modified equation, he demonstrated that edge-enhancement takes place.

There are various works which study the degenerate parabolic equations, and for which the entropy solution is used (see [3], [4], [5], [10]). The n-dimensional Neumann problem associated with the equation $u_t =$

$\operatorname{div}(a(u, Du))$ is studied in [3]. Here $a(z,\xi) = \nabla_\xi f(z,\xi)$, where f is a function with linear growth as $||\xi|| \to \infty$. For initial data $u_0 \in L^1(\Omega)$, the existence of the entropy solution is shown by using the Crandall-Ligget scheme and the uniqueness of the entropy solution is proved by means of Kruzhkov's technique of doubling variables.

Equation (1.1) is a particular case of the more general class of degenerate equations considered in [3]. In this work, we propose a simpler method, namely the method of variational inequalities for showing existence and uniqueness of the problem (1.1),(1.2),(1.3). The possibility of the use of variational inequalities for the study of discontinuous solutions is interesting. In [14], the method of variational inequalities was used to prove well-posedness of the evolution problem associated with the minimal surface equation:

$$u_t = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big), \quad (x,t) \in \Omega \times (0,T) \tag{2.3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2.4}$$

$$u|_{S_T} = \Phi, \tag{2.5}$$

where the initial data u_0 belongs to the Sobolev space $W^{1,2}(\Omega)$, and the boundary function $\Phi \in W^{1,1}(\Omega)$.

This work can be seen as a generalization of the Temam's work [14] for Neumann problems associated with more general divergence flows and with initial data from the BV space.

To conclude this Section we mention the works by Anzelotti [1, 2], who used the variational inequalities and BV spaces for the study of stationary problems. Combination, rather not trivial, of Anzelotti's results and Theorem 3.2 in the book of Brezis [9] leads to the statement which is close to Theorem 6.1 below. Nevertheless we point out that our approach is a direct one and simpler.

3 The Beltrami flow

In this section we review the Beltrami framework [17, 18] for image denoising. In this framework an image, and other local features, are represented as embedding maps of a Riemannian manifold into a higher dimensional space. The simplest example is a gray-level image. The graph of the brightness function is regarded as a 2D surface embedded in $\mathbb{R}^3$. We denote the map by $U : \Sigma \to \mathbb{R}^3$, where Σ is a two-dimensional surface, and we denote the local coordinates on it by (σ^1, σ^2). The map U is given in general by $(U^1(\sigma^1,\sigma^2), U^2(\sigma^1,\sigma^2), U^3(\sigma^1,\sigma^2))$. In our example we represent it as follows : $(U^1 = \sigma^1, U^2 = \sigma^2, U^3 = I(\sigma^1,\sigma^2))$, where $I(\cdot)$ is the brightness/intensity function.

On this surface we choose a Riemannian structure, namely, a metric. A metric is a positive definite and a symmetric 2-tensor that may be defined through the local distance measurements:

$$ds^2 = g_{11}(d\sigma^1)^2 + 2g_{12}d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2.$$

Cartesian coordinates are usually chosen in image processing. For these coordinates, we identify $\sigma^1 = x^1$ and $\sigma^2 = x^2$. Below we use the Einstein summation convention in which the above equation reads $ds^2 = g_{ij}dx^i dx^j$, where repeated indices are summed. We denote the elements of the inverse of the metric by superscripts $g^{ij} = (g^{-1})_{ij}$.

Once the image is defined as an embedding mapping of Riemannian manifolds it is natural to look for a measure on this space of embedding maps.

3.1 Polyakov Action: A measure on the space of embedding maps

Denote the image manifold and its metric by (Σ, g) and by (M, h) the space-feature manifold and its metric. Then the functional $S[U]$ attaches a real number to a map $U : \Sigma \to M$,

$$S[U^a, g_{ij}, h_{ab}] = \int dV \langle \nabla U^a, \nabla U^b \rangle_g h_{ab}$$

where dV is a volume element and $\langle \nabla U^a, \nabla U^b \rangle_g = (\partial_{x_i} U^a) g^{ij} (\partial_{x_j} U^b)$. This functional, for $m = 2$ (a two dimensional image manifold) and $h_{ab} = \delta_{ab}$, was first proposed by Polyakov [15] in the context of high energy physics, and the theory is known as *string theory*.

