The Uncertainty Principle: Group Theoretic Approach, Possible Minimizers and Scale-Space Properties

Chen Sagiv[1], Nir A. Sochen[1], and Yehoshua Y. Zeevi[2]

[1] Department of Applied Mathematics
University of Tel Aviv
Ramat-Aviv, Tel-Aviv 69978, Israel
chensagi@post.tau.ac.il,sochen@post.tau.ac.il
[2] Department of Electrical engineering
Technion - Israel Institute of Technology
Technion City, Haifa 32000, Israel
zeevi@ee.technion.ac.il

Abstract. The uncertainty principle is a fundamental concept in the context of signal and image processing, just as much as it has been in the framework of physics and more recently in harmonic analysis. Uncertainty principles can be derived by using a group theoretic approach. This approach yields also a formalism for finding functions which are the minimizers of the uncertainty principles. A general theorem which associates an uncertainty principle with a pair of self-adjoint operators is used in finding the minimizers of the uncertainty related to various groups.

This study is concerned with the uncertainty principle in the context of the Weyl-Heisenberg, the SIM(2), the Affine and the Affine-Weyl-Heisenberg groups. We explore the relationship between the two-dimensional affine group and the SIM(2) group in terms of the uncertainty minimizers. The uncertainty principle is also extended to the Affine-Weyl-Heisenberg group in one dimension. Possible minimizers related to these groups are also presented and the scale-space properties of some of the minimizers are explored.

1 Introduction

Various applications in signal and image processing call for deployment of a filter bank. The latter can be used for representation, de-noising and edge enhancement, among other applications. A key issue is the definition of the best filter bank for the application at hand. One possible criterion lends itself to using functions which achieve minimal uncertainty. For example, the Gaussian window minimizes the uncertainty of the combined representation of the signal in the time-frequency (or position - frequency) space. The short time Fourier transform, implementing a gaussian window function, is well known in signal processing as the Gabor transform. The minimal uncertainty quality, together with the fact that Gabor functions are tuned to orientation and scale, led to an intensive usage of Gabor functions and Gabor-Morlet wavelets in computer vision and image processing.

The Gabor transform can be viewed as a representation obtained by the action of the Weyl-Heisenberg group on a Gaussian window [31], or, alternatively, as a convolution of the signal with Gaussian-modulated complex exponentials (Gabor elementary functions (GEF) [10]). These GEF are equivalent to a family of canonical coherent states of the Weyl-Heisenberg group [16]. The Gaussian function appears as a pivot in scale-space theory as well, where its successive applications to images produce coarser resolution images [8].

The wavelet transform emerged as an important theoretical and applicative tool in signal and image processing, while it is rooted in several research domains, such as pure mathematics, physics and engineering. Specifically, Gabor wavelets which sample the frequency domain in a log-polar manner play an important role in texture representation and segmentation, evaluation of local features in images and other. Gabor wavelets can be considered as a sub-group of the family of canonical coherent states related to the Weyl-Heisenberg group. However, they are generated according to the operations of the affine group in one dimension, or the similitude group in two dimensions. Therefore, it is interesting to look for the canonical coherent states of the affine or the similitude groups. Moreover, it is interesting to investigate whether these minimizers have any scale-space like attributes, similar to those exhibited by the Gaussian function. It turns out that this problem does not have a single deterministic solution, similar to the one that exists in the case of the Weyl-Heisenberg group. Based on previous work of Dahlke and Maass [4] and of Ali, Antoine and Gazeau [1], one may conclude that the full significance of the scale-space properties of possible minimizers is not yet fully understood.

The motivation for this study comes from our previous studies on texture segmentation and representation [24–28]. A major concern encountered in dealing with these issues is the selection of an appropriate filter bank. In several studies Gabor-wavelets are chosen because they are believed to provide the best trade-off between spatial resolution and frequency resolution [2, 12, 20]. However, this is true in terms of the Weyl-Heisenberg group, i.e. with respect to Gabor-functions which sample the joint spatial-frequency space via constant-value translations. Gabor-wavelets can be generated by a logarithmic distortion of Gabor functions (the minimizers of the Weyl-Heisenberg group) [19] or alternatively by using multi-windows, so that a collection of the functions generated by both the Weyl-Heisenberg group and the affine group are considered [32]. As these Gabor-wavelets are generated using the affine group, the joint spatial-frequency space is sampled in an octave-like manner. The general question arises whether Gabor-wavelets provide the minimal combined uncertainty with respect to the affine group. Since the Gabor wavelets combine both time (position) and frequency translations, along with dilations, it seems that it may be related to the Affine-Weyl-Heisenberg (AWH) group. The canonical representation U of the AWH group on $L^2(R)$ is given by:

$$[U(b, \omega, a, \varphi)\psi](t) = \frac{1}{\sqrt{a}} e^{i\varphi} e^{i\omega t} \psi \left(\frac{t-b}{a} \right) \tag{1}$$

and the coefficients generated by the inner product $\langle f, U(x)\psi \rangle$ provide the Gabor-wavelets transform, if ψ is selected to be a Gaussian. Thus, searching for the minimizer of the uncertainty principle related to the AWH group, may provide a mother-wavelet which allows for maximal accuracy in the time-frequency-scale combined space. This may be significant in terms of optimal representations of signals. The applications are numerous yet one notable motivation is an affine invariant treatment of texture. Since one of the most important transformations in vision is the perspective transformation, which is well approximated in many cases by the affine group, it is of major interest to generalize the analysis from the Euclidean case to the Affine case. While we have a reason to believe that affine based transform can facilitate an invariant treatment of texture we believe that this issue deserves a separate publication.

The rest of this paper is organized as follows: First, we provide some review of background and related work. Next, we apply the uncertainty principle theorem to the Weyl-Heisenberg group in one and two-dimensions, to obtain the Gaussian function. Motivated by the need to define the minimizers for the uncertainty associated with the affine group, we follow the analysis of Dahlke and Maass [4] and that of Ali, Antoine and Gazeau[1], and apply the uncertainty theorem to the affine group in one and two dimensions. Moreover, we explore this issue in the context of the AWH group. We conclude by pointing out the scale-space properties of some of the obtained minimizers [29].

2 Background and Related Work

The uncertainty principle is a fundamental concept in the context of signal and information theory. It was originally stated in the framework of quantum mechanics, where it is known as the Heisenberg uncertainty principle. In this context it does not allow to simultaneously observe the position and momentum of a particle. In 1946, Gabor [10] has extended this idea to signal and information theory, and has shown that there exists a trade off between time resolution and frequency resolution for one-dimensional signals, and that there is a lower bound on their joint product. These results were later extended to $2D$ signals [3, 19].

The functions which attain the lower bound of the inequality defining the uncertainty principle have been the subject of ongoing research. In quantum mechanics they are regarded as a family of canonical coherent states generated by the Weyl-Heisenberg group. In information and signal theory, Gabor discovered that Gaussian-modulated complex exponentials provide the best trade-off for time resolution and frequency resolution.

A general theorem which is well known in quantum mechanics and harmonic analysis [9] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty

equation.
Theorem 1: Two self-adjoint operators, A and B obey the uncertainty relation:

$$\Delta A_\psi \Delta B_\psi \ \geq \frac{1}{2}|\langle[A,B]\rangle| \quad \forall\psi, \tag{2}$$

where $\Delta A_\psi, \Delta B_\psi$ denote the variances of A and B with respect to the signal ψ. The triangular parenthesis mean an average over the signal i.e. $\langle X\rangle = \int \psi^* X\psi$. The mean of the action of an operator P on a function ψ is denoted by: $\mu_P(\psi) = \langle P(\psi)\rangle$, and the commutator $[A,B]$ is given by: $[A,B] := AB - BA$. A function ψ is said to have minimal uncertainty if the inequality turns into an equality. This happens iff there exists an $\eta \in iR$ such that

$$(A - \mu_A)\psi = \eta(B - \mu_B)\psi. \tag{3}$$

This last relation yields a differential equation for each non-commuting couple of group generators.

The Weyl-Heisenberg and the affine groups are both related to well known transforms in signal processing: the windowed-Fourier and wavelet transforms. Both can be derived from square integrable representations of these groups. The windowed-Fourier transform is related to the Weyl-Heisenberg group, and the wavelet transform is related to the affine group. Deriving the infinitesimal generators of the unitary group representations, we obtain self-adjoint operators. Thus, the general uncertainty theorem [9] stated above provides a tool for obtaining uncertainty principles using these infinitesimal generators of the group representations. In the case of the Weyl-Heisenberg group, the canonical functions which minimize the corresponding uncertainty relation are Gaussian functions. The canonical functions which minimize the uncertainty relations for the affine group in one dimension and for the similitude group in two dimensions, were the subject of previous studies [4, 1].

