A Complete System of Measurement Invariants for Abelian Lie Transformation Groups

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Abstract. We present a complete system of functionally independent invariants for Abelian Lie transformation groups acting on an image. The invariants are based on measurements, given by inner product of predesigned functions and the image. We build on steerable filters and adopt a Lie theoretical approach that is applicable to any dimensionality. A complete characterization of Lie measurement invariants of a general irreducible component of the group, termed block invariants, is provided. We show that invariants for the entire group can be taken as the union of the invariants of its components. The system is completed by deriving invariants between components of the group, termed cross invariants.

1 Introduction

The problem of invariance to transformations has been studied extensively in the past. There are many different types of invariants in image processing and computer vision. We will present below invariants which are obtained from dense measurements of an image. For the fascinating subject of differential invariants the interested reader is referred to the excellent book by Olver [7]. Another direction of research is moment invariants. These invariants, to linear transformations, including translation, scaling and rotation, have been introduced by Hu [3] as early as 40 years ago. In this remarkable work the theories of algebraic invariants and moment invariants have been connected. Keren [4] applied algebraic invariants to models which are symbolically defined by implicit polynomials. A fundamental problem with moments is that they vanish for symmetric or antisymmetric images and hence information is lost. To correct this Palaniappan et al [8] proposed improved moment invariants. In this method acentric moments satisfying properties similar to the regular moments are used and their displacement depends on regular moments of the image. The invariants are constructed from these new moments. Zitová and Flusser [8] extended the scope of moment invariance to convolution, rotation, scaling, translation and contrast changes. Invariant distance metrics have also been a focus of research. Werman and Weinshall [11] proposed an invariants distance metric for 2D point sets. Simard et al [10] introduced the locally invariant tangent distance metric using a Lie theoretical approach. Semi-differential invariants were presented and studied in [6]. Kernel invariants were introduced by Segman et al [9] who showed a method to estimate the transformation between two images, considering an Abelian Lie transformation group model for the deformation between the images, by applying frequency domain techniques to the canonical coordinates of the group. Many uses of Lie groups, Lie algebras and their representations are found in the excellent book by Lenz [5]. The seminal work by Teo and Hel-Or [2] reformulated the problem of steerability, estimation and invariance of features on a unified Lie algebraic ground. It offers the basic differential equations which are satisfied by the measurement invariants as well. However, explicit expressions for the complete system of invariants, derived from these equations, have never been derived.

We present here, for the first time, a complete system of invariants[3]. The system of invariants we develop is based exclusively on inner products with a positive definite weighting function w, defined by $\langle \phi, \psi \rangle_w \doteq \int \phi \psi w$. The invariance is with respect to deformations modelled by an Abelian Lie transformation group. The restriction to a Lie group transformation is not a serious limitation, as most deformations considered in computer vision and pattern recognition, e.g. translation, rotation, scaling, projection and many more are Lie transformations groups.

The rest of this paper is organized as follows. In section 2 we describe the framework due to Teo and Hel-Or [2]. In section 3 we analyze irreducible components of Abelian Lie transformation groups by considering their conjugate generator. In section 4 we develop the complete system of differential invariants. In section 5 we discuss experimental results. We conclude in section 6.

2 Steerability and Equivariance

We consider conjugate generators of Lie transformation groups and their connection to the action of the group on measurements.

2.1 Lie Transformation Groups and Generators

A Lie transformation group $\mathrm{G}(\tau)$ is a set of transformations in a k dimensional parameter domain P parameterized by $\tau \doteq (\tau_1, \ldots, \tau_k)$ with a continuous group structure. Recall that a group structure implies the following. For any $\mathcal{T}(\alpha), \mathcal{T}(\beta), \mathcal{T}(\gamma) \in \mathrm{G}(\tau)$

- Closure: The composition $\mathcal{T}(\alpha\beta) \doteq \mathcal{T}(\alpha)\mathcal{T}(\beta)$ belongs to $\mathrm{G}(\tau)$,
- Existence of the identity: There exists the identity transformation $\mathcal{I} \in \mathrm{G}(\tau)$ such that $\mathcal{I}\mathcal{T}(\alpha) \equiv \mathcal{T}(\alpha)\mathcal{I} \equiv \mathcal{T}(\alpha)$,
- Existence of an inverse: There exists an inverse transformation $\mathcal{T}(\alpha)^{-1} \in \mathrm{G}(\tau)$ such that $\mathcal{T}(\alpha)\mathcal{T}(\alpha)^{-1} \equiv \mathcal{T}(\alpha)^{-1}\mathcal{T}(\alpha) \equiv \mathcal{I}$,
- Associativity: $(\mathcal{T}(\alpha)\mathcal{T}(\beta))\mathcal{T}(\gamma) \equiv \mathcal{T}(\alpha)(\mathcal{T}(\beta)\mathcal{T}(\gamma)) \doteq \mathcal{T}(\alpha\beta\gamma)$.

