On Affine Invariance in the Beltrami Framework for Vision

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Abstract

We use the geometric Beltrami framework to incorporate and explain some of the known invariant flows e.g. the equi-affine invariant flow. It is also demonstrated that the new concepts put forward in this framework enable us to construct new invariant flows for the case where the codimension is greater than one e.g. for color images and video.

1 Introduction

Invariance is an important issue in computer vision. Many works dealt with this question in the context of scale-space [21, 10] and in the context of the recognition of planar curves (Refs. [3, 16, 13, 12, 1, 6, 7, 14] and [8, 4, 5] may serve as pointers to the subject.)

We demand that our treatment of the image depends on the relevant data and is independent of irrelevant one. The relevant data is the *shape* of the image. The shape depends on the intrinsic geometry of the image surface, that is, its metric, and on the extrinsic geometry i.e. the embedding. It should not depend on the way one chooses to describe the shape. In other words, a geometrical object should not depend on the coordinate system in which it is described. It should be invariant under reparameterization. We show below that the Beltrami flow is invariant under the reparameterization of the image manifold. The requirement of reparameterization invariance singles out the pre-factor of the Euler-Lagrange equations in our gradient descent scheme. This is an invariance under a *passive* coordinate change.

Other possibility of interest is invariance under *active* coordinate transformations. We consider here the affine transformations for higher dimensional and codimensional images. These transformations act on the *embedding space*. We construct an equi-affine invariant flow for hypersurfaces and show possible ways to extend the affine invariance to embedding with codimension > 1. In particular we introduce an affine color invariant flow.

2 The Beltrami framework

Let us briefly review the Beltrami framework for non-linear diffusion in computer vision [18, 9].

We represent an image and other local features as an embedding maps of a Riemannian manifold in a higher dimensional space. The simplest example is a gray-level image which is represented as a 2D surface embedded in $\mathbb{R}^3$. We denote the map by $X : \Sigma \to \mathbb{R}^3$. Where Σ is a two-dimensional surface, and we denote the local coordinates on it by (σ^1, σ^2). The map X is given in general by $(X^1(\sigma^1, \sigma^2), X^2(\sigma^1, \sigma^2), X^3(\sigma^1, \sigma^2))$. In our example we represent it as follows $(X^1 = \sigma^1, X^2 = \sigma^2, X^3 = I(\sigma^1, \sigma^2))$. We choose on this surface a Riemannian structure, namely, a metric. The metric is a positive definite and a symmetric 2-tensor that may be defined through the local distance measurements:

$$ds^2 = g_{11}(d\sigma^1)^2 + 2g_{12}d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2.$$

We use below the Einstein summation convention in which the above equation reads $ds^2 = g_{\mu\nu}d\sigma^\mu d\sigma^\nu$ where repeated indices are summed over. We denote the inverse of the metric by $g^{\mu\nu}$.

2.1 Polyakov Action: A measure on the space of embedding maps

Denote by (Σ, g) the image manifold and its metric and by (M, h) the space-feature manifold and its metric, then the functional $S[X]$ attaches a real number to a map $X : \Sigma \to M$:

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int dV \langle \nabla X^i, \nabla X^j \rangle_g h_{ij}$$

where dV is a volume element and $\langle \nabla R, \nabla G \rangle_g = g^{\mu\nu}\partial_\mu R \partial_\nu G$. This functional, for $m = 2$ and $h_{ij} = \delta_{ij}$, was first proposed by Polyakov [15] in the context of high energy physics, and the theory known as *string theory*.

Using standard methods in the calculus of variations (see [17]), the Euler-Lagrange equations with respect to the embedding (assuming Euclidean embedding space) are:

$$-\frac{1}{2\sqrt{g}}h^{il}\frac{\delta S}{\delta X^l} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i).$$

The extension for non-Euclidean embedding space is treated in [18, 9, 19, 20]. Since $(g_{\mu\nu})$ is positive definite, $g \equiv \det(g_{\mu\nu}) > 0$ for all σ^μ. This factor is the simplest one that doesn't change the minimization solution while giving a reparameterization invariant expression. The operator that is acting on X^i is the natural generalization of the Laplacian from flat spaces to manifolds and is called *the second order differential parameter of Beltrami* [11], or for short *Beltrami operator*, and we will denote it by Δ_g.