In these studies, it was shown that there is no non-trivial canonical function which minimizes the uncertainty equation associated with the similitude group of R^2, $SIM(2) = R^2 \times (R^+ \times SO(2))$. Thus, there is no non-zero solution to the set of differential equations obtained for this group generators. Rather than using the original generators of the $SIM(2)$ group, Dahlke and Maass [4] used a different set of operators that includes elements of the enveloping algebra, i.e. polynomials in the generators of the algebra, to obtain the $2D$ isotropic Mexican hat as a minimzer. Ali, Antoine and Gazeau [1] noted a symmetry in the set of commutators obtained for the $SIM(2)$ group and derived a possible minimizer in the frequency domain for some fixed direction. Their solution is a real wavelet which is confined to some convex cone in the positive-half-plane of the frequency space and is exponentially decreasing inside.

The representation theory of the Affine-Weyl-Heisenberg group and its possible extensions/modifications have already been addressed in this context in the early $90's$. Torresnai [23] considered wavelets associated with representations of the AWH group, as well as associated with resolutions of the identity. He had also shown that the canonical representation of the AWH group is not square integrable, but can be regularized with some density function. This work was later extended to N-dimensional AWH wavelets[13]. Segman and Schempp[30] introduced ways to incorporate scale in the Heisenberg group with an intertwining operator and presented the resulting signal representations. More recently, Teschke[22] proposed a mechanism for construction of generalized uncertainty principles and their minimizing wavelets in anisotropic Sobolev spaces. He derived a new set of uncertainties by weakening the two operator relations and by introducing a multi-dimensional operator setting.

3 The Weyl-Heisenberg Group

The uncertainty principle related to the Weyl-Heisenberg group has a tremendous importance in two main fields: In quantum mechanics, the uncertainty principle prohibits the observer from exactly knowing the location and momentum of a particle. In signal processing, the uncertainty principle provides a limit on the localization of the signal in both time (or position) and frequency domains.

Let G be the Weyl-Heisenberg group,

$$G := \{(\omega, b, \tau) | b, \omega \in \mathbb{R}, \tau \in \mathbb{C}, |\tau| = 1\}, \tag{4}$$

with group law

$$(\omega, b, \tau) \circ (\omega', b', \tau') = \left(\omega + \omega', b + b', \tau\tau' e^{i\frac{(\omega b' - \omega' b)}{2}}\right). \tag{5}$$

We assume that the toral component, τ, of the group representation, is fixed. Let π be its canonical left action on $L^2(\mathbb{R})$; the coefficients generated by $\langle f, \pi(x)\psi\rangle$ are known as the windowed Fourier transform of the function f, with ψ being the window function. The windowed Fourier transform is defined by:

$$\langle f, \pi(x)\psi\rangle = (W_\psi f)(\omega, b) = \int f(x)\psi(x-b)e^{-i\omega x}dx \tag{6}$$

The Fourier transform is a tool of profound importance in signal processing and in quantum physics, where it is used for the study of coherent states. The Gaussian window function $\psi(x) = e^{-\frac{x^2}{2}}$ has an important role in the windowed Fourier analysis as it minimizes the Weyl-Heisenberg uncertainty principle.

Next, we review the derivation of the uncertainty principles for the Weyl-Heisenberg group in one and two dimensions using the uncertainty principle theorem. The reader may find the classical proofs of the uncertainty principle for the Weyl-Heisenberg group in the work of Gabor [10] for one-dimensional signals and in the work of Daugman [3] for two-dimensional signals.

3.1 The one-dimensional case

The unitary irreducible representation of the Weyl-Heisenberg group in $L^2(R)$ is given by: $[U(\omega, b, \tau)\psi](x) = \tau e^{\frac{-i\omega b}{2}} e^{i\omega x}\psi(x-b)$. If the toral component of the group representation is fixed, then the representation can be defined as: $[U(\omega, b)\psi](x) := e^{i\omega x}\psi(x-b)$. The following infinitesimal generators of the group can be defined as:

$$(T_\omega \psi)(x) := i\frac{\partial}{\partial \omega}[U(\omega, b)\psi](x)|_{\omega=0,b=0} = -x\psi(x) \tag{7}$$

$$(T_b \psi)(x) := i\frac{\partial}{\partial b}[U(\omega, b)\psi](x)|_{\omega=0,b=0} = -i\frac{d}{dx}\psi(x) \tag{8}$$

The one-dimensional uncertainty principle for the Weyl-Heisenberg group can be derived using the general uncertainty principle.
Corollary [9]: Let $T_\omega = -x$ and $T_b = -i\frac{\partial}{\partial x}$ be the infinitesimal operators of the Weyl-Heisenberg group. If $\psi \in L^2(\mathbb{R})$ we have: $\|(T_\omega - \mu_\omega)\psi\|_2 \|(T_b - \mu_b)\psi\|_2 \geq \frac{1}{4}\|\psi\|_2$, where: $\|\|_2$ is defined as: $\int \psi(x)\psi^*(x)dx$. Equality is obtained iff

$$\psi(x) = Ce^{-i\mu_b x}e^{-\frac{i}{2\eta}(x-\mu_\omega)^2}, \tag{9}$$

where $C = \left(\frac{i}{2\pi\eta}\right)^{\frac{1}{4}}$ and $\eta \in i\mathbb{R}^+$.

3.2 The two-dimensional case

The unitary irreducible representation of the Weyl-Heisenberg group in two dimensions is given by: $[U(\omega_1, \omega_2, b_1, b_2, \tau)\psi](x, y) = \tau e^{i(\omega_1 x + \omega_2 y)}\psi(\overrightarrow{u} - \overrightarrow{b})$, where $\overrightarrow{u} = (x, y)$, $\overrightarrow{b} = (b_1, b_2)$. The following infinitesimal generators of the group can be defined as:

$$(T_{\overrightarrow{\omega}}\psi)(\overrightarrow{u}) := i\frac{\partial}{\partial \overrightarrow{\omega}}[U\psi](\overrightarrow{u})|_{\overrightarrow{\omega}=0, \overrightarrow{b}=0} = -\overrightarrow{u}\psi(\overrightarrow{u}) \tag{10}$$

$$(T_{\overrightarrow{b}}\psi)(\overrightarrow{u}) := i\frac{\partial}{\partial \overrightarrow{b}}[U\psi](\overrightarrow{u})|_{\overrightarrow{\omega}=0, \overrightarrow{b}=0} = -i\overrightarrow{\nabla}\psi(\overrightarrow{u}), \tag{11}$$

where $\overrightarrow{\omega} = (\omega_1, \omega_2)$. The only non-vanishing commutators of these four operators are:

$$[T_{w_k}, T_{b_k}] = -i \quad , \quad k = 1, 2 \ . \tag{12}$$

Thus, an uncertainty principle can be obtained for translations in the spatial and frequency domains. This can be executed for each dimension separately. It is interesting to note that using the Weyl-Heisenberg group, there is no coupling between the x and y components. Thus attaining a certain accuracy in the x component does not affect the degree of accuracy in the y component.

If we derive the minimization equation, we simply get the result of the one-dimensional analysis for both x and y coordinates. The separability of the Weyl-Heisenberg group results in separable Gaussian functions as the minimizers of the combined uncertainty. This is, in fact, an inherent property of the Gaussian function.

4 The Affine Group

Let A be the affine group, and let π be its canonical left action on $L^2(\mathbb{R})$; the coefficients generated by $\langle f, \pi(x)\psi\rangle$ are known as the wavelet transform of a function f, where ψ is the mother wavelet, or template. The wavelet transform is defined by:

$$(W_\psi f)(a,b) = \int_R f(x)|a|^{-\frac{1}{2}}\overline{\psi\left(\frac{x-b}{a}\right)}dx, \tag{13}$$

where $\overline{x}$ denotes the complex conjugate of x.

4.1 The one-dimensional case

Let A be the affine group,

$$A := \left\{(a,b)|(a,b) \in \mathbb{R}^2, a \neq 0\right\} \tag{14}$$

with group law

$$(a,b) \circ (a',b') = (aa', ab' + b). \tag{15}$$

A unitary group representation is obtained by the action of A on $\psi(x)$:

$$[U(a,b)\psi](x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right) \tag{16}$$

In preparation for our extension of this approach to two-dimensions and other groups, we quote the main results presented in the work of Dahlke and Maass [4] for the one-dimensional affine group. First, the self-adjoint infinitesimal operators are calculated by computing the derivatives of the representation at the identity element:

$$\begin{aligned} T_a &= -i\left(\frac{1}{2} - x\frac{\partial}{\partial x}\right) \\ T_b &= -i\frac{\partial}{\partial x}. \end{aligned} \tag{17}$$

Using these operators, the affine uncertainty principle is given [4], and the following differential equation

$$(T_a - \mu_a)\psi(x) = \eta\left((T_b - \mu_b)\psi(x)\right), \tag{18}$$

which reads:

$$-\frac{1}{2}i\psi(x) - ix\psi'(x) - \mu_a\psi(x) = -i\eta\psi'(x) - \eta\mu_b\psi(x). \tag{19}$$

The solution to this equation is: $\psi(x) = c(x-\eta)^\alpha$, where $\alpha = -\frac{1}{2} - i\eta\mu_a + i\mu_b$, and some constraints on the value of α are imposed to guarantee that the obtained solution is in $L^2(\mathbb{R})$.