Note that commutativity is not implied.

In the following, we will consider only the two dimensional case. The extension to higher dimensions is straightforward. We denote $I(x, y)$ as the image and analyze the following Lie transformation groups.

- Uniform brightness scaling: $\mathrm{G}_\sigma(\sigma)$ is the group of transformations of the form $\mathcal{T}(\sigma)I(x, y) \doteq e^\sigma I(x, y)$,

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– y translation and y scaling: $\mathrm{G}_{\mu_y,\sigma_y}(\tau_y,\sigma_y)$ is the group of transformations of the form $\mathcal{T}(\tau_y,\sigma_y)I(x,y) \doteq I(x, e^{-\sigma_y}y - \tau_y)$,
– Rotation and uniform scaling: $\mathrm{G}_{\mu_\theta,\sigma_r}(\tau_\theta,\sigma_r)$ is the group of transformations of the form $\mathcal{T}(\tau_\theta,\sigma_r)I(x,y) \doteq I(e^{-\sigma_r}x', e^{-\sigma_r}y')$ where $x' \doteq x\cos\tau_\theta - y\sin\tau_\theta$ and $y' \doteq x\sin\tau_\theta + y\cos\tau_\theta$ are the rotated coordinates and r,θ are the polar coordinates.

Of fundamental importance to the analysis of Lie groups is the studying of the infinitesimal action of the group with respect to each of its parameters about the identity. The differential operators defined by

$$\mathcal{L}_i \doteq \frac{d}{d\tau_i}\mathcal{T}(\tau)\Big|_{\tau=0} \equiv \frac{\partial x}{\partial \tau_i}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau_i}\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau_i}\Big|_{\tau=0}, \quad i = 1,\dots,k \tag{1}$$

are called generators of the group. We follow the convention that the identity transformation corresponds to $\tau = 0$ for all group parameterizations hereafter. By equation (1) we have

$$\begin{aligned} \mathrm{G}_\sigma &\rightsquigarrow \mathcal{L}_\sigma \equiv \mathcal{I} \\ \mathrm{G}_{\mu_y,\sigma_y} &\rightsquigarrow \mathcal{L}_{\mu_y} \equiv -\partial_y, \quad \mathcal{L}_{\sigma_y} \equiv -y\partial_y \\ \mathrm{G}_{\mu_\theta,\sigma_r} &\rightsquigarrow \mathcal{L}_{\mu_\theta} \equiv x\partial_y - y\partial_x \equiv -\partial_\theta, \quad \mathcal{L}_{\sigma_r} \equiv -x\partial_x - y\partial_y \equiv -r\partial_r. \end{aligned} \tag{2}$$

The set of elements $\{\tau_i\mathcal{L}_i\}$ is called the *tangent space* - a linear vector space. The commutator, or the Lie bracket, defines a multiplication rule for the this space by $[\mathcal{L}_1,\mathcal{L}_2] \doteq \mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1$. The resulting algebra is called the Lie algebra associated with the Lie group. Each element of the group may be generated then from an element of the tangent space (i.e. the associated Lie algebra) using the *exponential map* $\mathcal{T}(\tau_i) \equiv e^{\tau_i\mathcal{L}_i}$ and $\mathcal{T}(\tau) \equiv e^{\tau_1\mathcal{L}_1+\cdots+\tau_k\mathcal{L}_k}$. The vanishing of the commutator for each pair of generators of a given group is equivalent to the commutativity, or Abelianity, of the group. It is easy to verify that $[\mathcal{L}_{\mu_y},\mathcal{L}_{\sigma_y}] \neq 0$ and hence $\mathrm{G}_{\mu_y,\sigma_y}$ is non-commutative, but $[\mathcal{L}_{\mu_\theta},\mathcal{L}_{\sigma_r}] \equiv 0$ and hence $\mathrm{G}_{\mu_\theta,\sigma_r}$ is commutative. The action of an Abelian group is separable by $\mathcal{T}(\tau) \equiv e^{\tau_1\mathcal{L}_1}\cdots e^{\tau_k\mathcal{L}_k}$ irrespective of the order of multiplication. We will hereafter exploit the separability of Abelian groups and treat them as compositions of single parameter groups each generated by a single generator of the group.