Keeping in mind the form of the map U, the elements of the induced metric for gray-scale images are

$$g_{ij} = \delta_{ij} + I_{x_i} I_{x_j}. \tag{3.1}$$

This leads to the fact that the functional S is actually the area of the image manifold,

$$S = \int \sqrt{g}\, d\sigma_1 d\sigma_2, \tag{3.2}$$

where $g = \det(g_{ij}) = g_{11}g_{22} - g_{12}^2$.

Using standard methods in the calculus of variations, the Euler-Lagrange equations with respect to the embedding (assuming a Euclidean embedding space) are (see [17] for explicit derivation) :

$$0 = \frac{1}{\sqrt{g}}\partial_{x_i}(\sqrt{g}g^{ij}\partial_{x_j}I). \tag{3.3}$$

Since (g_{ij}) is positive definite, $g \equiv \det(g_{ij}) > 0$ for all σ^i. This factor is the simplest that does not change the minimization solution, while giving a reparametrization invariant expression. The operator that is acting on I is the natural generalization of the Laplacian from flat spaces to manifolds, and is called the Laplace-Beltrami operator, denoted by Δ_g.

Then for the gray-level images the non-linear diffusion emerges as a gradient descent minimization:

$$I_t = \frac{(1 + I_y^2)I_{xx} - 2I_xI_yI_{xy} + (1 + I_x^2)I_{yy}}{(1 + I_x^2 + I_y^2)^2}.$$

Remark 3.1 *This equation is not in divergence form.*

In the following sections we will study the one dimensional version of this equation, namely

$$I_t = \frac{I_{xx}}{(1 + I_x^2)^2}. \tag{3.4}$$

4 Preliminaries

In this section, we introduce basic notation, definitions and results on the space $BV(\Omega)$. For general results on BV spaces, we refer the reader to [6], [20].

Definition 4.1 *Let Ω be a bounded subset of $\mathbb{R}^N$ and $u \in L^1(\Omega)$ be a real-valued function on Ω. We set*

$$\int_\Omega |Du| = \sup_{v \in C_0^1(\Omega, R^N)} \{\int_\Omega u\,div(v)\,dx : |v_i(x)| \leq 1, \forall x \in \Omega, 1 \leq i \leq N\},$$

where v is a vector valued function $v = (v_1, v_2, ..., v_N)$.

Definition 4.2 *A function $u \in L^1(\Omega)$ is in $BV(\Omega)$ if*

$$\int_\Omega |Du| < \infty \ .$$

Remark 4.1 *(see [20]) A function $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_i, \ldots, \mu_n$ such that $|D\mu_i|(\Omega) < \infty \ \ \forall i$ and*

$$\int_\Omega u D_i v \, dx = - \int_\Omega v \mu_i, \tag{4.1}$$

for all $v \in C_0^\infty(\Omega)$.

Remark 4.2 *If $u \in BV(\Omega)$, the distributional gradient Du is a vector valued measure.*

Definition 4.3 *Let u be a BV function. The BV norm is defined as*

$$||u||_{BV} = \int_\Omega |Du| + ||u||_{L^1(\Omega)}.$$

Properties of the BV spaces

We denote by $M(\Omega)$ the set of all bounded measures on Ω.

P1) *A weak * topology*

We will not use the norm defined above, since it does not possess good compactness properties. We will work with the $BV - w^*$ topology, defined as

$$u_j \rightharpoonup u \ (BV - w^*) \Leftrightarrow u_j \to u \text{ in } L^1(\Omega) \quad \text{and} \quad Du_j \rightharpoonup Du \ (\text{in} \quad M(\Omega))$$

where

$Du_j \rightharpoonup Du$ (in $\;M$) means $\int_\Omega \phi Du_j \to \int_\Omega \phi Du$ for all $\phi \in C_0(\Omega)$.

P2) *Compactness*

For every bounded sequence u_j in $BV(\Omega)$, there exists a subsequence u_{j_k} and a function u in $BV(\Omega)$ so that $u_{j_k} \rightharpoonup u \ (BV - w^*)$.