4.2 The two-dimensional case

This section is divided into two parts. In the first part we recall the results of Dahlke and Maass [4], and of Ali, Antoine and Gazeau [1], who analyze the $SIM(2)$ group. In the second part, we extend their findings to account for the full Affine group in two dimensions.

The $2D$ similitude group of $\mathbb{R}^2$, $SIM(2) = \mathbb{R}^2 \times (\mathbb{R}^+ \times SO(2))$. Consider the group $SIM(2)$ with group law $(a, b, \tau_\theta) \circ (a', b', \tau_{\theta'}) = (aa', b + a\tau_\theta b', \tau_{\theta+\theta'})$. The unitary representation of $SIM(2)$ in $L^2(\mathbb{R}^2)$ is given by:

$$[U(a,b,\theta)f](x,y) = \frac{1}{a} f\left(\tau_{-\theta}\left(\frac{x-b_1}{a}, \frac{y-b_2}{a}\right)\right), \tag{20}$$

where the rotation $\tau_\theta \in SO(2)$ acts on a vector (x, y) in the following way:

$$\tau_\theta(x,y) = (xcos(\theta) - ysin(\theta), xsin(\theta) + ycos(\theta)), \tag{21}$$

and $\theta \in [0, 2\pi)$. The self-adjoint infinitesimal operators are given by:

$$T_\theta = i(\overrightarrow{u}^{\perp})^t \cdot \nabla, \quad T_a = -i(1 + \overrightarrow{u}^t \cdot \nabla),$$
$$T_{\overrightarrow{b}} = -i\nabla.$$

where $(\overrightarrow{u}^{\perp})^t = (-y, x)$ These operators yield four non-zero commutators, which generate in turn a system of four differential equations. It turns out that there does not exist a non-zero solution to this system of differential equations. Therefore, Dahlke and Maass [4] find a solution for a different set of operators from the enveloping algebra. The solution they find is a minimizer to the uncertainty principles associated with the operators: T_a, T_θ and $T_b = T_{b1}^2 + T_{b2}^2$. A possible solution is the Mexican hat function: $\psi(x,y) = [2 - 2\beta r^2]e^{-\beta r^2}$, where $r = \sqrt{x^2+y^2}$. Ali, Antoine and Gazeau [1] observe that the relationships between T_a and T_{b_1}, and between T_θ and T_{b_2}, can be transformed into the relationships between T_a and T_{b_2}, and T_θ and T_{b_1} by a $\frac{\pi}{2}$-rotation. Thus, they define a new translation operator $T_b = T_{b_1}cos(\gamma) + T_{b_2}sin(\gamma)$, so that a minimizing function can be obtained for this new operator as well as for T_a and T_θ with respect to a fixed direction γ. The minimizer they obtain in the frequency space k_x, k_y is a function which vanishes outside some convex cone in the half-plane $k_x > 0$ and is exponentially decreasing inside:

$$\hat{\psi(k)} = c|\overrightarrow{k}|^s e^{-i\eta k_x}, \tag{22}$$

where $s > 0$ and $i\eta > 0$.

The Affine Group in 2D Let us explore the most straightforward representation of the Affine group. Define an invertible matrix $\mathbf{s} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$. Its determinant is $D = |s_{11}s_{22} - s_{21}s_{12}|$, $\overrightarrow{b} = (b_1, b_2)$ and $\overrightarrow{x} = (x, y)$. The representation corresponding to the action of the Affine group is accordingly given by:

$$[U(\mathbf{s}, \overrightarrow{b})\psi](\overrightarrow{x}) = \sqrt{\mathbf{D}}\psi\left(\mathbf{s}\left(\overrightarrow{x} - \overrightarrow{b}\right)\right). \tag{23}$$

Let us calculate the infinitesimal operators associated with: $s_{11}, s_{12}, s_{21}, s_{22}, b_1, b_2$:

$$\begin{aligned} T_{s_{11}}(x, y) &= i\left(\frac{1}{2} + x\frac{\partial}{\partial x}\right), \quad T_{s_{22}}(x, y) = i\left(\frac{1}{2} + y\frac{\partial}{\partial y}\right), \\ T_{s_{12}}(x, y) &= iy\frac{\partial}{\partial x}, \quad T_{s_{21}}(x, y) = ix\frac{\partial}{\partial y}, \\ T_{b_1}(x, y) &= -i\frac{\partial}{\partial x}, \quad T_{b_2}(x, y) = -i\frac{\partial}{\partial y}. \end{aligned} \tag{24}$$

As these operators were derived from a unitary representation, they are self-adjoint. The non-vanishing commutation relations are:

$$\begin{array}{lll} [T_{s_{11}}, T_{s_{12}}] = iT_{s_{12}}, & [T_{s_{11}}, T_{s_{21}}] = -iT_{s_{21}}, & [T_{s_{11}}, T_{b_1}] = iT_{b_1} \\ [T_{s_{12}}, T_{s_{22}}] = iT_{s_{12}}, & [T_{s_{12}}, T_{b_2}] = iT_{b_1}, & [T_{s_{21}}, T_{s_{22}}] = -iT_{s_{21}} \\ [T_{s_{21}}, T_{b_1}] = iT_{b_2}, & [T_{s_{22}}, T_{b_2}] = iT_{b_2}, & [T_{s_{12}}, T_{s_{21}}] = -i(T_{s_{11}} - T_{s_{22}}) \end{array}$$

Thus, of the fifteen possible commutation relations, we obtain nine uncertainty principles. It is interesting to note that the scaling in the x direction (s_{11}) is not constrained by the scaling in the y direction (s_{22}). The same goes for the x and y translations. Using the uncertainty theorem for self-adjoint operators, we obtain a set of differential equations, whose solution is the function which obtains the minimal uncertainty. A simultaneous solution for all equations necessarily imposes: $\psi \equiv 0$. Thus, we attempt to find possible solutions over sub-sets. We define new operators which are derived from the group's infinitesimal generators, and are elements of the enveloping algebra. First, we look at the linear combinations of the infinitesimal operators: $T_\theta = T_{s_{12}} - T_{s_{21}} = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$ and $T_{scale} = T_{s_{11}} + T_{s_{22}} = i + ix\frac{\partial}{\partial x} + iy\frac{\partial}{\partial y}$. We may consider these new operators as representing the total orientation and scale changes due to the operation of the affine group. Moreover, these operators, along with the translation operators, are identical to those obtained for the $SIM(2)$ group and, thus, we can easily implement the analysis offered for this group. It is also possible to use rotation invariant functions which can be presented by: $\psi(x, y) = g(\sqrt{x^2 + y^2})$ [4]. These are the minimizers of the following three operators, which are defined as polynomials in the existing six operators:

$$\begin{aligned} T_\theta &= T_{s12} - T_{s12}, \\ T_{scale} &= T_{s11} + T_{s22} = i\left(1 + r\frac{\partial}{\partial r}\right), \\ T_r &= T_{b_1}^2 + T_{b_2}^2 = \frac{1}{r} - \frac{\partial^2}{\partial r^2}. \end{aligned}$$

The equations to be solved are:

$$(T_\theta - \mu_\theta)g(r) = \eta_1(T_r - \mu_r)g(r) \tag{25}$$

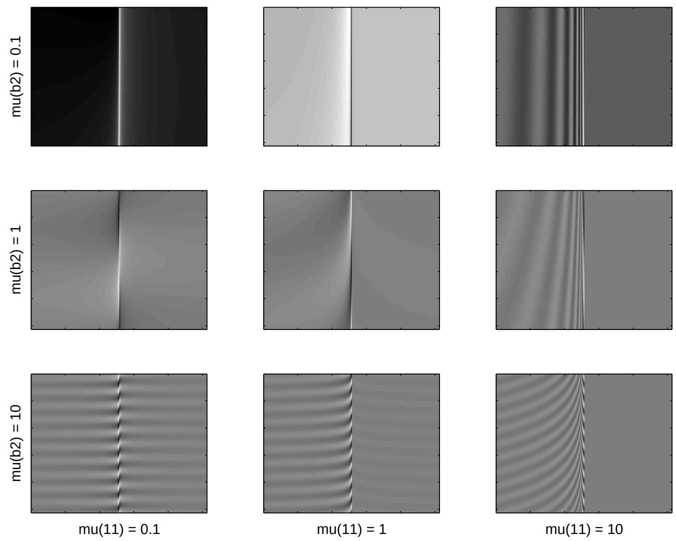

Fig. 1. The real part of the minimizer for the Affine group: $\psi(x,y) = x^{-i\mu_{11}-\frac{1}{2}} e^{i\mu_{b_2} y}$ which does not belong to L^2.