2.2 Action on Measurements

Let $\mathbf{b} \doteq (b_0,\dots,b_n)^t$ be a vector of measuring functions applied to the image $I(x,y)$. We denote the resulting vector of measurements, also called *features*, by

$$(f_0,\dots,f_n)^t \doteq \mathbf{f} \doteq \langle \mathbf{b}, I\rangle \equiv \iint \mathbf{b}I\,dxdy. \tag{3}$$

We define the *transformed features* as those resulting from applying a transformation $\mathcal{T}(\tau)$ to the image and then taking measurements, denoted

$$(\mathcal{T}(\tau)f_0,\dots,\mathcal{T}(\tau)f_n)^t \doteq (f_0(\tau),\dots,f_n(\tau))^t \doteq \mathbf{f}(\tau) \equiv \iint \mathbf{b}\mathcal{T}(\tau)I(x,y)\,dxdy. \tag{4}$$

Note that by our group parameterization convention $\mathbf{f} \doteq \mathbf{f}(0)$. It is possible to select measuring functions for which the transformed features may be computed exactly from the original features. For example, take G_{μ_x} as the group of x translations defined by $\mathcal{T}(\tau_x)I(x,y) \doteq I(x-\tau_x, y)$ and choose $\mathbf{b} \equiv (e^x, e^x + xe^x)$. Then

$$f_0(\tau_x) \equiv \iint e^x I(x-\tau_x, y)dxdy \equiv \iint e^{u+\tau_x} I(u,y)dudy \equiv e^{\tau_x} \iint e^u I(u,y)dudy \equiv e^{\tau_x} f_0 \tag{5}$$

and

$$\begin{aligned} f_1(\tau_x) &\equiv \iint (e^x + xe^x) I(x-\tau_x, y)dxdy \\ &\equiv \iint (e^{u+\tau_x} + (u+\tau_x)e^{u+\tau_x}) I(u,y)dudy \\ &\equiv e^{\tau_x} \iint (\tau_x e^u + e^u + ue^u) I(u,y)dudy \\ &\equiv e^{\tau_x}(\tau_x f_0 + f_1). \end{aligned} \tag{6}$$

Hence $\mathbf{f}(\tau_x)$ are linear in $\mathbf{f}$ in a manner solely dependent on τ_x. This property is fundamental and we formalize it next.

The set of all images attainable from the image $I(x,y)$ by applying a transformation from the group is called the *image orbit* and denoted $\mathtt{O}(\mathrm{G}(\tau), I) \doteq \{\mathcal{T}(\tau)I | \tau \in \mathtt{P}\}$. Similarly, the set of all features corresponding to the image orbit is called *feature orbit* and denoted $\mathtt{O}(\mathrm{G}(\tau), \mathbf{f}) \doteq \{\langle \mathbf{b}, \mathcal{T}(\tau)I \rangle | \tau \in \mathtt{P}\}$. The span of the feature orbit is called a *feature space*.

Definition 1 *A feature space, derived from the measuring functions* $\mathbf{B}$*, is called equivariant under* $\mathrm{G}(\tau)$ *if* $\mathbf{f}(\tau) \equiv \mathsf{A}(\tau)\mathbf{f}$ *where* A *is a matrix solely dependent on* τ*. In this case the latter relation is termed interpolation equation and* $\mathbf{b}$ *are called equivariant measuring functions [2].*

We also refer to the functions $\mathbf{b}$ as *steerable filters* [1] due to the possibility to steer, or transform, the filters outputs directly instead of steering the measuring functions and filtering again. This property is very useful to applications and saves valuable computation time. Since equivariance is a linear property then without loss of generality we will hereafter assume $\mathbf{b}$ are linearly independent, i.e. a basis. For G_{μ_x} we have

$$\begin{pmatrix} f_0(\tau_x) \\ f_1(\tau_x) \end{pmatrix} \equiv e^{\tau_x} \begin{pmatrix} 1 & 0 \\ \tau_x & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \quad \Rightarrow \quad \mathsf{A}_{\mu_x}(\tau_x) \equiv e^{\tau_x} \begin{pmatrix} 1 & 0 \\ \tau_x & 1 \end{pmatrix}. \tag{7}$$

Teo and Hel-Or [2] demonstrated that any generator $\mathcal{L}$ has a *conjugate generator* $\overline{\mathcal{L}}$ satisfying $\langle \phi, \mathcal{L}\psi \rangle \equiv \langle \overline{\mathcal{L}}\phi, \psi \rangle$. Since a Lie transformation group $\mathrm{G}(\tau)$ is generated by its generators using the exponential map, the conjugate generators generate an isomorphic *conjugate group* $\overline{\mathrm{G}}(\tau)$ satisfying $\langle \phi, \mathcal{T}(\tau)\psi \rangle \equiv \langle \overline{\mathcal{T}}(\tau)\phi, \psi \rangle$. This useful property of measurements allows us to treat the action of the group on an unknown image as equivalent to an action on a known measuring function. It comes as no surprise that conjugate generators directly connect equivariance of a feature space with respect to the group to its equivariance with respect to the components of the group, and we formalize this in the following Theorem.

