COFINAL TYPES OF ULTRAFILTERS OVER MEASURABLE CARDINALS

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Abstract. We develop the theory of cofinal types of ultrafilters over measurable cardinals and establish its connections to Galvin's property. We generalize fundamental results from the countable to the uncountable, but often in surprisingly strengthened forms, and present models with varying structures of the cofinal types of ultrafilters over measurable cardinals.

0. Introduction

This article develops the theory of cofinal types of ultrafilters over measurable cardinals. While cofinal, or Tukey, types of ultrafilters over ω have been studied extensively (see, for instance, [11, 13, 14, 15, 16, 17, 18, 49, 50, 51, 52, 56, 58, 59]), this is the first substantive investigation of cofinal types of ultrafilters over measurable cardinals.

Given two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, a function $f : Q \to P$ is *cofinal* if for every cofinal subset $X \subseteq Q$, its f-image $f''X$ is cofinal in P. A function $g : P \to Q$ is *unbounded* if for every unbounded subset $X \subseteq P$, $g''X$ is unbounded in Q. Schmidt showed in [60] that there is a cofinal map from Q to P if and only if there is an unbounded map from P to Q. We say that P is *Tukey reducible to* Q, and write $P \leq_T Q$, if there is a cofinal map $f : Q \to P$ or, equivalently, an unbounded map $g : P \to Q$. When both $P \leq_T Q$ and $Q \leq_T P$, then we say that they are *Tukey equivalent*, and write $P \equiv_T Q$. This forms an equivalence relation, and the equivalence classes are called *Tukey types*. We regard ultrafilters as directed partial orders under reverse inclusion. A cofinal subset of an ultrafilter U with respect to reversed inclusion is also called a *filter base*.

The Tukey order was introduced by Tukey in [64] to extend the notion of Moore-Smith convergence in topology to the more general setting of nets, or directed partial orders. Two directed partial orders are *cofinally similar* exactly when there is a third directed partial order into which they each embed as a cofinal subset. In particular, for ultrafilters Tukey equivalence is the same as cofinal similarity; hence Tukey types are also called *cofinal types*.

Tukey reduction has been seen to provide insightful classifications of partially ordered sets in contexts where isomorphism is too strong a notion to produce any meaningful results. This has been seen in work of Isbell [35], Todorcevic [62, 63],

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Fremlin [21], and Solecki–Todorcevic [61], to mention a few, as well as a large collection papers on ultrafilters over countable base sets beginning with Milovich's paper [50]. The majority of results on cofinal types of ultrafilters over ω fall into one of the following categories:

(1) Finding conditions under which Fubini products are cofinally equivalent to cartesian products [16];
(2) Canonizations of cofinal maps into continuous or at least finitary maps which imply cofinal types of certain ultrafilters have cardinality $\mathfrak{c}$ (rather than $2^{\mathfrak{c}}$) [16, 58, 11, 13, 15, 14];
(3) Other conditions guaranteeing an ultrafilter is not of the maximum cofinal type [50, 16, 58, 11, 13];
(4) Embeddings of certain partial orders [16, 58], including long lines [59, 49] and $\mathcal{P}(\omega)/\text{fin}$ [56], into the cofinal types of p-points in various models of ZFC;
(5) Finding conditions under which Tukey reduction implies Rudin-Keisler or even Rudin-Blass reduction [58, 14];
(6) Finding the exact structure of the cofinal types, including the precise structure of the Rudin-Keisler classes inside them, below Ramsey ultrafilters [58], below certain p-points satisfying weak partition relations [17, 18, 15], and below non-p-points forced by $\mathcal{P}(\omega^k)/\text{fin}^{\otimes k}$ [13].

We extend some results from each of the categories (1)–(4) to ultrafilters on measurable cardinals. In contrast to the situation on ω, we will pinpoint the exact structure of the cofinal types in certain models with measurable cardinals, establishing results on ω lying in category (6) but not exact analogues of them. Our results will be discussed in Subsection 0.2.

The present investigation began with a realization of the deep connection between the Galvin property (Definition 0.2) and the non-maximality of cofinal types of ultrafilters (Theorem 2.4). The Galvin property for ultrafilters on measurable cardinals has been studied extensively in recent years, and a major motivation for our work was to develop a finer understanding of conditions guaranteeing the Galvin property through the development of the Tukey theory of ultrafilters on measurable cardinals.