$$(T_\theta - \mu_\theta)g(r) = \eta_2(T_{scale} - \mu_{scale})g(r) \tag{26}$$

$$(T_r - \mu_r)g(r) = \eta_3(T_{scale} - \mu_{scale})g(r). \tag{27}$$

Naturally, the motivation for defining these new operators is the rotation invariance property of T_θ, i.e. $T_\theta g(r) = 0$. Thus, instead of seven equations to be solved, we are left with only three. We can simply select $\eta_1 = \eta_2 = 0$, and are left with:

$$-g''(r) - \frac{1}{r}g'(r) - \mu_r g = -\eta_3 i(g(r) + rg'(r)) - \eta_3\mu_{scale}g. \tag{28}$$

As already mentioned, a possible solution of this equation is the Mexican hat function. Another possible solution, in the spirit of [1], can be obtained by observing that the set of commutators:

$$[T_{s_{11}}, T_{s_{12}}],\ [T_{s_{11}}, T_{s_{21}}],\ [T_{s_{11}}, T_{b_1}],\ [T_{s_{12}}, T_{s_{21}}],\ [T_{s_{12}}, T_{b_2}]$$

transforms under $\frac{\pi}{2}$-rotation into the complementary set of commutators:

$$[T_{s_{22}}, T_{s_{21}}],\ [T_{s_{22}}, T_{s_{12}}],\ [T_{s_{22}}, T_{b_2}],\ [T_{s_{21}}, T_{s_{12}}],\ [T_{s_{21}}, T_{b_1}].$$

If the commutator relation between $T_{s_{21}}$ and $T_{s_{12}}$ is ignored, we may obtain the following set of differential equations:

$$i\left(\frac{\psi(x,y)}{2} + x\psi_x(x,y)\right) - \mu_{11}\psi(x,y) = \eta_1(iy\psi_x(x,y) - \mu_{12}\psi(x,y))$$

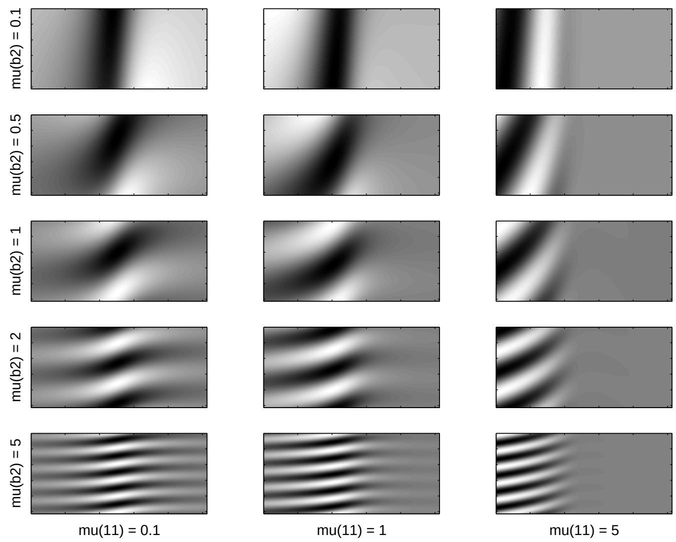

Fig. 2. The real part of the minimizer for the sub-Affine group: $\psi(x,y) = (\eta_3 + x)^{-\frac{1}{2}-i\mu_{11}+i\eta_3\mu_{b1}}e^{i\mu_{b_2}y}$ which belongs to L^2.

$$i\left(\frac{\psi(x,y)}{2} + x\psi_x(x,y)\right) - \mu_{11}\psi(x,y) = \eta_2(ix\psi_y(x,y) - \mu_{21}\psi(x,y))$$
$$i\left(\frac{\psi(x,y)}{2} + x\psi_x(x,y)\right) - \mu_{11}\psi(x,y) = \eta_3(-i\psi_x(x,y) - \mu_{b_1}\psi(x,y))$$
$$-i\psi_y(x,y) - \mu_{b_2}\psi(x,y) = \eta_4(iy\psi_x(x,y) - \mu_{12}\psi(x,y)), \quad (29)$$

where $\mu_{ij} = \mu_\psi(T_{s_{ij}})$. Selecting all η's to be zeros, a possible solution for this system is: $\psi(x,y) = x^{-i\mu_{11}-\frac{1}{2}}e^{i\mu_{b_2}y}$. The real part of this solution is depicted in Fig. (1). This solution, however, does not belong to $L^2(R^2)$ in terms of both x and y. If we restrict our analysis to the differential equations which relate $T_{s_{11}}$ to T_{b_1} and $T_{s_{12}}$ to T_{b_2}:

$$i\left(\frac{\psi(x,y)}{2} + x\psi_x(x,y)\right) - \mu_{11}\psi(x,y) = \eta_3(-i\psi_x(x,y) - \mu_{b_1}\psi(x,y))$$
$$-i\psi_y(x,y) - \mu_{b_2}\psi(x,y) = \eta_4(iy\psi_x(x,y) - \mu_{12}\psi(x,y)), \quad (30)$$

then, under the selection of η_3 to be non-zero, we may obtain a solution of the form $\psi(x,y) = (\eta_3 + x)^{-\frac{1}{2}-i\mu_{11}+i\eta_3\mu_{b1}}e^{i\mu_{b_2}y}$. The solution may become square integrable with respect to the variable x if we select: $|\eta_3| \geq \frac{1}{2\mu_{b_1}}$. This solution is not square integrable in terms of the variable y, although it is periodic. It is shown in Fig. (2) for a selection of $\eta_3 = i$ and $\mu_{b_1} = 1$.

5 The Affine Weyl-Heisenberg Group

The AWH group is generated by time (or spatial coordinate) and frequency translations, and time (or spatial coordinate) dilations. The AWH group can be viewed as the extension of the affine group, incorporating frequency translations or, alternatively, as the extension of the Weyl-Heisenberg group by dilations. Its canonical representation in $L^2(R)$ fails, however, to be square integrable, but can be regularized in an appropriate way, by the introduction of a density function [23].

5.1 The one-dimensional case

The unitary irreducible representation of the AWH group in $L^2(R)$ is given by:

$$[U(\omega, a, b)\psi](t) = \frac{1}{\sqrt{a}} e^{i\omega t} \psi\left(\frac{t-b}{a}\right). \tag{31}$$

Following are the infinitesimal generators of the group:

$$\begin{aligned} T_a(t) &:= i\frac{\partial U}{\partial a}|_{a=1,b=0,\omega=0} = -i\left(\frac{1}{2} + t\frac{\partial}{\partial t}\right) \\ T_b(t) &:= i\frac{\partial U}{\partial b}|_{a=1,b=0,\omega=0} = -i\frac{\partial}{\partial t} \\ T_\omega(t) &:= i\frac{\partial U}{\partial \omega}|_{a=1,b=0,\omega=0} = -t \end{aligned} \tag{32}$$

Next, we calculate the commutation relations between the four operators. The non-zero commutation relations are given by:

$$[T_a, T_b] = iT_b, [T_a, T_\omega] = -iT_w, [T_b, T_\omega] = -i \tag{33}$$

Using the uncertainty theorem, the following set of differential equations is derived:

$$\begin{aligned} -i\psi'(t) - \mu_b\psi(t) &= \eta_1\frac{i\psi(t)}{2} - \eta_1 it\psi'(t) - \eta_1\mu_a\psi(t) \\ -t\psi(t) - \mu_\omega\psi(t) &= \eta_2\frac{i\psi(t)}{2} - i\eta_2 t\psi'(t) - \eta_2\mu_a\psi(t) \\ -i\psi'(t) - \mu_b\psi(t) &= -\eta_3 t\psi(t) - \eta_3\mu_\omega\psi(t), \end{aligned} \tag{34}$$

The solution of this set of equations is the minimizer of the uncertainty of the AWH group. However, there is no non-trivial solution for these equations. The first equation brings us back to the one-dimensional affine group, whose solution was already discussed. The third equation is the same one obtained for the one-dimensional Weyl-Heisenberg group. If we solve the second equation, which relates the scaling and frequency translations, we obtain a polynomial solution which is not in L^2. In order to find a minimizing function for the uncertainty principle for the AWH group, we substantiate the work of Torresani [23], which provides the permitted relationships between scale and frequency.

6 A Gabor-wavelet type subgroup of the Affine Weyl-Heisenberg Group

In his work, Torresani [23] considers a subgroup of the AWH, where frequency translations are functions of the scale parameter. This sub-group is represented by G_λ. He proves that the relationship between the scale a and the frequency $\Omega(a)$ has the following form: $\Omega_\lambda(a) = \lambda[\frac{1}{a} - 1]$, where $\lambda \in R$. This reciprocal relations are in agreement with the structure of the Gabor wavelets, where the frequency depends on the scale, so that smaller scales are related to higher frequencies and vise-versa. The canonical action of G_λ on $L^2(R)$ is inherited from that of the AWH group:

$$[U(b,a)\psi](t) = [U(b,\Omega_\lambda(a),a,0)\psi](t) = \frac{1}{\sqrt{a}} e^{i\lambda t(\frac{1}{a}-1)} \psi\left(\frac{t-b}{a}\right).$$

This representation is then proved to be square integrable [23].