Theorem 1 *A feature space* $\mathtt{F}$*, derived from the measuring functions* $\mathbf{b}$*, is equivariant with respect to the group* $\mathrm{G}(\tau)$ *if and only if* $\overline{\mathcal{T}}(\tau)\mathbf{b} \equiv \mathsf{A}(\tau)\mathbf{b}$ *or equivalently if and only if* $\overline{\mathcal{L}}_i\mathbf{b} \equiv \mathsf{B}_i\mathbf{b}$ *for some matrices* B_i *for* $i = 1, \ldots, k$*. In the latter case* $\mathsf{A}(\tau) \equiv e^{\tau_k \mathsf{B}_k + \cdots + \tau_1 \mathsf{B}_1}$*. The matrices* B_i *are called the representation matrices,* A *is called the interpolation matrix and* $\mathtt{F}$ *is called an equivariant measuring space (EMS) [2].*

For G_{μ_x} we have

$$\mathsf{B}_{\mu_x} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathsf{A}_{\mu_x}(\tau_x) \equiv e^{\tau_x \mathsf{B}_{\mu_x}} \equiv e^{\tau_x} \begin{pmatrix} 1 & 0 \\ \tau_x & 1 \end{pmatrix}. \tag{8}$$

An EMS is a linear vector space. This is evident from the linearity of the interpolation equation. Let $\mathbf{b}_1, \mathbf{b}_2$ be EMS bases and P be a regular matrix P then using a similar argument we find that $\mathsf{P}\mathbf{b}_1$, the direct sum $\mathbf{b}_1 \oplus \mathbf{b}_2$ and the Kronecker product $\mathbf{b}_1 \otimes \mathbf{b}_2$ also span EMSs. This leads to the following fundamental Theorem [2].

Theorem 2 *Let* $\mathbf{b}_1, \mathbf{b}_2$ *be two equivariant measuring space bases of size* n *with respect to the same single-parameter Lie transformation group* G_τ *with* conjugate generator $\overline{\mathcal{L}}$*, and let* $\overline{\mathcal{L}}\mathbf{b}_1 = \mathsf{B}_1\mathbf{b}_1, \overline{\mathcal{L}}\mathbf{b}_2 = \mathsf{B}_2\mathbf{b}_2$ *for some matrices* $\mathsf{B}_1, \mathsf{B}_2$*. Then* $\mathbf{b}_1 = \mathsf{P}\mathbf{b}_2 \iff \mathsf{B}_1 = \mathsf{P}\mathsf{B}_2\mathsf{P}^{-1}$ *for any* $n \times n$ *non-singular matrix* P*.*

Hence *similar* matrices correspond to the same feature space, and the representation matrices may be categorized by their *Jordan form*. We will therefore hereafter assume that without loss of generality B is in *block Jordan form* i.e. B is an $m \times m$ matrix with eigenvalue λ of multiplicity m along its main diagonal, 1s along its lower secondary diagonal and zero otherwise. In the above example, B_{μ_x} is in block Jordan form. If otherwise B has more than one Jordan block then it corresponds to a direct sum of spans, each associated with a Jordan block representation matrix of B for which the assumption holds.

3 Conjugate Generators

In this section we analyze conjugate generators, formulate standard bases for their EMSs and demonstrate results for standard Lie transformation groups.

Transformation	Integration Domain	Operator	Generator	Conjugate Generator
brightness scaling	$(-\infty,\infty)$	$e^{\tau}I(x)$	$\mathcal{I}$	$\mathcal{I}$
x-translation	$(-\infty,\infty)$	$I(x-\tau)$	$-\partial_x$	∂_x
x-scaling	$(-\infty,\infty)$	$I(e^{-\tau}x)$	$-x\partial_x$	$\mathcal{I}+x\partial_x$
x-projective	$(-\infty,\infty)$	$I(\frac{x}{1+\tau x})$	$-x^2\partial_x$	$2x+x^2\partial_x$
rotation	$[-\pi,\pi]$	$I(\theta-\tau)$	$-\partial_\theta$	∂_θ
uniform scaling	$(-\infty,\infty)$	$I(e^{-\tau}r)$	$-r\partial_r$	$2\mathcal{I}+r\partial_r$

Table 1. Conjugate Generators

3.1 Deriving Conjugate Generators

To derive conjugate generators we need only two rules [2]. The *multiplicative rule* is $\langle\phi, c\psi\rangle \equiv \langle c\phi,\psi\rangle$ for any function c and the *derivative rule* for x is

$$\begin{aligned}\langle\phi,\partial_x\psi\rangle &\equiv \iint \phi(\partial_x\psi)dxdy \\ &\equiv \int \phi\psi dy \mid_{-\infty}^{\infty} - \iint (\partial_x\phi)\psi dxdy \\ &\equiv \langle -\partial_x\phi,\psi\rangle.\end{aligned} \tag{9}$$

This rule applies similarly to y. Note that the vanishing of the boundary term is by assumption that the image is bounded. In case of a compact transformation the required assumption is that image is periodic with respect to the generator, which is generally the case, as well as the basis. Considering variables different than the integration variables x and y, one must account for the Jacobian of the variables change. For example, the derivative rule for r becomes

$$\begin{aligned}\langle\phi,\partial_r\psi\rangle &\equiv \iint \phi(\partial_r\psi)rdrd\theta \\ &\equiv \int \phi\psi rdrd\theta \mid_{-\infty}^{\infty} - \iint (\partial_r\phi r)\psi drd\theta \\ &\equiv 0 + \iint ((\partial_r\phi)r+\phi)\psi drd\theta \\ &\equiv \langle -(\partial_r + r^{-1})\phi,\psi\rangle.\end{aligned} \tag{10}$$

Another way to handle generators of parameters other than x and y is to express them in x and y before applying the rules. Table 1 summarizes the conjugate generators of common transformations derived by applying these rules. Note that the results for the y transformations are similar to those of x.