0.1. **The Galvin property.** In 1973, F. Galvin proved the following surprising result [20]:

Theorem 0.1 (Galvin's Theorem)**.** *Suppose that $\kappa^{<\kappa} = \kappa > \aleph_0$. Then for every normal filter[1] U over κ, the Galvin property holds: for every sequence of sets $\langle A_i \mid i < \kappa^+\rangle \in [U]^{\kappa^+}$, there is an $I \in [\kappa^+]^\kappa$ such that $\bigcap_{i\in I} A_i \in U$.*

For example, if $\kappa = \omega_1$ and GCH holds (so $\omega_1^{<\omega_1} = \omega_1$), then the club filter Cub_{ω_1} is normal and Galvin's Theorem asserts that for every collection of ω_2 many clubs there is a subcollection of size ω_1 whose intersection is still a club.

Although Abraham and Shelah [1] and later Benhamou, Garti, and Poveda [7] show that the assumption that $\kappa^{<\kappa} = \kappa$ cannot be dropped from Galvin's Theorem, the normality assumption and the property obtained in Galvin's theorem were recently improved by Benhamou and Gitik [9, 4], and by Garti [22, 23]. In this paper we will provide certain strengthenings of those results.

[1] U is normal if $\triangle_{\alpha<\kappa} F_\alpha := \{\beta < \kappa \mid \forall \alpha < \beta\, (\beta \in F_\alpha)\} \in U$ for all $\langle F_\alpha \mid \alpha < \kappa\rangle \subseteq U$.

A filter satisfying the property described in Galvin's theorem is called a *Galvin filter*. Galvin filters turned out to be useful for several applications described below, which brought the Galvin property back to the front of the stage. Let us use notation from [7] to allow some flexibility with certain parameters in Galvin's property:

Definition 0.2 (The Galvin property)**.** For an ultrafilter U over κ and two cardinals $\mu \leq \lambda$ we denote by $\mathrm{Gal}(U, \mu, \lambda)$ the assertion that

$$\forall \langle A_i \mid i < \lambda \rangle \in [U]^\lambda \ \exists I \in [\lambda]^\mu \ \cap_{i \in I} A_i \in U$$

A *Galvin ultrafilter* is an ultrafilter satisfying $\mathrm{Gal}(U, \kappa, 2^\kappa)$.

Before diving into deeper set-theoretic water, let us elaborate on some applications of Galvin's property.

The use of filters and ultrafilters to establish theorems both in finite and infinitary combinatorics has a long history. Paradigmatic examples of this are the *Pigeonhole Principle*, *Ramsey's Theorem*, *Arrow's Impossibility Theorem*, or the partition relation $\kappa \to (\kappa)^2_2$ for measurable cardinals, among many others. See [43] for additional interesting examples. It turns out that Galvin's property characterizes some *polarized partition relations*. Hajnal [30] proved that if κ is a measurable cardinal then $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_\theta$ for every $\theta < \kappa$.[2] A modern proof of Hajnal's result using $\mathrm{Gal}(U, \kappa, \kappa^+)$ can be found in [8, Proposition 3.3]. This latter proof actually yields the enhanced polarized relation saying that the second component of the monochromatic set is a member of U (i.e., $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{U}_\theta$). Moreover, that argument is more flexible than other stronger versions of Galvin's property, such as the ones considered in [7], entailing stronger forms of the polarized partition relation. But most interestingly, the above implication is reversible, namely, $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{U}_2$ implies $\mathrm{Gal}(U, \kappa, \kappa^+)$[3].

Other interesting connections between Galvin's property and infinite combinatorics arise in the context of *prediction principles*; especially in the study of the *diamond-like* ones (such as *Devlin-Shelah weak diamond* in [22, 2] and *the superclub principle* in [23], see also [24]) More recently, the Galvin property has been used by Benhamou, Garti and Poveda [6] to prove consistently new instances of the partition relation $\lambda \to (\lambda, \omega + 1)$ and the classical question of [20] about possible generalizations (both in ZFC and ZF+AD) of the *Erdös-Dushnik-Miller theorem*.

The range of applicability of Galvin's property has not been limited to the borders of infinite combinatorics. There is also a recently-opened avenue of research in *Prikry-type forcings* where Galvin's property is being used to classify the intermediate models of the Prikry and Magidor generic extensions [9], it is equivalent to the density of old sets in Prikry extensions [27, 7], and violating Galvin's property of U is necessary for the corresponding Prikry forcing to project onto Cohen forcing $\mathrm{Add}(\kappa, \kappa^+)$ [10]. As a final application, we mention that the Galvin property serves as a yardstick principle in canonical inner models to separate certain large cardinal notions [4].

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[2] This stands for the following statement: For every coloring $c\colon \kappa^+ \times \kappa \to \theta$ there is $A \in [\kappa^+]^\kappa$ and $B \in [\kappa]^\kappa$ such that $c \restriction (A \times B)$ is constant.