The non-linear diffusion or scale-space equation emerges as a gradient descent minimization:

$$X_t^i = \frac{\partial}{\partial t}X^i = -\frac{1}{2\sqrt{g}}h^{il}\frac{\delta S}{\delta X^l} = \Delta_g X^i$$

3 Passive Transformations

3.1 Image Manifold Reparameterization

In order to prove reparametrization invariance, we need to understand how the metric transforms under coordinate change. This is easy to figure out since distances on the image do not depend on the coordinate system. Therefore, denote by x and $\tilde{x}$ the old and new coordinates systems, then we notice that

$$ds^2 = g_{\mu\nu}d\sigma^\mu d\sigma^\nu = \tilde{g}_{\mu\nu}d\tilde{\sigma}^\mu d\tilde{\sigma}^\nu \tag{1}$$

from which we deduce

$$\tilde{g}_{\mu\nu} = g_{\gamma\delta}\frac{d\sigma^\gamma}{d\tilde{\sigma}^\mu}\frac{d\sigma^\delta}{d\tilde{\sigma}^\nu} \tag{2}$$

We introduce now some notations that will help us below. Define a totally antisymmetric symbol as follows

$$\epsilon_{i_1 i_2 \ldots i_d} = (-1)^{s(i_1,\ldots,i_d)} \tag{3}$$

where $s(i_1,\ldots,i_d)$ is the number of basic permutations needed to bring $(i_1,\ldots i_d)$ to the form $(1,2,\ldots d)$. Although s is not well defined (there are many ways with different number of basic permutations to do that) its parity is invariant. $\epsilon_{i_1 i_2 \ldots i_d} = 0$ if any of the indices appears twice.

It is convenient to use this symbol to give an explicit expression for the determinant of a matrix. Take a matrix

$$A = \begin{pmatrix} A_1^1 & A_1^2 & \ldots & A_1^d \\ A_2^1 & A_2^2 & \ldots & A_2^d \\ \vdots & \vdots & \ddots & \vdots \\ A_d^1 & A_d^2 & \ldots & A_d^d \end{pmatrix}. \tag{4}$$

The determinant is defined as follows:

$$\det A = \epsilon_{i_1\ldots i_d}A_1^{i_1}A_2^{i_2}\ldots A_d^{i_d} = \epsilon^{i_1\ldots i_d}A_{i_1}^1 A_{i_2}^2\ldots A_{i_d}^d, \tag{5}$$

where summation is assumed on indices that appear twice.

Take the determinant of a 2x2 matrix for example

$$\begin{aligned} &\det\begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \epsilon_{ij}A_1^i A_2^j \\ = \;& \epsilon_{11}A_1^1A_2^1 + \epsilon_{12}A_1^1A_2^2 + \epsilon_{21}A_1^2A_2^1 + \epsilon_{22}A_1^2A_2^2 \\ = \;& A_1^1A_2^2 - A_1^2A_2^1. \end{aligned} \tag{6}$$

It can be easily proved that

$$\epsilon_{i_1 i_2\cdots i_n}A_{p_1}^{i_1}A_{p_2}^{i_2}\cdots A_{p_n}^{i_n} = \det(A)\epsilon_{p_1p_2\ldots p_n}. \tag{7}$$

The object to consider in a coordinate change is the matrix whose elements are $\partial\sigma^\mu/\partial\tilde{\sigma}^\nu$. The Jacobian is the determinant of this matrix:

$$J = \det(\partial\sigma^\mu/\partial\tilde{\sigma}^\nu) = \epsilon^{\mu_1\mu_2\cdots\mu_d}\frac{\partial\sigma^1}{\partial\tilde{\sigma}^{\mu_1}}\frac{\partial\sigma^2}{\partial\tilde{\sigma}^{\mu_2}}\cdots\frac{\partial\sigma^d}{\partial\tilde{\sigma}^{\mu_d}} \tag{8}$$

Using the identity Eq. (7) we get

$$\epsilon^{\mu_1\mu_2\cdots\mu_d}\frac{\partial\sigma^{\nu_1}}{\partial\tilde{\sigma}^{\mu_1}}\frac{\partial\sigma^{\nu_2}}{\partial\tilde{\sigma}^{\mu_2}}\cdots\frac{\partial\sigma^{\nu_d}}{\partial\tilde{\sigma}^{\mu_d}} = J\epsilon^{\nu_1\nu_2\ldots\nu_d}. \tag{9}$$