3.2 Deriving Standard Bases

To determine a basis of a feature spaces with respect to a generator, termed *fundamental feature space*, we need to solve the equivariance equation. Solutions of the equivariance

Transformation	Jacobian	Solution	Constraints	Standard Basis
brightness scaling	1	I_1	$m = 1, \lambda = 1$	any
x-translation	1	$e^{\mathsf{B}x}$	$m \in \mathbb{N}, \lambda \in \mathbb{C}$	$\overrightarrow{\sum_{n=0}^{m-1}} \frac{1}{n!} x^n e^{\lambda x}$
x-scaling	e^τ	$x^{\mathsf{B}-\mathsf{I}}$	$m \in \mathbb{N}, \lambda \in \mathbb{C}$	$\overrightarrow{\sum_{n=0}^{m-1}} \frac{1}{n!} \ln^n(x) x^{\lambda-1}$
x-projective	$(-1+\tau x)^{-2}$	$x^{-2} e^{-x^{-1}\mathsf{B}}$	$m \in \mathbb{N}, \lambda \in \mathbb{C}$	$\overrightarrow{\sum_{n=0}^{m-1}} \frac{(-1)^n}{n!} x^{-n-2} e^{-\lambda x^{-1}}$
rotation	1	$e^{\mathsf{B}\theta}$	$m = 1, \lambda \in i\mathbb{Z}$	$e^{\lambda\theta}$
uniform scaling	$e^{2\tau}$	$r^{\mathsf{B}-2\mathsf{I}}$	$m \in \mathbb{N}, \lambda \in \mathbb{C}$	$\overrightarrow{\sum_{n=0}^{m-1}} \frac{1}{n!} \ln^n(r) r^{\lambda-2}$

Table 2. Equivariance solutions and row space for B in block Jordan form with eigenvalue λ and multiplicity m

equation $\overline{\mathcal{L}}\mathbf{b}(x) = \mathsf{B}\mathbf{b}(x)$ for $\mathbf{b}(x)$ yield an EMS basis with respect to the generator $\mathcal{L}$. They may be obtained using simple ODE techniques and are of the form $\mathbf{b}(x) = \mathsf{M}(x)\mathbf{b}(0)$, where $\mathsf{M}(x)$ is a matrix solely dependent on x, and with arbitrary $\mathbf{b}(0)$ as initial condition. For G_{μ_x} we have

$$\mathsf{M}_{\mu_x}(x) \equiv e^x \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \tag{11}$$

The corresponding row space of M is used as the *standard basis*. For G_{μ_x} this is exactly $\mathbf{b} \equiv (e^x, e^x + xe^x)$. Solutions and their corresponding row spaces, i.e. the standard bases, are summarized in table 2. All integration are $dxdy$ with a regular inner product. The Jacobian of the transformation, defined by $\det \partial x_i / \partial x'_j$ where x_i are the original variables and x'_j are the transformed variables, is also listed in table 2 for reference. $\overrightarrow{\sum}$ is the cumulative sum vector, e.g. $\overrightarrow{\sum_{i=1}^{n}} a_i \equiv (a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_n)$. If a w weighted inner product is given simply multiply the bases by w^{-1}. The results stay the same since $\langle \phi, \psi \rangle \equiv \langle w^{-1}\phi, \psi \rangle_w$. No singularity is introduced since the weighting function w is positive definite by definition.

A fundamental issue arises here. Two isomorphic Lie algebras, or tangent spaces, are not guaranteed to yield the same solution. This may happen due to global constraints, i.e. when the corresponding Lie transformation groups are not isomorphic. For example the fundamental feature spaces of compact and non-compact single parameter groups are different. The integration domain of a non-compact group is invariant under the variables change of the transformation involved, i.e. it remains $(-\infty, \infty)$, but not so for a compact group. The solution for a compact group must be periodic with respect to the group parameter. This is the case for rotation in table 2 where the periodicity implies that the multiplicity must be 1 and the eigenvalue must be an integer imaginary number. Also note that further constraints apply to prevent the basis functions from introducing singularities.