P3) *Approximation* (Ziemer, [20])

Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_n\} \in C^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \int_\Omega |u_n - u| \, dx = 0$$

and

$$\lim_{n\to\infty}\int_\Omega |Du_n|\,dx = \int_\Omega |Du|.$$

P4) *Convex functions of measures*

The next issue concerns the functional

$$\int_\Omega G(Du), \quad u \in BV(\Omega),$$

where G is a convex function with linear growth at infinity.

Due to the convexity assumption, the asymptote G_∞ of G exists, that means

$$G_\infty(\xi) = \lim_{t\to\infty} \frac{G(t\xi)}{t} \quad \text{exists.}$$

Given a measure μ on Ω, its Lebesgue decomposition is :

$$\mu = hdx + \mu^s$$

Here h is the $|\mu|$-measurable function of modulus 1, i.e. $\mu = h|\mu|$ and μ^s is singular with respect to the Lebesgue measure μ.

We define $G(\mu)$ (see [12]) by setting

$$G(\mu) = G \circ hdx + G_\infty(\mu^s). \tag{4.2}$$

(here, the notation $\circ$ means composition of two functions).

Below we use the following lemma (Lemma 1.1, [12]):

For every $\phi \in C(\bar{\Omega})$, $\phi \geq 0$,

$$\int_\Omega \phi G(\mu) = \sup_{v\in\mathcal{D}_G(C_0^\infty)} \{\int_\Omega v\phi\mu - \int_\Omega G^*(v)\phi\,dx\}, \tag{4.3}$$

where G^*, the conjugate of G is defined as

$$G^*(\xi) = \sup_{y\in\mathbb{R}}\{y\cdot\xi - G(y)\},$$

and $\mathcal{D}_G(C_0^\infty)$ is:

$$\mathcal{D}_G(C_0^\infty) = \{v \in C_0^\infty(\Omega), G^*\circ v \in L^1(\Omega)\}. \tag{4.4}$$

Taking $\phi = 1$, $\mu = Du$ in (4.3), and using (4.1), for $u \in BV(\Omega)$, one gets

$$\int_\Omega G(Du) = \sup_{v\in\mathcal{D}_G(C_0^\infty)} \{-\int_\Omega u\mathrm{div}v\,dx - \int_\Omega G^*(v)\,dx\}. \tag{4.5}$$

5 Definition and motivation of a weak solution

In this section we start with the one-dimensional Beltrami flow. Adding the initial and boundary conditions to eq. (3.4), we arrive at the following problem:

$$u_t = \frac{u_{xx}}{(1+u_x^2)^2}, \quad (x,t) \in \Omega \times R^+ \tag{5.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{5.2}$$

$$u_x|_{x=a} = u_x|_{x=b} = 0, \quad t \in R^+, \tag{5.3}$$

where

$$\Omega = (a,b). \tag{5.4}$$

Note that equation (5.1) is degenerate if we allow u to be discontinuous. In order to include the solutions with discontinuities, we use the space BV for the definition of the weak solution.

Let $T > 0$ be some fixed number. For $\tau \in [0,T]$ define $Q_\tau = (0,\tau) \times \Omega$. Note that below we use the two BV spaces, $BV(\Omega)$ and $BV(Q_T)$, where Ω is given in (5.4).

First, note that we can write the equation (5.1) in the following divergence form:

$$u_t = (g(u_x))_x \tag{5.5}$$

with

$$g(s) = \int^s \frac{1}{(1+t^2)^2}\,dt = \frac{1}{2}\big(\frac{s}{1+s^2} + \arctan s\big). \tag{5.6}$$

Equation (5.5) with g as in (5.6) is a particular case of equation (1.1) for $n = 1$.

Below, following [14], we give the definition of the weak solution. Assume first, that the solution u is sufficiently smooth to justify the following calculations.

For smooth functions $v \in C^2(0,T,C^2(\Omega))$ we multiply equation (5.1) by $v - u$, integrate by parts, and obtain

$$\int_{Q_T} u_t(v-u)\,dx\,dt + \int_{Q_T} g(Du)(Dv - Du)\,dx\,dt = 0. \tag{5.7}$$

Let G denote the primitive of the function g. In our case

$$G(s) = \frac{1}{2} s \arctan s. \tag{5.8}$$

From the convexity, it follows: $G(Dv) - G(Du) \geq g(Du)(Dv - Du)$.
Therefore equation (5.7) becomes :

$$\int_{Q_T} u_t(v-u)\,dx\,dt + \int_{Q_T} (G(Dv) - G(Du))\,dx\,dt \geq 0. \tag{5.9}$$

We are thus led to the following definition of a weak solution.