The determinant g transforms as follows:

$$\begin{aligned} \tilde{g} &= \epsilon^{\mu_1\mu_2\cdots\mu_d}\tilde{g}_{1\mu_1}\tilde{g}_{2\mu_2}\cdots\tilde{g}_{d\mu_d} \\ &= \epsilon^{\mu_1\mu_2\cdots\mu_d}g_{\gamma_1\delta_1}\frac{d\sigma^{\gamma_1}}{d\tilde{\sigma}^1}\frac{d\sigma^{\delta_1}}{d\tilde{\sigma}^{\mu_1}}\cdots g_{\gamma_d\delta_d}\frac{d\sigma^{\gamma_d}}{d\tilde{\sigma}^d}\frac{d\sigma^{\delta_d}}{d\tilde{\sigma}^{\mu_d}} \\ &= J\epsilon^{\delta_1\delta_2\ldots\delta_d}g_{\gamma_1\delta_1}\frac{d\sigma^{\gamma_1}}{d\tilde{\sigma}^1}\cdots g_{\gamma_d\delta_d}\frac{d\sigma^{\gamma_d}}{d\tilde{\sigma}^d} \\ &= Jg\epsilon_{\gamma_1\gamma_2\ldots\gamma_d}\frac{d\sigma^{\gamma_1}}{d\tilde{\sigma}^1}\frac{d\sigma^{\gamma_2}}{d\tilde{\sigma}^2}\cdots\frac{d\sigma^{\gamma_d}}{d\tilde{\sigma}^d} = (J^2)g. \end{aligned} \tag{10}$$

The derivatives transform as a vector

$$\tilde{\partial}_\mu = \frac{d\sigma^\mu}{d\tilde{\sigma}^\nu}\partial_\nu. \tag{11}$$

Putting everything together, we find

$$\begin{aligned} \tilde{\Delta}_g I^i &= \frac{1}{\sqrt{\tilde{g}}}\tilde{\partial}_\mu(\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\tilde{\partial_\nu}I^i) \\ &= \frac{1}{J\sqrt{g}}\frac{\partial\sigma^\alpha}{\partial\tilde{\sigma}^\mu}\partial_\alpha(J(\sqrt{g}\frac{\partial\tilde{\sigma}^\mu}{\partial\sigma^\beta}\frac{\partial\tilde{\sigma}^\nu}{\partial\sigma^\gamma}g^{\beta\gamma}\frac{\partial\sigma^\delta}{\partial\tilde{x}^\nu}\partial_\delta I^i) \\ &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu I^i) = \Delta_g I^i \end{aligned} \tag{12}$$

where we used the fact that

$$\frac{\partial}{\partial\sigma^\alpha}J = \frac{\partial}{\partial\tilde{\sigma}^\alpha}\frac{\partial\tilde{\sigma}^\beta}{\partial\sigma^\gamma} = 0,$$

and where for color image $i = 1, 2, 3$.

3.2 Embedding Space Reparameterization

Next we see that a passive transformation of the embedding space, i.e. a reparameterization thereof, leaves the Polyakov action invariant.

Denote

$$\tilde{h}_{ij} \equiv h_{ij}(\tilde{X}^1, \cdots, \tilde{X}^n). \tag{13}$$

Since h_{ij} is a metric it transforms under a coordinate transformation as follows:

$$\tilde{h}_{ij} = \frac{\partial X^k}{\partial \tilde{X}^i}\frac{\partial X^l}{\partial \tilde{X}^j} h_{kl} \tag{14}$$

We also need the fact that

$$\partial_\mu \tilde{X}^i = \partial_\mu X^k \frac{\partial \tilde{X}^i}{\partial X^k}. \tag{15}$$

Then the Polyakov action is invariant

$$\begin{aligned} S[g_{\mu\nu}, \tilde{X}^i, \tilde{h}_{ij}] &= \int dag^{\mu\nu} \partial_\mu \tilde{X}^i \partial_\nu \tilde{X}^j \tilde{h}_{ij} \\ &= \int dag^{\mu\nu} \partial_\mu X^k \frac{\partial \tilde{X}^i}{\partial X^k} \partial_\mu X^l \frac{\partial \tilde{X}^j}{\partial X^l} \frac{\partial X^r}{\partial \tilde{X}^i} \frac{\partial X^s}{\partial \tilde{X}^j} h_{rs} \\ &= S[g_{\mu\nu}, X^i, h_{ij}] \end{aligned} \tag{16}$$

where $da = d\sigma^1 d\sigma^2 \sqrt{g}$ is the area element.