Theorem 1 shows that a transformed feature of the standard basis, e.g. $\mathbf{f}(\tau_x)$ with respect to translation, is connected to the original features $\mathbf{f}$ via e.g.

$$f_i(\tau_x) = e^{\lambda\tau_x} \sum_{n=0}^{i} \frac{1}{n!} \tau_x^n f_{i-n}(0). \tag{12}$$

This is the *standard interpolation equation*. Other transformations have similar connections. One may verify that our example A_{μ_x} adheres to this equation.

4 Measurements Invariants

In this section we derive measurements invariants for B in block Jordan form and then extend to B in general Jordan form.

4.1 Block Invariants

An invariant $h(\mathbf{f})$ is a function of the features, which is a constant in the feature space when restricted to the orbit of the group. As was shown above it is enough to check that the invariant vanishes under the action of the conjugate generator:

$$\overline{\mathcal{L}}h(\mathbf{f}) \equiv (\mathsf{B}\mathbf{f})^t \nabla h(\mathbf{f}) = 0 \tag{13}$$

where $\nabla h(\mathbf{f})$ stands for $(\partial_{f_1}, \ldots, \partial_{f_n})^t h(\mathbf{f})$. Trivially, any function of the invariants is also an invariant and we are interested in non-trivial and functionally independent maximal set of invariants. For G_{μ_x} we have

$$f_0 \partial_{f_0} h + (f_0 + f_1)\partial_{f_1} h = 0. \tag{14}$$

We term these as *block invariants*, since B is in block Jordan form. Generally, this PDE is difficult to solve, as the general technique of characteristic ODE for solving it involves $m-1$ repeated integrations for $m-1$ invariants. However, under our standard condition on B this PDE is tractable. The characteristic ODE for G_{μ_x} is

$$\frac{df_0}{f_0} = \frac{df_1}{f_0 + f_1}. \tag{15}$$

In general, this is a list of terms for $f_0, \ldots, f_{m-1}$, all equal. The method we applied to solve this is to equate each term to the left most term. At each step $i = 1, \ldots, m-1$ an invariant h_i is solved, f_i is expressed as a function of f_0 and $h_1, \ldots, h_i$ and substituted into the next equation. This is possible due to the block Jordan form of B and produces simple expressions since $h_1, \ldots, h_i$ are constant. For G_{μ_x} we have

$$h_1 \equiv \ln f_0 - \frac{f_1}{f_0} \quad \Rightarrow \quad f_1 \equiv f_0 \ln f_0 - f_0 h_1. \tag{16}$$

For the general case, it can be verified that the solutions, for $\lambda = 0$, i.e. $\mathsf{B}_{i,j} \equiv \delta_{i,j+1}$, are

$$\begin{aligned} & h_1 \equiv f_0, \\ \text{and} \quad & \\ & h_i \equiv \frac{f_0^2}{i!}\left(\frac{f_1}{f_0}\right)^i - \sum_{n=1}^{i-2} \frac{h_{i-n}}{n!}\left(\frac{f_1}{f_0}\right)^n - f_0 f_i \\ \text{for} \quad & i > 1\ , \end{aligned} \tag{17}$$

and the solutions for $\lambda \neq 0$, i.e. $\mathsf{B}_{i,j} \equiv \lambda\delta_{i,j} + \delta_{i,j+1}$, are

$$h_i \equiv \frac{\lambda}{i!}\Big(\frac{\ln f_0}{\lambda}\Big)^i - \sum_{n=1}^{i-1} \frac{h_{i-n}}{n!}\Big(\frac{\ln f_0}{\lambda}\Big)^n - \frac{\lambda f_i}{f_0}. \tag{18}$$

4.2 Cross Invariants

Previously, we assumed B is in block Jordan form, i.e. it has a single block, and we analyzed the block for its standard basis functions and invariants. We now lift this assumption and derive further invariants. For B with s Jordan blocks we take the block invariants of each block separately as invariants. This is justified by correspondence of each block to a separate equivariant feature space. Indeed each feature space is closed under the linear action of the conjugate generator described by the separate representation block of B. Therefore the feature space corresponding to B is a direct sum of the feature spaces corresponding to its Jordan blocks. However, it is clear that additional invariants exist using features of more than one block. We term these as *cross invariants*. The method we applied to derive the invariants from equation (13) is to take the term in the characteristic ODE with the lowest degree for each block of the pair and equate them.

Let λ, Λ be eigenvalues of a pair of blocks and denote $f_0, \dots, f_m$ and $F_0, \dots, F_M$ the features of the corresponding blocks. If $\lambda, \Lambda \neq 0$ then we have

$$\frac{df_0}{\lambda f_0} \equiv \frac{dF_0}{\Lambda F_0} \quad \Rightarrow \quad h \equiv \Lambda \ln f_0 - \lambda \ln F_0, \tag{19}$$

and if $\lambda = 0$ then the we have

$$\frac{df_1}{\lambda f_0} \equiv \frac{dF_0}{\Lambda F_0} \quad \Rightarrow \quad h \equiv \ln F_0 - \frac{f_1}{f_0}. \tag{20}$$

It should be clear that the set of cross invariants corresponding to all possible pair of blocks is redundant. It suffices to take $s-1$ functionally independent invariants. For example, if we denote $\lambda_1, \dots, \lambda_s$ the eigenvalues of B then the set of cross invariants corresponding to the pairs (λ_1, λ_i) with $i = 2, \dots, s$ is functionally independent and maximal. To see this notice that we equate s characteristic ODE terms for the cross invariants which correspond to one PDE constraint over s differentials.