6.1 The uncertainty principle for G_λ

First, we derive the self-adjoint differential operators which are associated with the G_λ group. For ease of presentation, we look at the following representation:

$$[U(b,a)\psi](t) = \sqrt{a} e^{ikat} \psi(a(t-b)).$$

The two associated self-adjoint operators are defined by:

$$T_a(t) = e^{ikt}\left(-kt + \frac{i}{2} + it\frac{\partial}{\partial t}\right)$$
$$T_b(t) = -ie^{ikt}\frac{\partial}{\partial t}. \tag{35}$$

The associated differential equation is:

$$(T_a - \mu_a)\psi(t) = \eta(T_b - \mu_b)\psi(t), \tag{36}$$

explicitly given by:

$$e^{ikt}\left(-kt\psi(t) + \frac{i}{2}\psi(t) + it\psi'(t)\right) - \mu_a\psi(t) = -\eta i e^{ikt}\psi'(t) - \eta\mu_b\psi(t). \tag{37}$$

After rearranging the terms, we obtain:

$$ds = \frac{d\psi(t)}{\psi(t)} = \frac{(-i)\left(kt + (\mu_a - \eta\mu_b)e^{-ikt} - \frac{i}{2}\right)}{t+\eta}. \tag{38}$$

This integral may be well defined if the integration bounds are finite (e.g. some finite t_0 and the variable t), but not otherwise. The solution is thus given by: $\psi = const * e^s$, where

$$s = \int_{t_0}^{t} \frac{-i\left(kq + (\mu_a - \eta\mu_b)e^{-ikq} - \frac{i}{2}\right)}{q+\eta} dq.$$

The integration of the terms $\int_{t_0}^{t}\frac{dq}{q+\eta}$ and $\int_{t_0}^{t}\frac{q}{q+\eta}dq$ presents no analytical difficulty, while the calculation of $\int_{t_0}^{t}\frac{e^{-ikq}}{q+\eta}dq$ is not analytically defined. We, therefore, must use some approximations to be presented in the next section. The solution is given by:

$$s = -ik\left((t-t_0) - \eta log(\frac{\eta+t}{\eta+t_0})\right) - \frac{1}{2}log(\frac{\eta+t}{\eta+t_0}) - i(\mu_a - \eta\mu_b)H(t), \quad (39)$$

where $H(t) = \int_{t_0}^{t}\frac{e^{-ikq}}{q+\eta}dq$. Thus, the solution for $\psi(t)$ is:

$$\psi(t) = e^{ikt_0}(\eta+t_0)^{\frac{1}{2}-ik\eta}e^{-ikt}(\eta+t)^{ik\eta-\frac{1}{2}}e^{-iAH(t)}, \quad (40)$$

where $A = \mu_a - \eta\mu_b$. In order that the solution will belong to $L^2(R)$, $Im(\eta) > \frac{1}{2k}$ if $k > 0$ or, $Im(\eta) < \frac{1}{2k}$ if $k < 0$.

Our main interest in this approximation is derived from the need to explore the behavior of the function which provides the minimum value for the AWH uncertainty rule, and to assess the validity of this approximation. Next, we elaborate on the numerical approximations of the complex exponential integral we have to solve.

6.2 The Complex Exponential Integral

The integral $H(t) = \int_{t_0}^{t}\frac{e^{-ikq}}{q+\eta}dq$ should be calculated for both t and q being real. Following the change of variables, $w = ik(q+\eta)$, we obtain: $H(z) = e^{ik\eta}\int_{z_0}^{z}\frac{e^{-w}}{w}dw$, where $z_0 = ik(t_0+\eta)$ and $z = ik(t+\eta)$.

The following approximation can be obtained for small values of z using the Taylor expansion:

$$H(z) = e^{ik\eta}\int_{z_0}^{z}\frac{e^{-w}}{w}dw = e^{ik\eta}\left(ln(z) + \Sigma_{s=1}^{\infty}\frac{(-1)^s z^s}{ss!} - ln(z_0) - \Sigma_{s=1}^{\infty}\frac{(-1)^s(z_0)^s}{ss!}\right).$$

Thus, inserting this expression into our function, we obtain:

$$\begin{aligned}\psi(t) &= C_1(t_0)e^{-ikt}(t+\eta)^{ik\eta-\frac{1}{2}}\left(ik(t+\eta)\right)^{-i(\mu_a-\eta\mu_b)e^{ik\eta}}e^{-i(\mu_a-\eta\mu_b)e^{ik\eta}\Sigma_{s=1}^{\infty}(-1)^s\frac{(ik(t+\eta))^s}{ss!}}\\ &= C_1(t_0)e^{-ikt}(t+\eta)^{ik\eta-\frac{1}{2}}\left(ik(t+\eta)\right)^{-i(\mu_a-\eta\mu_b)e^{ik\eta}}\\ &\quad exp\left\{i(\mu_a-\eta\mu_b)\left(ik(t+\eta)\right)e^{ik\eta}\right\}exp\left\{-i(\mu_a-\eta\mu_b)\frac{\left(ik(t+\eta)\right)^2}{2*2!}e^{ik\eta}\right\}\ldots, \quad (41)\end{aligned}$$

where

$$C_1(t_0) = e^{ikt_0}(t_0+\eta)^{\frac{1}{2}-ik\eta}\left(ik(t_0+\eta)\right)^{i(\mu_a-\eta\mu_b)e^{ik\eta}}e^{ie^{ik\eta}(\mu_a-\eta\mu_b)\Sigma_{s=1}^{\infty}(-1)^s\frac{(ik(t_0+\eta))^s}{ss!}}$$

Evaluating the exponential integral in the case of large values of z, we can use asymptotic approximation via successive integration by parts to obtain:

$$H(z) = e^{ik\eta}\int_{z_0}^{z}\frac{e^{-w}}{w}dw = e^{ik\eta-z}\left\{\frac{1}{z} - \frac{1}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \ldots\right\} - e^{ik\eta-z_0}\left\{\frac{1}{z_0} - \frac{1}{z_0^2} + \frac{2!}{z_0^3} - \frac{3!}{z_0^4} + \ldots\right\},$$

where the general term in the series has the form $\frac{(-1)^{n+1}(n-1)!}{z^n}$ for an arbitrary n. Inserting this into the expression for $\psi(t)$ we obtain:

$$\psi(t) = C_2(t_0)e^{-ikt}(t+\eta)^{ik\eta-\frac{1}{2}}exp\left\{-i(\mu_a - \eta\mu_b)e^{-ikt}V(t,\eta)\right\}),$$

where

$$C_2(t_0) = e^{ikt_0}(t_0+\eta)^{\frac{1}{2}-ik\eta}exp\left\{i(\mu_a - \eta\mu_b)e^{-ikt_0}V(t_0,\eta)\right\},$$

and

$$V(t,\eta) = \frac{1}{(ik(t+\eta))} - \frac{1}{(ik(t+\eta))^2} + ... + \frac{(-1)^{n+1}(n-1)!}{(ik(t+\eta))^n}.$$

A plot of the absolute value of this function is depicted in Fig. (3), and Fig. (4), where it is plotted on a logarithmic scale.

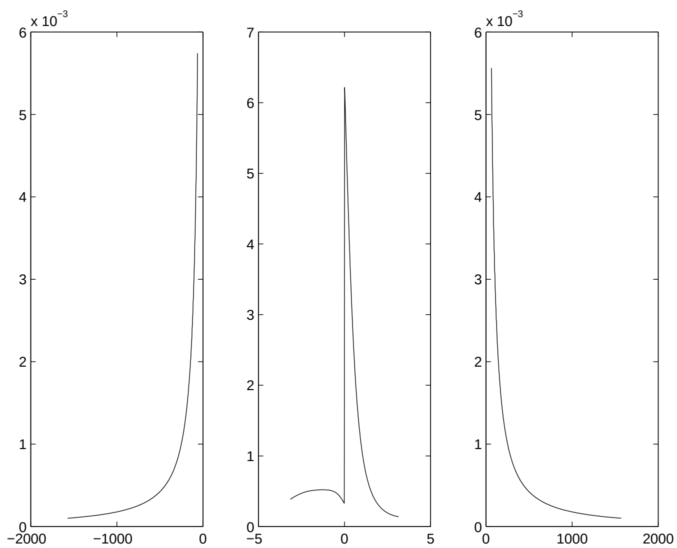

Fig. 3. The behavior of the absolute value of a possible minimizing function of the AWH uncertainty. The right and left figures demonstrate the behavior of this function according to the asymptotic expansion in $\pm\infty$. The center figure demonstrates the behavior close to zero.