4 Affine Invariance: General consideration

A generic affine transformation in n-dimensional space is

$$T_{\text{affine}}[X^i] = \tilde{X}^i = A^i_j X^j + B^i$$

where A^i_j is a non-singular constant matrix and B^i is a constant shift vector [2]. Denote an infinitesimal transformation of the image along the diffusion (scale) parameter by $T_t[X^i]$ and the affine transformation by $T_{\text{affine}}[X^i]$. Affine invariance is a commutation relation between the transformations:

$$T_t[T_{\text{affine}}[X^i]] = T_{\text{affine}}[T_t[X^i]]. \tag{17}$$

It means that in the invariant flow the following is true

$$\partial_t \tilde{X}^i = \widetilde{\partial_t X^i}, \tag{18}$$

and it implies via the Beltrami framework that

$$\Delta_{\tilde{g}} A^i_j X^j = \Delta_{\tilde{g}} \tilde{X}^i = \widetilde{\Delta_g X^i} = A^i_j \Delta_g X^j. \tag{19}$$

It is clear now that the condition

$$(g_{\mu\nu}) = (\tilde{g}_{\mu\nu})$$

is sufficient to insure invariance with respect to affine transformations. Note that each one of the metric's elements is affine invariant, not only the determinant.

5 Codimension$=1$

5.1 The Curve Affine Flow

We construct an affine invariant metric. Let the coordinates of $\mathbb{R}^2$ be $X^i \;\; i = 1, 2$ and the curve is parameterized by σ. the line element is

$$ds^2 = g(\sigma) d\sigma^2. \tag{20}$$

Theorem
The following expression:

$$f = \epsilon_{ij} X^i_\sigma X^j_{\sigma\sigma} \tag{21}$$

is equi-affine invariant.
Proof:
Denote the transformed coordinates of $\mathbb{R}^2$ by tilde

$$\tilde{X}^i = A^i_j X^j, \tag{22}$$

where $\det A = 1$. Then

$$\begin{aligned} \tilde{f} &= \epsilon_{ij} \tilde{X}^i_\sigma \tilde{X}^j_{\sigma\sigma} = \epsilon_{ij} A^i_k X^k_\sigma A^j_l X^l_{\sigma\sigma} \\ &= \det A \epsilon_{kl} X^k_\sigma X^l_{\sigma\sigma} = f \end{aligned} \tag{23}$$

where we used Eq. (22) in the second equality, the identity Eq. (7) in the third and the fact that $\det A = 1$ for equi-affine transformation in the last equality. **Q.E.D**

Under a reparameterization of the curve $\sigma \to \hat{\sigma}(\sigma)$ it transforms as

$$\hat{f} = \epsilon_{ij} X^i_{\hat{\sigma}} X^j_{\hat{\sigma}\hat{\sigma}} = (\frac{\partial \sigma}{\partial \hat{\sigma}})^3 \epsilon_{ij} X^i_\sigma X^j_{\sigma\sigma} = (\frac{\partial \sigma}{\partial \hat{\sigma}})^3 f. \tag{24}$$

Define

$$g = f^{2/3} \tag{25}$$

then obviously g is invariant under equi-affine transformation and transforms as

$$\hat{g} = (\frac{\partial \sigma}{\partial \hat{\sigma}})^2 g, \tag{26}$$

under reparameterization of the curve. It follows that g is an equi-affine invariant metric for the curve. Consequently the Beltrami flow is equi-affine invariant

Theorem
The flow

$$\begin{aligned} X^i_t &= \Delta_g X^i = \frac{1}{\sqrt{g}} \partial_\sigma \sqrt{g} g^{-1} \partial_\sigma X^i \\ &= \frac{1}{\sqrt{g}} \partial_\sigma \frac{1}{\sqrt{g}} \partial_\sigma X^i = X^i_{ss} \end{aligned} \tag{27}$$

is equi-affine invariant:

$$\tilde{X}^i_t = A^i_j \Delta_g X^j \tag{28}$$

Proof:

$$\tilde{X}^i_t = \Delta_{\tilde{g}} \tilde{X}^j = \Delta_g A^i_j X^j = A^i_j \Delta_g X^j \qquad (29)$$