[3] Given a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$, the desired intersection of size κ is given by the homogeneous rectangle guaranteed by the assumption $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{U}_2$, for the coloring $c : \kappa^+ \times \kappa \to 2$ defined by $c(\alpha, \beta) = 1$ iff $\beta \in A_\alpha$.

0.2. **Main results.** The first significant observation of this paper is that the Galvin property is deeply connected to the Tukey order.

Theorem 2.4. *Let $\lambda \leq \kappa$ be cardinals, where λ is inaccessible, and let U be a λ-complete ultrafilter over κ. Then $(U, \supseteq) \equiv_T ([2^\kappa]^{<\lambda}, \subseteq)$ if and only if $\neg\,\mathrm{Gal}(U, \lambda, 2^\kappa)$. In particular, if κ is a measurable cardinal then the non principal κ-complete ultrafilters over κ which are Tukey-top[4] are exactly those for which $\neg\,\mathrm{Gal}(U, \kappa, 2^\kappa)$ holds.*

Isbell [35] and Juhász [39] independently constructed, in ZFC, ultrafilters which are Tukey-top among directed partial orders of size continuum. It turns out that this construction generalizes to uncountable cardinals if one assumes the filter extension property. Moreover, it is possible to combine this construction with the methods from [4] using $\diamondsuit(\kappa)$ to obtain a q-point which is a Tukey-top. This asserts the conjecture from [4]:

Corollary 2.25. *If κ is κ-compact, then there is a Tukey-top ultrafilter among the κ-complete ultrafilters which is a q-point.*

Section 3 maps out the connections between the spectra of characters for filter bases, the Tukey orders with respect to both $\supseteq$ and $\supseteq^*$, and the Galvin property with respect to both of these orders. Also, the analog $\mathrm{Gal}^*(U, \kappa, \lambda)$ of $\mathrm{Gal}(U, \kappa, \lambda)$ is introduced (see Definition 3.14), which guarantees that from any given λ-many sets in U there are κ-many of them which have a pseudo intersection in U. We then use it to extend and improve some results of Milovich in [50], including:

Theorem 3.17. *Let κ be either ω or a measurable cardinal. For any κ-complete ultrafilter U over κ, $(U, \supseteq^*) \equiv_T ([2^\kappa]^{<\kappa}, \subseteq)$ if and only if $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$.*

Theorem 3.18. *Suppose that κ is κ-compact. Then there is a q-point ultrafilter U such that $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$.*

Given ultrafilters $U \geq_T V$ on κ, a priori there are 2^{2^κ} many cofinal maps from U to V. A valuable mechanism which unlocked much of the Tukey theory for ultrafilters over ω was the discovery by Dobrinen and Todorcevic [16] that any cofinal map from a p-point U into an ultrafilter V is actually continuous upon restriction to a suitably chosen member of U. In Theorem 4.1, we derive the analogue of this theorem for p-points over measurable cardinals, and derive stronger forms of continuity for normal ultrafilters (Corollary 4.2).

Furthermore, Theorem 4.4 shows that cartesian products of finitely many p-points over a measurable cardinal κ also have continuous cofinal maps. Whereas such a theorem for ultrafilters over ω is only useful for rapid p-points (see Section 4 of [16]), surprisingly Theorem 4.4 becomes a central tool for investigating cofinal types of Fubini products of p-points, due to Theorem 5.6 below. The results in Section 4 are used throughout the rest of the paper to bound the number of Tukey-predecessors of a p-point, and to obtain models where there are many Tukey-incomparable normal ultrafilters.

In Section 5 we turn our attention to the Tukey-class of sums, products and powers of ultrafilters. A number of surprising differences between the Tukey theory of ultrafilters over measurable cardinals versus ω have emerged. On ω, Dobrinen

[4]Namely, belongs to the maximum Tukey class among κ-directed closed posets of cardinality 2^κ.

and Todorcevic [16] proved that if U is a rapid p-point, then for any ultrafilter V, $V \cdot U \equiv_T V \times U$ and in particular any Fubini iterate of U is Tukey equivalent to U. Moreover, Dobrinen and Todorcevic presented a non-rapid p-point U such that $U <_T U^2$. On measurable cardinals we have found the situation to be quite different:

Theorem 5.6. *Given κ measurable, for every two κ-complete ultrafilters U, V over κ, $U \cdot V \equiv_T U \times V$.*

Theorem 5.7. *Given κ measurable, for any κ-complete ultrafilter U over κ, $U^n \equiv_T U$.*

In particular, we can conclude that the class of κ-complete Galvin ultrafilters is closed under finite Fubini powers.