7 Scale-Space Nature of the Uncertainty Principle Minimizers

As has already been shown, the Gaussian function is the minimizer of the uncertainty related to the Weyl-Heisenberg group. It also has an important role

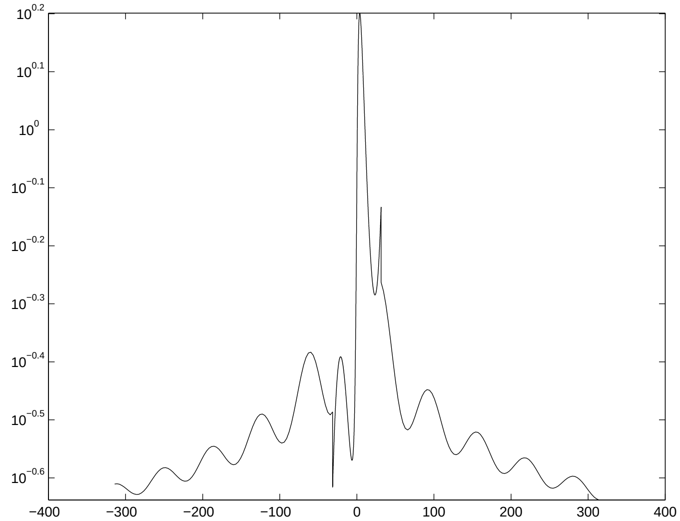

Fig. 4. The behavior of the absolute value of the possible minimizing function of the AWH uncertainty, shown in Fig. (3), plotted on a logarithmic scale.

in the framework of scale-space [21]. Application of Gaussian functions with different values of variance result in smoother versions of the original image, where the degree of smoothness is determined by the standard deviation of the Gaussian. Moreover, successive applications of two Gaussian functions with parameters $t_1 = \frac{1}{2}\sigma_1^2$ and $t_2 = \frac{1}{2}\sigma_2^2$, are equivalent to application of a Gaussian with $t = t_1 + t_2$. Thus, the Gaussian functions with the parameter $t = \frac{1}{2}\sigma^2$ form a semi-group with respect to convolution.

The concept of linear and non-linear scale-space is important in image processing, in terms of representation of images, image denoising, features extraction and image analysis. Therefore, we would like to explore whether functions which are minimizers of uncertainty principle encompass scale-space like attributes, and thus may be used in image interpretation. This mathematical curiosity is rooted in a deeper question: is the Gaussian function really so unique, or is it one in many other functions that may posses attributes such as: smoothness, separability, self-similarity (in time and frequency), scale-space generation, minimizers of an uncertainty principle and being the kernel (Green function) of a heat-like (diffusion) equation. This section serves as an appetizer, and provides evidence that minimizers of uncertainty principles related to groups other than the Weyl-Heisenberg, also posses scale-space generation properties.

In this study we have considered the minimizers of the uncertainties related to the $SIM(2)$ and the AWH group. We now proceed to present some preliminary results, indicating that there are scale-space attributes to minimizers of uncertainty relations, other than the Gaussian function [29].

The solution offered by Dahlke and Maass for the minimizer with respect to the SIM(2) group is scale-space by nature. The minimizer that they found is the Mexican hat function: $\psi(x, y) = \beta(1-\beta r^2)\exp(-\beta r^2)$, where $r := \sqrt{x^2 + y^2}$. Its Fourier transform is $\pi^2 k^2 \exp(-\frac{\pi^2 k^2}{\beta})$. Clearly, if we define $\beta = 1/t$ then the semi-group property is trivially satisfied with t as the semi-group parameter. Note that this is a scale-space of an *edge detector* and not of the image smoothness as usual. It is in fact an element of the jet-space of the traditional Gaussian scale-space. It is interesting to note the similarity with the scale-space generated by the complex diffusion operator [11], as well as the study of α-scale-spaces [5, 6] and the Poisson Scale-Space [7].

The rest of this section is devoted to exploring the scale-space nature of the minimizer given by Ali, Antoine and Gazeau for the uncertainty related to the $SIM(2)$ group [1]. Their solution is given in the the wave number (frequency) space (k_x, k_y). It is a function which vanishes outside some convex cone in the half-plane $k_x > 0$ and is exponentially decreasing inside:

$$\hat{\psi(k)} = c|\overrightarrow{k}|^s e^{-rk_x}, \tag{42}$$

where $s = i\eta\langle P_1\rangle > 0$, $\eta \in i\mathbb{R}$, $\langle P_1\rangle$ is the mean value of the translation operator in the k_x direction, and $r = i\eta > 0$. The one-dimensional equivalent of this solution is known as the Cauchy wavelets [14, 18]: $\hat{\psi(\xi)} = c\xi^s e^{-r\xi}$ for $\xi \geq 0$ where $\hat{\psi(\xi)} = 0$ for $\xi < 0$, and $s > 0$. The characteristic responses of the one- and two-dimensional filters are depicted in Fig. (5,6) and Fig. (7,8), respectively, in both the Fourier and time/position domains. It is quite obvious, from the mere definition of the function, that successive applications of the filters with two values of either s or r correspond to a single application of an effective parameter. Moreover, this function has the following properties. The term $|\overrightarrow{k}|^s = (k_x^2+k_y^2)^{\frac{s}{2}}$ in frequency space is actually equivalent (up to a sign) to a power of the Laplacian operator $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^{\frac{s}{2}}$ in the spatial space and, thus, can be considered as an edge enhancement operator. The term e^{-rk_x} can be considered as a smoothing operator in the x direction.

Applications of the Cauchy wavelets to a rectangular pulse function (Fig. 9) yields the following results: as s increases, the edges become more pronounced, while as r increases, the signal becomes smoother (Fig. 10).

We next apply the two-dimensional minimizer filter to a test image of a clown, symmetrizing the filter as follows: $\hat{\psi}(\hat{k}) = c|\overrightarrow{k}|^s e^{-r|k_x|}$. When the value of r is kept constant, increasing the value of s results in a progressive edge enhancement (Fig. 11). When the value of s is kept constant, increasing the value of r results in a motion blurring effect in the x-direction (Fig. 12). To conclude, we have shown in this section that the minimizers of the uncertainty related to the $SIM(2)$ group posses intrinsic scale-space generation properties. Further research is required in order to tackle the more general question regarding the existence of "Gaussian-like" functions for other groups, in the context of the issues discussed at the beginning of this section.

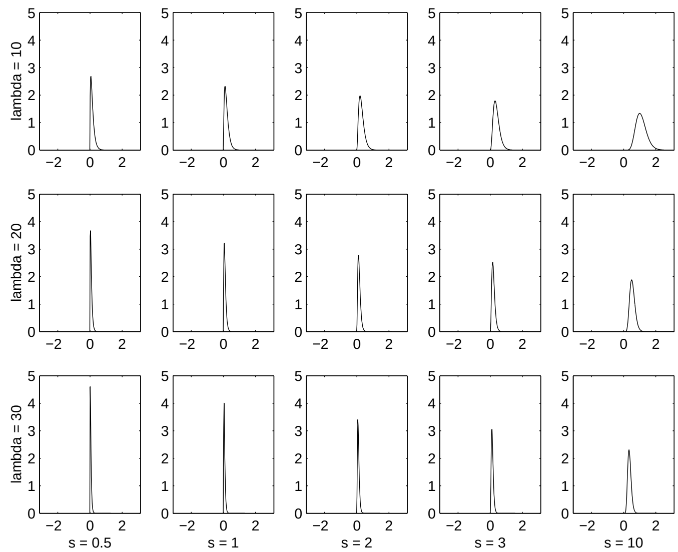

Fig. 5. The one-dimensional Cauchy wavelets in the frequency domain given by: $\hat{\psi(\xi)} = c\xi^s e^{-r\xi}$ for $\xi \geq 0$ where $\hat{\psi(\xi)} = 0$ for $\xi < 0$, and $s > 0$. We present the different functions obtained for different values of s and r.

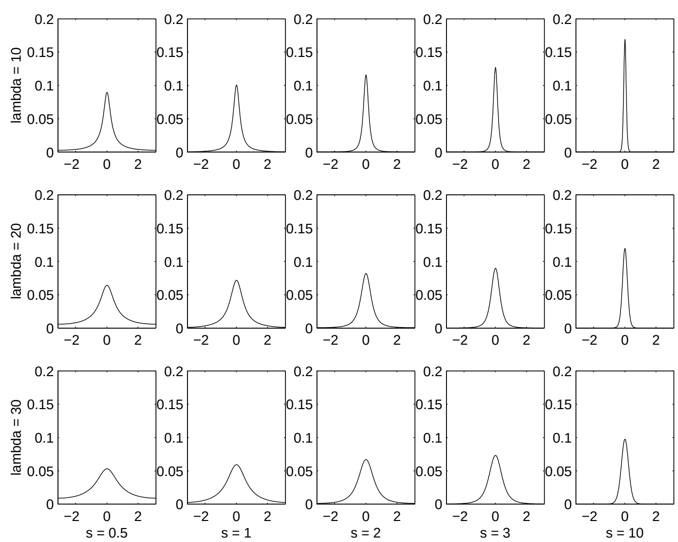

Fig. 6. The one-dimensional Cauchy wavelets in the time domain. This is a numerical approximation obtained by taking the inverse Fourier transform of the function. The functions depend on both s and r. As r increases, the size of the window increases, thus it may be associated with a higher degree of smoothing, while as s increases the window becomes smaller.