Q.E.D

Explicitly we get

$$\vec{X}_t = \frac{1}{g} X^i_{ss} - \frac{(\partial_s g)}{2g^2} X^i_s = a\vec{T} + b\vec{N}, \qquad (30)$$

where we use the reparameterization freedom to work with the Euclidean arclength. In this parameterization $\vec{X}_s = \vec{T}$ the tangent vector and $\vec{X}_{ss} = \kappa\vec{N}$ where N is the normal and κ is the curvature. Since $\vec{X}_{ss} = \kappa\vec{N}$ for the Euclidean arc length and since $g = \kappa^{2/3}$ we get $b = \kappa^{1/3}$. The $\vec{T}$ term affect the parameterization of the curve but not its shape and can be ignored.

5.2 The Surface Affine Flow

Following the general considerations, in Section 4, we construct an affine invariant metric for the flow of a surface in $\mathbb{R}^3$. The coordinates of the embedding space are $X^i \;\; i = 1, 2, 3$. The Riemannian surface is parameterized by the local coordinates σ^1, σ^2.

Theorem

The expression

$$f_{\mu\nu} = 2\epsilon_{ijk} X^i_{\sigma^1} X^j_{\sigma^2} X^k_{\mu\nu} = \epsilon^{\theta\phi}\epsilon_{ijk} X^i_\theta X^j_\phi X^k_{\mu\nu} \qquad (31)$$

is equi-affine invariant.

Proof:

The proof is similar to the one we gave for affine curve evolution:

$$\begin{aligned} \tilde{f}_{\mu\nu} &= \epsilon_{ijk} \tilde{X}^i_{\sigma^1} \tilde{X}^j_{\sigma^2} \tilde{X}^k_{\mu\nu} = \epsilon_{ijk} A^i_l X^l_{\sigma^1} A^j_m X^m_{\sigma^2} A^k_n X^n_{\mu\nu} \\ &= \det A \epsilon_{ijk} X^i_{\sigma^1} X^j_{\sigma^2} X^k_{\mu\nu} = f_{\mu\nu} \end{aligned} \qquad (32)$$

where we used Eq. (22) in the second equality, the identity Eq. (7) in the third and the fact that $\det A = 1$ for equi-affine transformation in the last equality. **Q.E.D**

A metric should also transform in a specific way under a change in the reparameterization $\sigma \to \hat{\sigma}(\sigma)$. Denote by f the determinant of $f_{\mu\nu}$ then f and $f_{\mu\nu}$ transform as follows

$$\begin{aligned} \hat{f}_{\mu\nu} &= \frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu}\frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu}\epsilon^{\theta\phi}\frac{\partial\sigma^a}{\partial\hat{\sigma}^\theta}\frac{\partial\sigma^b}{\partial\hat{\sigma}^\phi}\epsilon_{ijk} X^i_a X^j_b X^k_{\lambda\rho} \\ &= \frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu}\frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu} J\epsilon^{ab}\epsilon_{ijk} X^i_a X^j_b X^k_{\lambda\rho} = J\frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu}\frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu}\hat{f}_{\lambda\rho} \\ \hat{f} &= J^4 f \end{aligned} \qquad (33)$$

It is clear that $g_{\mu\nu} = f_{\mu\nu}/f^{1/4}$ satisfies both the equi-affine and the metric transformation rules. It follows that the Beltrami flow is equi-affine invariant.

5.3 Hypersurfaces in $\mathbb{R}^{n+1}$

We construct, along the same lines, an affine invariant metric for higher dimensional hypersurfaces i.e. manifolds with codimension1. The notations are similar to the two-dimensional case. Let $X^i \;\; i = 1, \ldots, n+1$ be the coordinates of $\mathbb{R}^{n+1}$, and $\sigma^1, \ldots, \sigma^n$ the local coordinates of the n-dimensional Riemannian manifold embedded in $\mathbb{R}^{n+1}$.

Theorem

The line element $g_{\mu\nu} = f_{\mu\nu}/f^{1/(n+2)}$, where

$$f_{\mu\nu} = \epsilon^{\rho_1\rho_2\ldots\rho_n}\epsilon_{i_1 i_2 \ldots i_n i_{n+1}} X^{i_1}_{\rho_1} X^{i_2}_{\rho_2} \cdots X^{i_n}_{\rho_n} X^{i_{n+1}}_{\mu\nu} \qquad (34)$$

is equi-affine invariant.