Definition 5.1 *A function $u \in L^\infty(Q_T) \bigcap L^\infty((0,T), BV(\Omega)) \bigcap \{u; Du \in M(Q_T)\}$ is called a weak solution of the problem (5.5),(5.2),(5.3) if $u_t \in L^2(Q_T)$,$u(x,0) = u_0(x)$, and u satisfies*

$$\int_{Q_T} u_t(v-u)\,dx\,dt + \int_{Q_T} (G(Dv) - G(Du)) \geq 0 \tag{5.10}$$

for all $v \in L^\infty(Q_T) \bigcap \{v; Dv \in M(Q_T)\}$, where Du is the distributional gradient in space only and $G(Du)$ and $G(Dv)$ should be understood as functions of measure as were defined in (4.2).

6 Main result

In this section we prove the existence and uniqueness of the weak solution to the problem (5.5), (5.2),(5.3).

Theorem 6.1 *Suppose $u_0 \in L^\infty(\Omega) \bigcap BV(\Omega)$. Then there exists a unique weak solution of the problem (5.5), (5.2), (5.3).*

Proof.
a) **Existence**
Let $\epsilon > 0$ and consider the following approximating problem :

$$(P_\epsilon) \qquad \begin{aligned} u_t &= (g(u_x))_x, \quad (x,t) \in (a,b) \times (0,T) && (6.1)\\ u_x|_{x=a} &= u_x|_{x=b} = 0, \quad t \in (0,T) && (6.2)\\ u(x,0) &= u_0^\epsilon(x), \quad x \in (a,b) && (6.3) \end{aligned}$$

where the function g is defined in (5.6).

The regularizing initial data are chosen such that $u_0^\epsilon \in C^\infty(\bar{\Omega})$, $(u_0^\epsilon)'(a) = (u_0^\epsilon)'(b) = 0$,

$$||u_0^\epsilon - u_0||_{L^\infty(\Omega)} \to 0, \text{ as } \epsilon \to 0, ||u_0^\epsilon||_{L^\infty(\Omega)} \leq ||u_0||_{L^\infty(\Omega)} + 1 = m_0 \qquad (6.4)$$

and

$$\int_\Omega |(u_0^\epsilon)'| dx \leq C(\Omega) \int_\Omega |Du_0|. \qquad (6.5)$$

The existence of such a sequence u_0^ϵ follows from the assumptions of the Theorem 6.1.

It is well known [13] that there exists a unique classical solution u^ϵ of problem P_ϵ. We shall establish some a priori estimates for the sequence $\{u^\epsilon\}$.

Lemma 6.1 *a) The sequence $\{u^\epsilon\}$ is uniformly bounded in $L^\infty(Q_T)$ and the sequence u_t^ϵ is uniformly bounded in $L^2(Q_T)$.*

b) The sequence $\{u^\epsilon\}$ is uniformly bounded in $BV(Q_T)$ and in $L^\infty((0,T), BV(\Omega))$.

c) The sequence $\{u^\epsilon\}$ converges in the space $L^\infty((0,T), L^2(\Omega))$ and the sequence $\{u^\epsilon(t,\cdot)\}$ converges in the space $L^2(\Omega)$ for all $t \in [0,T]$.

Proof.

a) By the maximum principle, we have

$$u^\epsilon(x,t) \leq m_0, \quad \text{for all} \quad (x,t) \in Q_T.$$

Next we multiply equation (6.1) by u_t^ϵ and get

$$\int_{Q_\tau} (u_t^\epsilon)^2 \, dx \, dt + \int_{Q_\tau} g(u_x^\epsilon) u_{xt}^\epsilon \, dx \, dt = 0.$$

$$\int_{Q_\tau} (u_t^\epsilon)^2 \, dx \, dt + \int_\Omega G(u_x^\epsilon)|_{t=\tau} \, dx = \int_\Omega G(u_{0,x}^\epsilon) \, dx. \qquad (6.6)$$

By (5.8), we have

$$\int_\Omega G(u_{0,x}^\epsilon) \, dx \leq C \int_\Omega |u_{0,x}^\epsilon| \, dx \leq \bar{C}(\Omega) \int_\Omega |Du_0|. \qquad (6.7)$$

In inequality (6.7) we used (6.5).