4.3 A Complete System of Invariants

We have shown above that any EMS of an Abelian Lie transformation group has a standard basis for which the representation matrix B is in Jordan form. The block invariants and cross invariants derived are together a functionally independent and maximal set. This can be seen through simple counting of degrees of freedom. Assume that we have s blocks where the ith block has degree m_i, i.e. there are m_i features it corresponds to. For the ith block take $m_i - 1$ functionally independent block invariants. In addition take $s-1$ functionally independent cross invariants as described above. It should be clear that the block and cross invariants together are functionally independent. This totals $s - 1 + \sum_{i=1}^{s}(m_i - 1) = -1 + \sum_{i=1} m_i$ functionally independent invariants for $\sum_{i=1}^{s} m_i$ features. Furthermore, we note that any other basis for the EMS is linearly connected to the standard basis and thus any invariant defined on the features of this measuring basis is functionally dependent on our standard invariants. We therefore conclude that this system of invariants is complete for Abelian Lie transformation groups.

5 Results

We have used slices of the images D2,D18,D23,D88,D91 from the Brodatz texture database of size 128×128 displayed in Figure 1 to test the theory. Deformations of different transformation groups have been applied to the images. We have experimented with the x-,y-translation, x-,y-scaling, rotation and uniform scaling transformations and simulated them using a bilinear method. Measurements in the form of a regular inner product for selected standard bases have been taken and the corresponding invariants have been calculated. We have used one block of multiplicity $m = 8$ for non-compact transformations when testing features and block invariants. We have used 8 blocks of multiplicity $m = 1$ for compact and non-compact groups when testing cross invariants.

Equation (12) was verified against the features of the multiplicity 8 blocks as displayed in Figure 2. All relative errors were small and independent of the degree of the basis function or the amount of transformation. We assume that the errors are due to the interpolation.

We verified that the invariants are constant for deformations of an image and are different for different images, as demonstrated in Figure 3. Figures 4, 5 and 6 show the average relative errors of the block and cross invariants. All relative errors are below 6%, and most of them are orders of magnitude less.

6 Conclusions

We have presented a complete system of invariants to deformations modelled by a Lie transformation group. The invariants are functions of standard features. These features are given as measurements, integral inner product of functions and the image, using a standard basis derived from the transformation group. The system of invariants is applicable to any Abelian Lie transformation group acting on the image, and the image space may be of any dimensionality.

We have shown that an Abelian Lie transformation group is separable. Using Theorem 2 all feature spaces that are equivariant with respect to a given generator have been categorized and standard bases for these spaces have been derived. Inner product with a weighting function w may be accommodated by multiplying the standard bases by w^{-1}. The simple structure of the PDE derived from the Jordan form of the representation with respect to the standard basis has allowed us to arrive at the block invariants. The system of invariants has been completed with the cross invariants, functions of features from different blocks. The separability of Abelian groups has allowed us to take the direct sum of standard bases as a basis for the entire group, and the union of

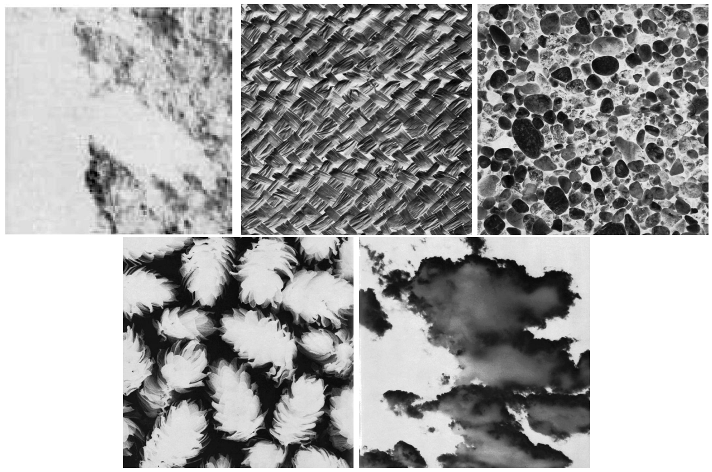

Fig. 1. The slices of the images D2,D18,D23,D88,D91 (left to right) used in the experiments

the corresponding invariants as invariants for the entire group. The results apply to any dimensionality by the use of a Lie theoretical approach.