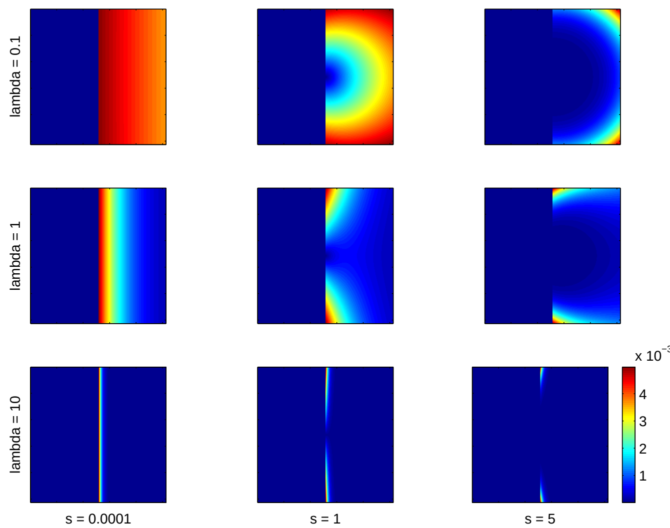

Fig. 7. The two-dimensional solution of Ali, Antoine and Gazeau [1]in the frequency domain given by: $\psi(\hat{\hat{k}}) = c|\overrightarrow{k}|^s e^{-rk_x}$ where $s > 0$, $r > 0$ and $k_x > 0$. We present the different functions obtained for different values of s and r.

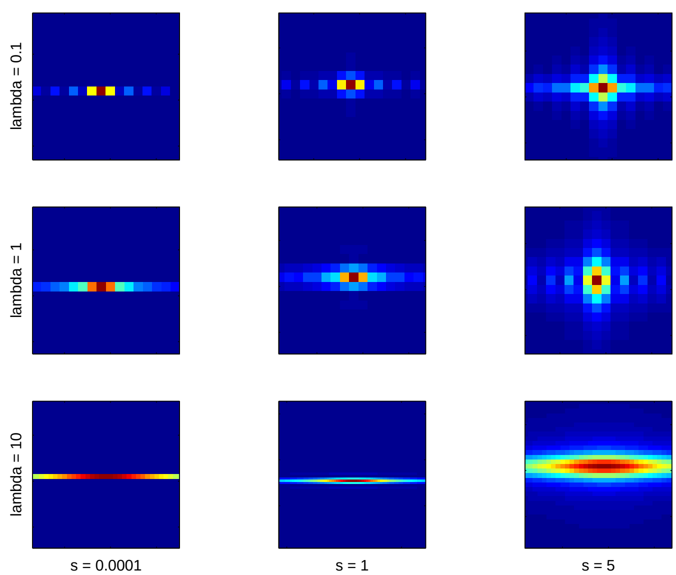

Fig. 8. The $2D$ solution of Ali et al [1]in the spatial domain. This is a numerical approximation obtained by taking the inverse Fourier transform of the function. As r increases, the size of the window increases (a higher degree of smoothing). As s increases the window becomes smaller.

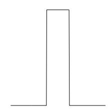

Fig. 9. A one-dimensional rectangular pulse function.

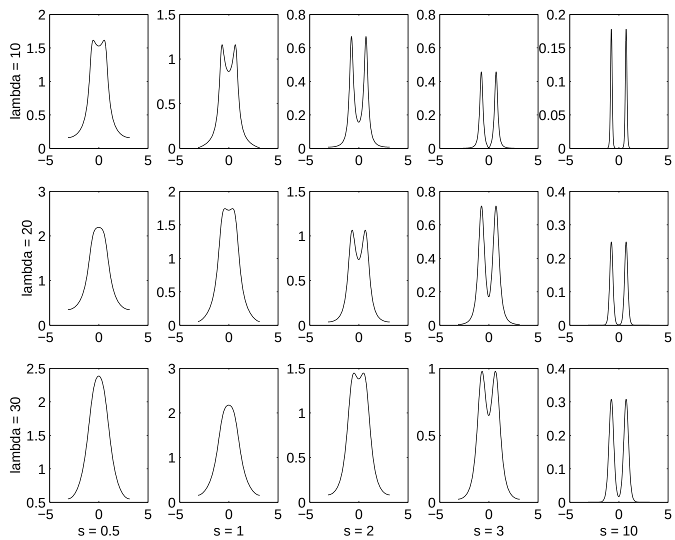

Fig. 10. When the 1D Cauchy wavelets are applied to a rectangular pulse, the larger s is the more noticeable the edges are (left to right). The larger r is the smoother the edges become (up to bottom).

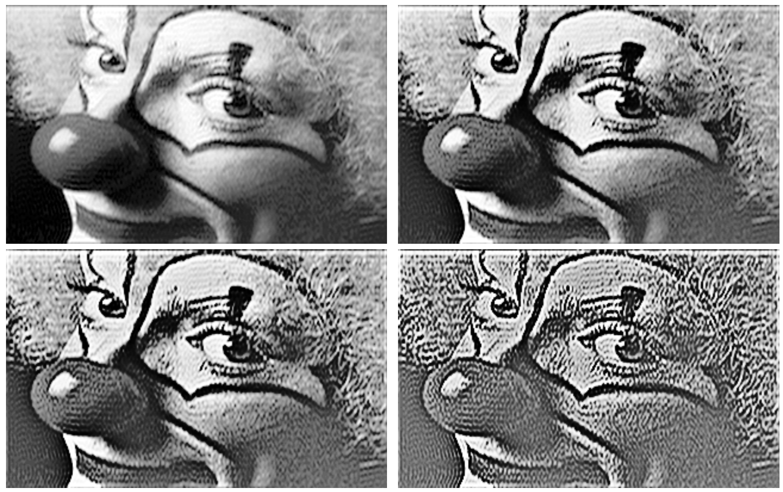

Fig. 11. For a constant value of $r = 0.00001$, increasing the value of s the value of s is increased: $0.01, 0.2, 0.5, 1$ (up left to bottom right), results in an edge enhancement effect.

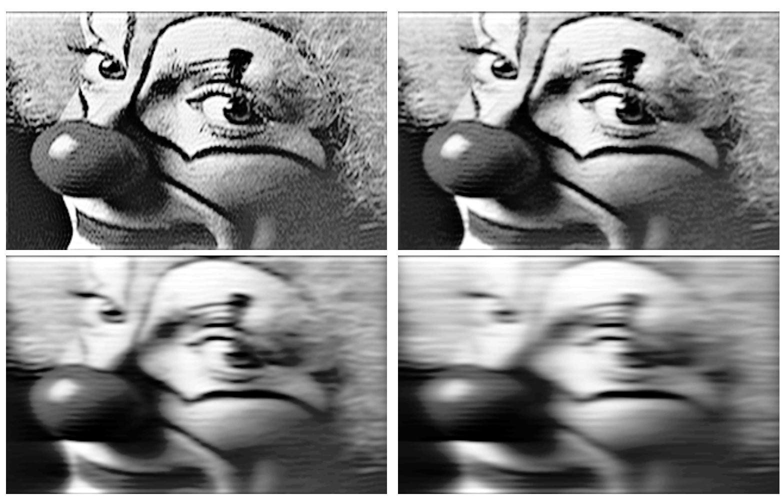

Fig. 12. For a constant value of $s = 0.2$, increasing the value of r: $0.001, 0.01, 0.05, 0.1$ (up left to bottom right) results in an effect of motion-blurring in the x-direction.

8 Discussion and Conclusions

The use of Gabor wavelets for texture analysis and synthesis is frequently justified with the well-known fact that Gabor functions provide the best combined time-frequency resolution. This fact can be easily derived using the basic uncertainty theorem for self-adjoint operators. Moreover, it can be easily extended to higher dimensions, as for the Weyl-Heisenberg group, we always obtain a Gaussian solution. Dahlke and Maass, as well as Ali, Antoine and Gazeau presented an extension of this notion to other groups: the affine group in one dimension and the $SIM(2)$ group in two dimensions. It turned out that finding the unique function that simultaneously minimizes the uncertainties in these cases is impossible.