Taking $\tau = T$, it follows from (6.6) and (6.7) that

$$u_t^\epsilon \in L^2(Q_T). \tag{6.8}$$

Thus u_t^ϵ is uniformly bounded in $L^2(Q_T)$.

b) Relations (6.6) and (6.7) lead also to

$$\int_\Omega G(u_x^\epsilon)\,dx \le \bar{C}(\Omega)\int_\Omega |Du_0|, \quad \forall\, t \in [0,T]. \tag{6.9}$$

We have that there exist constants $\alpha > 0$ and $\beta > 0$ so that $s \arctan s > \alpha s - \beta$. Then

$$\int_\Omega G(u_x^\epsilon) \ge \alpha \int_\Omega |u_x^\epsilon|\,dx - \beta, \quad \forall\, t \in [0,T]. \tag{6.10}$$

From (6.9) and (6.10) it follows that

$$\int_\Omega |u_x^\epsilon|\,dx \le C_1 \int_\Omega |Du_0| + C_2, \quad \forall\, t \in [0,T]. \tag{6.11}$$

(Here $C_1 = \frac{\bar{C}}{\alpha}$ and $C_2 = \frac{\beta}{\alpha}$.)

The inequality (6.11) implies

$$||u^\epsilon(t,\cdot)||_{BV(\Omega)} < C_0, \quad \forall\, t \in [0,T], \tag{6.12}$$

where C_0 does not depend on ϵ and on t, that means the sequence u^ϵ is uniformly bounded in $L^\infty((0,T), BV(\Omega))$.

From (6.8) and (6.11) we get

$$||u^\epsilon||_{BV(Q_T)} < C, \tag{6.13}$$

where C does not depend on ϵ.

Thus we showed that u^ϵ is uniformly bounded in $BV(Q_T)$.

c) Next, we show that the sequence $u^\epsilon(t,\cdot)$ converges in the space $L^2(\Omega)$ for all $t \in [0,T]$. Consider u^{ϵ_m} and u^{ϵ_n} that satisfy (6.1) and (6.4).

Multiply the difference of the equations (6.1) for u^{ϵ_m} and u^{ϵ_n} by the difference $u^{\epsilon_m} - u^{\epsilon_n}$ to obtain :

$$\frac{1}{2}\int_\Omega (u^{\epsilon_m} - u^{\epsilon_n})^2\,dx|_{t=\tau} + \int_{Q_\tau} (g(u_x^{\epsilon_m}) - g(u_x^{\epsilon_n}))(u_x^{\epsilon_m} - u_x^{\epsilon_n})\,dx\,dt = \frac{1}{2}\int_\Omega (u_0^{\epsilon_m} - u_0^{\epsilon_n})^2\,dx. \tag{6.14}$$

Since $g(s)$ is a monotone increasing function, the following integral is nonnegative:

$$\int_{Q_\tau} (g(u_x^{\epsilon_1}) - g(u_x^{\epsilon_2}))(u_x^{\epsilon_1} - u_x^{\epsilon_2})\, dx\, dt \geq 0. \tag{6.15}$$

From (6.14), (6.15) and since τ is arbitrary in $[0,T]$, we conclude

$$||u^{\epsilon_m} - u^{\epsilon_n}||_{L^2(\Omega)} \to 0 \quad \text{for all} \quad t \in [0,T] \quad \text{as} \quad \epsilon_m, \epsilon_n \to 0,$$

which means the sequence $u^{\epsilon_n}(t,\cdot)$ converges in $L^2(\Omega)$ for all $t \in [0,T]$.

Moreover, u^{ϵ_n} converges in $L^\infty(0,T,L^2(\Omega))$.

□

Next, we pass to the limit $\epsilon \to 0$.

By Lemma 6.1, there exists $u \in L^\infty(0,T,L^2(\Omega))$ such that

$$||u^\epsilon(t,\cdot) - u(t,\cdot)||_{L^2(\Omega)} \to 0, \text{ for all}\, t \in [0,T] \text{ and } ||u^\epsilon - u||_{L^2(Q_T)}\,, \text{as}\, \epsilon \to 0. \tag{6.16}$$

Moreover, $u \in L^\infty(Q_T)$.