Proof:

$$\begin{aligned} & \tilde{f}_{\mu\nu} = \epsilon^{\rho_1\rho_2\ldots\rho_n}\epsilon_{i_1 i_2 \ldots i_n i_{n+1}} \tilde{X}^{i_1}_{\rho_1} \tilde{X}^{i_2}_{\rho_2} \cdots \tilde{X}^{i_{n+1}}_{\mu\nu} \\ = \;& \epsilon^{\rho_1\rho_2\ldots\rho_n}\epsilon_{i_1 i_2 \ldots i_{n+1}} A^{i_1}_{j_1} X^{j_1}_{\rho_1} A^{i_2}_{j_2} X^{j_2}_{\rho_2} \cdots A^{i_{n+1}}_{j_{n+1}} X^{j_{n+1}}_{\mu\nu} \\ = \;& \det A \epsilon^{\rho_1\rho_2\ldots\rho_n}\epsilon_{i_1 i_2 \ldots i_n i_{n+1}} X^{i_1}_{\rho_1} X^{i_2}_{\rho_2} \cdots X^{i_{n+1}}_{\mu\nu} \\ = \;& f_{\mu\nu} \end{aligned} \qquad (35)$$

where we used Eq. (22) in the second equality, the identity Eq. (7) in the third and the fact that $\det A = 1$ for equi-affine transformation in the last equality. **Q.E.D**

6 Codimension > 1

The results in the Section 5 are well known [12]. They were rephrased in a language that facilitates generalizations. Several examples for such possible generalizations follow.

6.1 curves

Let us analyze first the equi-affine flow of a curve embedded in $\mathbb{R}^3$. The Cartesian Coordinates are $X^i \;\; i = 1, 2, 3$. The curve is parameterized by σ. The Serret-Frenet structure equations are

$$\frac{\partial}{\partial s}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} \qquad (36)$$

where $\vec{T}$, $\vec{N}$ and $\vec{B}$ are the tangent, normal and binormal unit vectors respectively. They form a right hand frame at point σ on the curve. κ is the curvature of the curve and τ is its torsion.

Theorem

The flow

$$\vec{X}_t = (\frac{\kappa}{\tau})^{1/3}\vec{N} \qquad (37)$$

is equi-affine invariant.

Proof:

Clearly the following expression

$$g(\sigma) = (\epsilon_{ijk} X^i_\sigma X^j_{\sigma\sigma} X^k_{\sigma\sigma\sigma})^{1/3} \quad (38)$$

is an equi-affine invariant metric.

Since $\vec{X}_s = \vec{T}$ and $\vec{X}_{ss} = \vec{T}_s = \kappa\vec{N}$ it follows that

$$\vec{X}_{sss} = \kappa_s \vec{N} + \kappa \vec{N}_s = -\kappa^2 \vec{T} + \kappa_s \vec{N} + \kappa\tau \vec{B}.$$

Using this identity we find

$$g(s) = \left((\vec{X}_s \times \vec{X}_{ss}) \cdot \vec{X}_{sss} \right)^{1/3} = \left(\kappa^2 \tau\right)^{1/3} \quad (39)$$

where s is the Euclidean arclength.

The Beltrami flow, based on this metric, is obviously an equi-affine invariant flow. Its explicit form is

$$\vec{X}_t = \frac{1}{g}\vec{X}_{ss} - \frac{(\partial_s g)}{2g^2}\vec{X}_s = \frac{1}{(\kappa^2\tau)^{1/3}}\kappa\vec{N} - \frac{g_s}{2g^2}\vec{T}. \quad (40)$$

The assertion follows from the fact that the $\vec{T}$ term changes only the parameterization but not the shape of the curve. **Q.E.D**

The generalization to a n-dimensional curve is straightforward. The equi-affine invariant metric is

$$g(s) = \left(\epsilon_{i_1 i_2 \ldots i_n} \frac{\partial X^{i_1}}{\partial s} \frac{\partial^2 X^{i_2}}{\partial s^2} \cdots \frac{\partial^n X^{i_n}}{\partial s^n} \right)^{\frac{4}{n(n+1)}}. \quad (41)$$