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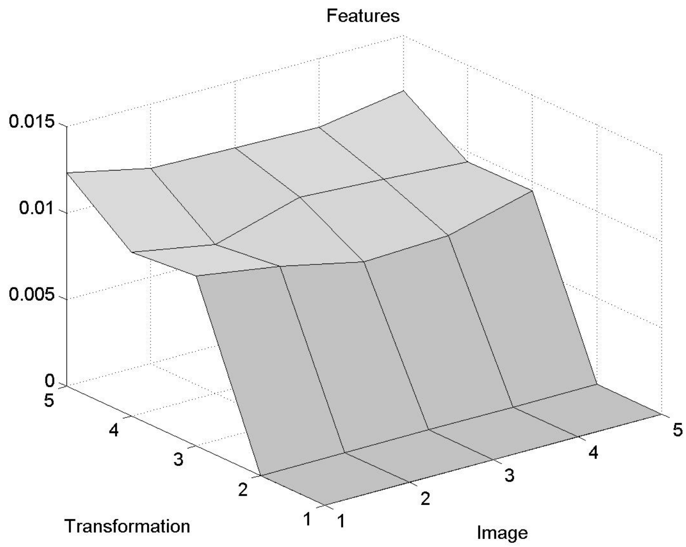

Fig. 2. Average relative error of features by equation (12). The images D2,D18,D23,D88,D91 are labelled (1)–(5), and the transformations are labelled: (1) x-translation with $\lambda = 1$, (2) y-translation with $\lambda = 0$, (3) x-scaling with $\lambda = 3$, (4) y-scaling with $\lambda = 3$, (5) uniform scaling with $\lambda = 4$

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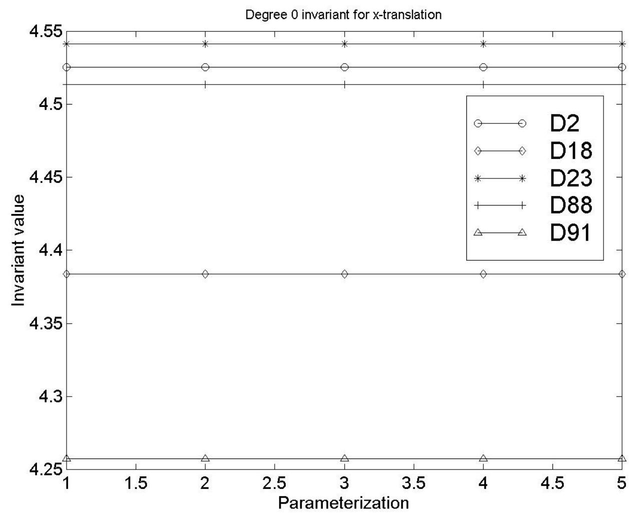

Fig. 3. The value of degree 0 invariant for D2,D18,D23,D88,D91 under x-translation with various parameterizations. The error is orders of magnitude less than this similarity

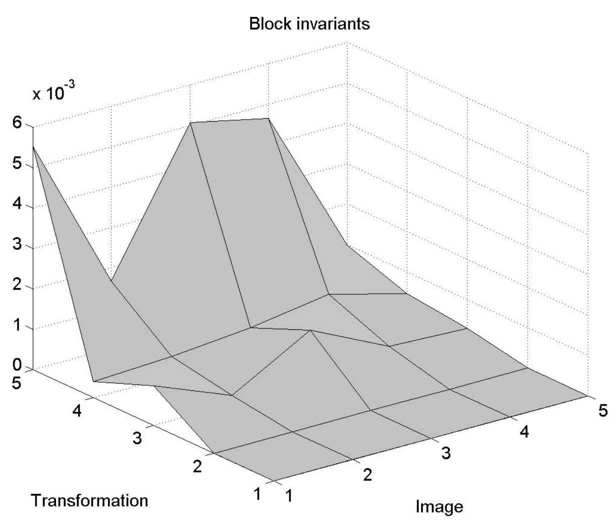

Fig. 4. Average relative error of block invariants by equation (17), (18). The images D2,D18,D23,D88,D91 are labelled (1)–(5), and the transformations are labelled: (1) x-translation with $\lambda = 1$, (2) y-translation with $\lambda = 0$, (3) x-scaling with $\lambda = 3$, (4) y-scaling with $\lambda = 3$, (5) uniform scaling with $\lambda = 4$

Fig. 5. Average relative error of cross invariants by equation (19). The images D2,D18,D23,D88,D91 are labelled (1)–(5), and the transformations are labelled: (1) x-translation (2) y-translation (3) x-scaling (4) y-scaling (5) rotation with $\lambda = 1, \ldots, 7$ and $\Lambda = 2, \ldots, 8$

Fig. 6. Average relative error of cross invariants by equation (20). The images D2,D18,D23,D88,D91 are labelled (1)–(5), and the transformations are labelled: (1) x-translation (2) y-translation (3) x-scaling (4) y-scaling (5) uniform scaling with $\lambda = 0$ and $\Lambda = 1, \ldots, 6$