By (6.16) we also have

$$||u^\epsilon(t,\cdot) - u(t,\cdot)||_{L^1(\Omega)} \to 0, \text{ for all}\, t \in [0,T]\,, \text{as}\, \epsilon \to 0. \tag{6.17}$$

and

$$||u^\epsilon - u||_{L^1(Q_T)}\,, \text{as}\, \epsilon \to 0. \tag{6.18}$$

Since $||u_t^\epsilon||_{L^2(Q_T)} \leq C$, we can extract a subsequence still denoted by $\{u^\epsilon\}$ such that :

$$u_t^\epsilon \rightharpoonup u_t \quad \text{in} \quad L^2(Q_T) \quad \text{and} \quad u_t \in L^2(Q_T). \tag{6.19}$$

Relation (6.19) implies that the limit function satisfies

$$u(x,0) = u_0(x).$$

Next, we show that the limit function u has the property that $u \in BV(Q_T)$. By (6.13) and use of Property P2) we can extract a subsequence $\{u^{\epsilon_i}\}$ and find a function $\eta \in BV(Q_T)$ such that

$$u^{\epsilon_i}(x,t) \rightharpoonup \eta(x,t)\, (w^* \,\text{in}\, BV(Q_T)). \tag{6.20}$$

This implies that $u^{\epsilon_i} \to \eta$ in $L^1(Q_T)$. Taking (6.18) into consideration, we have that also $u^{\epsilon_i} \to u$ in $L^1(Q_T)$. Therefore

$$\eta = u. \tag{6.21}$$

We thus obtained that $u \in BV(Q_T)$. By the definition of the BV spaces, Du is a bounded measure on Q_T, i.e. $Du \in M(Q_T)$.

We now prove that for all $t \in [0,T]$, $u(t,\cdot) \in BV(\Omega)$.

From (6.20) and (6.21) we have

$$u^{\epsilon_i}(x,t) \rightharpoonup u(x,t) \quad (w^* \,\text{in}\, BV(Q_T)). \tag{6.22}$$

By (6.12)

$$||u^{\epsilon_i}(t,\cdot)||_{BV(\Omega)} < C, \quad \forall\, t \in [0,T], \tag{6.23}$$

where C does not depend on ϵ and t.

Fix $t_0 \in [0,T]$. By Property P2) we can again extract a subsequence of $\{u^{\epsilon_i}\}$, which we denote by $\{u^{\epsilon_j}\}$ such that

$$u^{\epsilon_j}(t_0,\cdot) \rightharpoonup U(t_0,\cdot) \quad (w^* \,\text{in}\, BV(\Omega)), \tag{6.24}$$

where the function $U(t_0,\cdot) \in BV(\Omega)$.

By the same reasoning as above, we have $u^{\epsilon_j}(t_0,\cdot) \to U(t_0,\cdot)$ in $L^1(\Omega)$. Obviously, $U = u$.

We thus obtain that, for all $t \in [0,T]$,

$$u(t,\cdot) \in BV(\Omega). \tag{6.25}$$

Next we prove the following lemma, which is similar to Proposition 2.2, in [3].

Lemma 6.2 *Let $u \in BV(\Omega)$ and G is a convex function with linear growth at infinity. Then the functional $\int_\Omega G(Du)$ is lower semi-continuous with respect to the L^1 convergence.*

Proof.

We use Property P4), (4.5). For a function $u \in BV(\Omega)$ and G a convex function with linear growth at infinity, we have

$$\int_\Omega G(Du) = \sup_{v \in \mathcal{D}_G(C_0^\infty)} \{-\int_\Omega u \,\text{div}\, v \,dx - \int_\Omega G^*(v)\,dx\}.$$

with G^* and $\mathcal{D}_G(C_0^\infty)$ defined as above in the section Preliminaries.

By Property P3), there exists a sequence $\{u^n\} \in C^\infty(\Omega) \bigcap BV(\Omega)$ such that $u^n \to u$ in $L^1(\Omega)$.