6.2 Surfaces

For codimension greater then 1 we proceed by a case by case analysis. Consider first the case of two-dimensional surface embedded in $\mathbb{R}^4$. Define

$$f_{\mu\nu} = \epsilon^{ab}\epsilon^{\lambda\rho}\epsilon_{i_1 i_2 i_3 i_4} X^{i_1}_a X^{i_2}_b X^{i_3}_{\lambda\mu} X^{i_4}_{\rho\nu}. \quad (42)$$

Obviously $f_{\mu\nu}$ is equi-affine invariant. Under reparametrization $f_{\mu\nu}$ and its determinant f transform as

$$\begin{aligned} \hat{f}_{\mu\nu} &= J^2 \epsilon^{ab}\epsilon^{\lambda\rho}\epsilon_{i_1 i_2 i_3 i_4} X^{i_1}_a X^{i_2}_b X^{i_3}_{\lambda\mu} X^{i_4}_{\rho\nu} \\ \hat{f} &= J^6 f. \end{aligned} \quad (43)$$

It is easy to show that $g_{\mu\nu} = f_{\mu\nu}/f^{1/3}$ transforms properly under a change of local coordinates and is equi-affine invariant.

Next we consider the embedding of a two-dimensional surface in $\mathbb{R}^5$ which is of special interest since this is the representation of a colored image in the Beltrami framework. In this case we can construct the following equi-affine invariant expression

$$f_{\mu\nu} = \epsilon^{ab}\epsilon^{ce}\epsilon^{d\lambda}\epsilon^{f\rho}\epsilon_{i_1 i_2 i_3 i_4 i_5} X^{i_1}_a X^{i_2}_b X^{i_3}_{cd} X^{i_4}_{ef} X^{i_5}_{\lambda\rho\mu\nu} \quad (44)$$

Since

$$\hat{f}_{\mu\nu} = J^4 \frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\mu}\frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\nu} f_{\rho\lambda} \quad (45)$$

obviously, the determinant $f = \det(f_{\mu\nu})$ transforms as $\hat{f} = J^{10} f$ and

$$g_{\mu\nu} = f_{\mu\nu}/f^{2/5} \quad (46)$$

transforms as a metric tensor.

Note that the coordinates of the embedding space are i.e. (x, y, R, G, B). The flow that we present is invariant under shifts trivially, and also under the full five-dimensional group $SL(5, \mathbb{R})$, the group of 5×5 matrices with determinant $= 1$. This means that the flow is invariant under combined transformations of the spatial and the color spaces!

6.3 Volumetric data

Volumetric medical images and movies are examples of three-dimensional manifolds embedded in a higher dimensional spatial-feature space. Equi-affine invariant metric for a three-dimensional manifold embedded in $\mathbb{R}^5$ is defined in terms of

$$f_{\mu\nu} = \epsilon^{a_1 a_3 a_5}\epsilon^{a_2 a_4 a_6}\epsilon_{i_1 i_2 i_3 i_4 i_5} X^{i_1}_{a_1} X^{i_2}_{a_2} X^{i_3}_{a_3 a_4} X^{i_4}_{a_5 \mu} X^{i_5}_{a_6 \nu}.$$

The invariant metric is $g_{\mu\nu} = f_{\mu\nu}/f^{1/4}$. Color movie is an example for three-dimensional manifold embedded in $\mathbb{R}^6$. Once again the invariant metric is $g_{\mu\nu} = f_{\mu\nu}/f^{1/4}$, where

$$\begin{aligned} f_{\mu\nu} = \;& \epsilon^{a_1 a_2 a_3}\epsilon^{a_4 a_5 a_6}\epsilon_{i_1 i_2 i_3 i_4 i_5 i_6} \\ & X^{i_1}_{a_1} X^{i_2}_{a_2} X^{i_3}_{a_3} X^{i_4}_{a_4\lambda} X^{i_5}_{a_5\lambda} X^{i_6}_{a_5\mu\nu}. \end{aligned}$$

7 Concluding Remarks

We studied here the question of non-linear diffusion flows which are invariant under groups of transformation. The Beltrami viewpoint, which separates between the image manifold and the embedding space, makes it easier to notice the difference between passive and active transformations. We analyse the conditions on the metric in order to construct an invariant flow. We are able to generalize results from codimension 1 to higher codimension and in particular to construct an equi-affine invariant flow for color images. The flow is invariant under spatial and illumination transformations at the same time.

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