One of the declared justifications for using Gabor wavelets in image processing is that Gabor functions are the minimizers of the uncertainty of the Weyl-Heisenberg group. However, these filters are not minimizers of the uncertainty principle related to the affine group and the wavelet transform. Some intuitive understanding of this phenomenon can be achieved by looking at the presentations of the Weyl-Heisenberg group and the affine group. The unitary irreducible representation of the two-dimensional Weyl-Heisenberg group in $L^2(R^2)$, is given by:

$$[U(\omega_1, \omega_2, b_1, b_2, \tau)\psi](x, y) = \tau e^{i\omega_1 x + i\omega_2 y}\psi(\overrightarrow{u} - \overrightarrow{b}),$$

where $\overrightarrow{u} = (x, y)$, $\overrightarrow{b} = (b_1, b_2)$. The unitary irreducible representation of the two-dimensional affine group in $L^2(R^2)$ is given by:

$$u = D\psi\left(s^{-1}\left(\overrightarrow{x} - \overrightarrow{b}\right)\right).$$

Thus, a noticeable difference between the two representations is the fact that the x and y components are independent of each other in the Weyl-Heisenberg representation, while in the affine representation there is a coupling between the x and y variables. This may also be the reason for having a multi-dimensional minimizer for the Weyl-Heisenberg group, and not for the affine group.

In this study we focused our efforts on finding possible solutions for the minimizers of the affine and AWH groups. We applied the results of Dahlke and Maass [4] and of Ali, Antoine and Gazeau [1] to the two-dimensional affine group, and showed that solutions can be found for a sub-set of the affine group, or when elements of the enveloping algebra are involved. We also presented a possible candidate for the minimizer of the AWH group in one dimension, where a Gabor-wavelet type subgroup is considered.

Moreover, the scale-space properties of some of the minimizers have been considered. We examined the minimizer offered by Ali, Antoine and Gazeau, and found that modifying the function's parameters results in either edge enhancement or motion-like blurring.

Our preliminary results point to the need to further explore the attributes of the uncertainty minimizers, obtained in this study, as well as their scale-space properties. Gabor wavelets are still an important tool when considering the joint time (spatial) frequency uncertainty. Nevertheless, using these functions cannot guarantee the maximal joint accuracy.

References

1. S.T. Ali, J.P. Antoine and J.P. Gazeau, "Coherent States, Wavelets and Their Generalizations", Springer-Verlag, 2000.
2. A.C. Bovik and M. Clark and W.S. Geisler "Multichannel Texture Analysis Using Localized Spatial Filters", *IEEE Transactions on PAMI*, 12(1), 1990, 55-73.
3. J.G. Daugman, "Uncertainty relation for resolution in space, spatial frequency, and orientation optimized by two-dimensinal visual cortical filters", *J. Opt. Soc. Amer.* 2(7), 1985, 1160-1169.
4. S. Dahlke and P. Maass, "The Affine Uncertainty Principle in One and Two Dimensions", *Comput. Math. Appl.*, 30(3-6), 1995, 293-305.
5. R. Duits, M. Felsberg, L. Florack, B. Platel, "α-Scale-Spaces on a Bounded Domain", in: Proceedings of Scale Space Methods in Computer Vision, LNCS 2695, Springer, 2003, 494-510.
6. R. Duits, L. Florack, J. de Graaf, and B. Ter Haar Romeny, "On the axioms of scale space theory", *Journal of Mathematical Imaging and Vision*, 20, 2004, 267298.
7. M. Felsberg and G. Sommer, "Scale-adaptive filtering derived from the Laplace equation", in: Pattern Recognition, volume 2032 of LNCS, B. Radig and S. Florczyk (Eds.), 95106, Springer, Berlin, 2001.
8. L.M.J. Florack, B.M. ter Haar Romeny, J.J. Koenderink and M.A. Viergever, "Linear Scale-Space", *Journal of Mathematical Imaging and Vision*, 4, 1994, 325-351.
9. G. Folland, "Harmonic Analysis in Phase Space", *Princeton University Press*, Princeton, NJ, USA, 1989.
10. D. Gabor, "Theory of communication",*J. IEEE* , 93, 1946, 429-459.
11. G. Gilboa, N. Sochen, Y. Y. Zeevi, "Image Enhancement and Denoising by Complex Diffusion Processes", IEEE Transactions on Pattern Analysis and Machine Intelligence (PAMI), Vol. 25, No. 8, pp. 1020-1036, 2004.
12. A. Jain and F. Farrokhnia, "Unsupervised texture segmentation using Gabor filters", *Pattern Recognition*, 24, 1991, 1167-1186.
13. C. Kalisa and B. Torresani "N-dimensional affine Weyl-Heisenberg wavelets", *Annales de L'Instut Henri Poincare, Physique theorique*, 59(2), 1993, 201-236.
14. J.R. Klauder, "Path Integrals for Affine Variables", in: Functional Integration, Theory and Applications, 101-119, J.P. Antoine and E. Tirapegui (Eds.), Plenum Press, New York and London, 1980.
15. J. R. Klauder and B.S. Skagerstam, "Coherent States Applications in Physics and Mathematical Physics", World Scientific, Singapore, 1985.
16. T.S. Lee, "Image Representation using 2D Gabor-Wavelets", *IEEE Transactions on PAMI*, 18(10), 1996, 959-971.
17. S. Marcelja, "Mathematical description of the response of simple cortical cells", *J. Opt. Soc. Amer.*, 70, 1980, 1297-1300.
18. Th. Paul and K. Seip, "Wavelets in Quantum Mechanics", in: Wavelets and their Applications, 303-322, M.B. Ruskai, G. Beylkin, R. Coifman, I. Daubechies, S. Mallat, Y. Meyer and L. Raphael (Eds.), Jones and Bartlett, Boston, 1992.
19. M. Porat and Y.Y. Zeevi, "The generalized Gabor scheme of image representation in biological and machine vision", *IEEE Transactions on PAMI*, 10(4), 1988, 452-468.

20. M. Porat and Y.Y. Zeevi, “Localized texture processing in vision: Analysis and synthesis in the gaborian space“, *IEEE Transactions on Biomedical Engineering*, 36(1), 1989, 115-129.
21. L. M. J. Florack, B. M. ter Haar Romeny, J. J. Koenderink, and M. A. Viergever, “Linear scale-space“, *Journal of Mathematical Imaging and Vision*, 4(4), 1994, 325-351.
22. G. Teschke, “Construction of Generalized Uncertainty Principles and Wavelets in Bessel Potential Spaces“, *International Journal of Wavelets, Multiresolution and Information Processing*, 3(2), 2005.
23. B. Torresani, “Wavelets associated with representationsn of the affine Weyl-Heisenberg group“, *Journal of Mathematical Physics*, 32(5), 1991, 1273-1279.
24. C. Sagiv, N. Sochen, and Y.Y. Zeevi , “Gabor Space Geodesic Active Contours“, G. Sommer, Y.Y. Zeevi (Eds.), Algebraic Frames for the Perception-Action Cycle, *Lecture Notes in Computer Science*, Vol. 1888, Springer, Berlin, 2000.
25. C. Sagiv, N.A. Sochen, Y.Y. Zeevi, “Geodesic Active Contours Applied to Texture Feature Space“, M. Kerckhove (Ed.), Proceedings of Scale-Space and Morphology in Computer Vision, Lecture Notes in Computer Science, Vol. 2106, Springer, Berlin, 344-352, 2001.
26. C. Sagiv, N.A. Sochen, Y.Y. Zeevi, “Gabor Feature Space Diffusion via the Minimal Weighted Area Method“, M. Figueiredo, J. Zerubia, A.K. Jain (Eds.), Proceedings of Energy Minimization Methods in Computer Vision and Pattern Recognition, *Lecture Notes in Computer Science* Vol. 2134, Springer, Berlin, 621-635, 2001.
27. C. Sagiv, N.A. Sochen, Y.Y. Zeevi, “Texture segmentation via a diffusion-segmentation scheme in the Gabor feature space“, M. Chantler (Ed.), Proceedings of Texture 2002, The 2nd international workshop on texture analysis and synthesis, 123-128, Copenhagen, 2002.
28. C. Sagiv, N.A. Sochen, Y.Y. Zeevi, “Integrated Active Contours for Texture Segmentation“, Accepted to IEEE Transactions on Image Processing.
29. C. Sagiv, N.A. Sochen, Y.Y. Zeevi, “Scale-Space Generation via Uncertainty Principles“, R. Kimmel, N. Sochen, J. Weickert (Eds.), Proceedings of Scale Space and PDE Methods in Computer Vision 2005, Hofgeismar, Germany, 351-362.
30. J. Segman and W. Schempp, “Two ways to incorporate scale in the Heisenberg group with an intertwining operator“, *Journal of Mathematical Imaging and Vision*, 3(1), 1993, 79-94.
31. J. Segman and Y. Y. Zeevi, “Image Analysis by Wavelet-Type Transform: Group Theoretic Approach“, *J. Mathematical Imaging and Vision*, 3, 1993, 51-75.
32. M. Zibulski and Y.Y. Zeevi, “Analysis of multiwindow Gabor-type schemes by frame methods“, *Applied and Computational Harmonic Analysis*, 4, 1997, 188-221.