Then for any function $v \in \mathcal{D}_G(C_0^\infty)$ the following holds :

$$\int_\Omega -u \operatorname{div} v \, dx - \int_\Omega G^*(v) \, dx = \lim_{n\to\infty} \Big[\int_\Omega -u^n \operatorname{div} v \, dx - \int_\Omega G^*(v) \, dx \Big] \leq$$

$$\leq \liminf_{n\to\infty} \sup_{v \in \mathcal{D}_G(C_0^\infty)} \{ - \int_\Omega u^n \operatorname{div} v \, dx - \int_\Omega G^*(v) \, dx \} = \liminf_{n\to\infty} \int_\Omega G(Du^n) \, dx.$$

We thus obtained

$$\int_\Omega -u \operatorname{div} v \, dx - \int_\Omega G^*(v) \, dx \leq \liminf_{n\to\infty} \int_\Omega G(Du^n) \, dx. \tag{6.26}$$

Now in (6.26) we take the supremum over all $v \in \mathcal{D}_G(C_0^\infty)$ and get :

$$\sup_{v \in \mathcal{D}_G(C_0^\infty)} \Big(\int_\Omega -u \operatorname{div} v \, dx - \int_\Omega G^*(v) \, dx \Big) \leq \liminf_{n\to\infty} \int_\Omega G(Du^n) \, dx.$$

Therefore we obtained that if $u^n \to u$ in $L^1(\Omega)$, then

$$\int_\Omega G(Du) \leq \liminf_{n\to\infty} \int_\Omega G(Du^n) \, dx. \qquad \Box$$

Next, by using (6.17) and (6.25) and Lemma 6.2, we have

$$\liminf_{\epsilon\to 0} \int_\Omega G(Du^\epsilon) \, dx \geq \int_\Omega G(Du), \quad \forall t \in [0,T]. \tag{6.27}$$

By (5.8) and (6.11), we have

$$\int_\Omega G(Du^\epsilon) \, dx \leq \tilde{C}, \quad \forall t \in [0,T], \tag{6.28}$$

where $\tilde{C} = \tilde{C}(C_0)$ and the constant C_0 was defined in (6.12).
Therefore, (6.27) and (6.28) lead to :

$$u \in L^\infty((0,T), BV(\Omega)). \tag{6.29}$$

Until now we proved that u satisfies
$u \in L^\infty(Q_T) \bigcap L^\infty((0,T), BV(\Omega)) \bigcap \{u; Du \in M(Q_T)\}$.
Next, we show that u fulfills the variational inequality (5.10).

After integrating on $[0,T]$ in (6.27), we get :

$$\liminf_{\epsilon\to 0}\int_{Q_T} G(Du^\epsilon)\,dx\,dt \geq \int_{Q_T} G(Du). \tag{6.30}$$

We proceed now as in Section 5.1 and multiply eq. (5.1) by the function $v-u^\epsilon$, where v is a smooth function $v\in C^2(\bar{Q_T})$. After integration we get

$$\int_{Q_T} u_t^\epsilon(v-u^\epsilon)\,dx\,dt + \int_{Q_T} (G(Dv)-G(Du^\epsilon))\,dx\,dt \geq 0. \tag{6.31}$$

Now we have to show that (6.31) holds also for v as in Definition 5.1.

Let X be the space defined by $X=\{v\Big|v\in L^1(Q_T), Dv\in M(Q_T)\}$.

The space X_G is the space X equipped with the topology defined by the distance :

$$d(u,w)=|u-w|_{L^1(Q_T)}+\Big|\int_{Q_T}|Du|-\int_{Q_T}|Dw|\Big|+\Big|\int_{Q_T}G(Du)-\int_{Q_T}G(Dw)\Big|.$$

By [12] (Theorem 2.2, p. 689) and [19] (Theorem 2.2, p. 404), the space $C^\infty(Q_T)$ is dense in X_G. Then there exists a sequence of smooth functions $\{v^n\}_{n\in\mathbf{N}}$ such that

$$\int_{Q_T} G(Dv^n)\,dx\,dt \to \int_{Q_T} G(Dv), \text{as}\, n\to\infty, \tag{6.32}$$

and

$$\int_{Q_T} |v^n-v|\,dx\,dt \to 0, \text{as}\, n\to\infty. \tag{6.33}$$

For the smooth sequence $\{v^n\}_{n\in\mathbf{N}}$ we have :

$$\int_{Q_T} u_t^\epsilon(v^n-u^\epsilon)\,dx\,dt + \int_{Q_T} (G(Dv^n)-G(Du^\epsilon)\,dx\,dt \geq 0. \tag{6.34}$$

We have

$$\Big|\int_{Q_T} u_t^\epsilon(v^n-v)\,dx\,dt\Big| \leq \sup_{Q_T}|u_t^\epsilon||v^n-v||_{L^1(Q_T)}. \tag{6.35}$$

Using (6.32),(6.33) and (6.35), we pass to the limit as $n\to\infty$ in (6.34) and obtain

$$\int_{Q_T} u_t^\epsilon(v-u^\epsilon)\,dx\,dt + \int_{Q_T} G(Dv) - \int_{Q_T} G(Du^\epsilon)\,dx\,dt \geq 0, \tag{6.36}$$

for all $v \in L^\infty(Q_T)\bigcap\{v; Dv \in M(Q_T)\}$.

Using (6.19) and (6.30) we can pass to the lower limit ($\epsilon \to 0$) in the last inequality :

$$\int_{Q_T} u_t(v-u)\,dx\,dt + \int_{Q_T} (G(Dv) - G(Du)) \geq 0, \tag{6.37}$$

for all $v \in L^\infty(Q_T)\bigcap\{v; Dv \in M(Q_T)\}$.

Thus the existence of a weak solution is proved.

b) **Uniqueness**

Suppose there are two weak solutions u_1 and u_2 to our problem (5.1),(5.2), (5.3) satisfying

$$u_1(x,0) = u_2(x,0). \tag{6.38}$$

We use inequality (5.10) first for $u = u_1, v = u_2$, and then for $u = u_2, v = u_1$.

Then the following inequalities hold:

$$\int_{Q_T} \frac{\partial u_1}{\partial t}(u_2 - u_1)\,dx\,dt + \int_{Q_T} G(Du_2) \geq \int_{Q_T} G(Du_1). \tag{6.39}$$

$$\int_{Q_T} \frac{\partial u_2}{\partial t}(u_1 - u_2)\,dx\,dt + \int_{Q_T} G(Du_1) \geq \int_{Q_T} G(Du_2). \tag{6.40}$$

Adding the above inequalities we get

$$\int_{Q_T} \frac{\partial (u_1 - u_2)}{\partial t}(u_2 - u_1)\,dx\,dt \geq 0$$

Thus

$$\int_0^T \frac{d}{dt}\Big(\int_\Omega (u_1 - u_2)^2\,dx\Big)\,dt \leq 0.$$

By (6.38) we obtain :

$$||u_1(\cdot,T) - u_2(\cdot,T)||_{L^2(\Omega)} = 0.$$

As T is arbitrary, the uniqueness follows.

$\square$

7 Extensions and comments

In the same manner as we proceeded for the one dimensional case, we can extend the result of existence and uniqueness to the following Neumann n- dimensional problem:

$$u_t = \operatorname{div}(g(Du)), \quad (x,t) \in \Omega \times (0,T) \tag{7.1}$$
$$u(x,0) = u_0(x), \quad x \in \Omega \tag{7.2}$$
$$\frac{\partial u}{\partial \nu}|_{S_T} = 0. \tag{7.3}$$

where Ω is a bounded set in $\mathbb{R}^n$, $u_0 \in BV(\Omega) \bigcap L^\infty(\Omega)$, $g(\xi) = \nabla_\xi G(\xi)$, and the function G satisfies:

H1) G is a convex function;

H2) G satisfies the following linear growth condition:

$$C_1||s|| - D_1 \leq G(s) \leq M(||s|| + 1),$$

where C_1, D_1 and M are some positive constants.

That means that the existence and uniqueness in any dimension n for the class of equations (7.1), where the flux does not depend explicitly on x or u, with assumptions H1), H2) can be proved by the method of variational inequalities.

8 Conclusions

Motivated by problems from image processing we apply, in this paper, the variational inequalities technique to prove the well-posedness of a class of degenerate equations with discontinuous initial data. The weak solution belongs to the BV space, which allows discontinuities. This space is a good model for images.

The method of variational inequalities provides a relatively short and simple proof of existence and uniqueness of the solution in the BV space.

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