In an attempt to approximate the class of Galvin ultrafilters (i.e. non-Tukey-top), we define in Section 6 the notions of *basic* and *basically generated* ultrafilter. These notions generalize the notions for ultrafilters on ω from [16] with the slight modification of uniformity. We prove that basic is equivalent to p-point (Propositions 6.3 and 6.4) and give a shorter proof of a result in [9] that p-point implies Galvin. More generally, we prove that p-point implies basically generated which in turn implies Galvin (Theorem 6.8). Finally, we prove that the class of basically generated ultrafilters is closed under sums (Theorem 6.10). This in particular implies that sums (and limits) of p-points are Galvin, recovering the main result from [4].

In Section 7, we present several classical models and study the Tukey-order in those models. The first model is $L[U]$, in which Kunen [44] proved that every σ-complete ultrafilter W over κ is Rudin-Keisler equivalent to a finite Fubini power of U. The results of Section 6 imply that:

Theorem 7.2. *In $L[U]$, the σ-complete ultrafilters over κ form a single Tukey class which is not the Tukey-top one, and is the union of ω-many Rudin Keisler equivalence classes.*

We then prove that the Mitchell order is orthogonal to the reverse Tukey order (Theorem 7.5) and use it to construct models with chains of Tukey types of various lengths. Finally, we consider the situation in the Kunen-Paris [47] model where there are $2^{(2^\kappa)}$-many distinct normal ultrafilters and use our results from Section 4 to prove:

Theorem 7.10. *Assume GCH and that κ is a measurable cardinal. Then there is a generic extension where GCH holds and there is a Tukey-chain of order type $\omega + 1$.*

One problem we did not settle in this paper, is the problem of minimality in the Tukey order. In [58], Todorcevic proved that Ramsey ultrafilters are minimal in the Tukey order. We conjecture that a similar situation holds for uncountable cardinals and we call that conjecture *the minimality conjecture*. Given the minimality conjecture, we use Goldberg's *Ultrapower Axiom* (UA) [29] to fully characterize the Tukey order in models of UA below a μ-measurable cardinal. In particular, this characterization will hold true for the Mitchell models $L[\vec{U}]$:

Theorem 7.19. *Assume UA and the General Minimality Conjecture. Suppose that κ is a measurable cardinal with $o(\kappa) < 2^{(2^\kappa)}$. Then the Tukey types of ultrafilters on κ is isomorphic to $([o(\kappa)]^{<\omega}, \subseteq)$.*

1. Definitions and preliminaries

We begin by reviewing some standard notation used throughout. Let κ be a cardinal and X be any set. Then $[X]^\kappa = \{Y \in P(X) \mid |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \in P(X) \mid |Y| < \kappa\}$. Let κ be regular. For two subsets of κ, we write $X \subseteq^* Y$ to denote that $X \setminus Y$ is bounded in κ. Similarly, for $f, g : \kappa \to \kappa$ we denote $f \leq^* g$ if there is $\alpha < \kappa$ such that for every $\alpha \leq \beta < \kappa$, $f(\beta) \leq g(\beta)$. We say that $C \subseteq \kappa$ is *closed and unbounded at κ (club)*, if it is a closed subset with respect to the order topology on κ and unbounded in the ordinals below κ. The *club filter* is the filter:

$$\mathrm{Cub}_\kappa := \{X \subseteq \kappa \mid \exists C \subseteq X,\ C \text{ is a club at } \kappa\}.$$

Given two partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$, their *(cartesian) product* is denoted by $(P, \leq_P) \times (Q, \leq_Q) = (P \times Q, \leq_{P\times Q})$ where $(p, q) \leq_{P\times Q} (p', q')$ iff both $p \leq_P p'$ and $q \leq_Q q'$.

1.1. **Background and basic results for the Tukey order on ultrafilters.** Let U and V denote ultrafilters on an infinite cardinal κ. We shall assume throughout that all ultrafilters are uniform. Unless explicitly stated otherwise, each ultrafilter is assumed to be partially ordered by reverse inclusion. A map $f : U \to V$ is *monotone* if $f(A) \supseteq f(B)$ whenever $A \supseteq B$. The following simple but useful fact is a straightforward generalization of Fact 6 in [16].

Fact 1.1. *Let U and V be ultrafilters on any infinite cardinal κ. If $V \leq_T U$, then this is witnessed by a monotone cofinal map.*

The following connection between the Tukey and Rudin-Keisler orders is well-known: Given ultrafilters U, V over base sets X, Y, respectively, we say that U is *Rudin-Keisler below V*, and write $U \leq_{RK} V$, if there is a function $f : Y \to X$ such that $f_*(V) = U$, where

$$f_*(V) := \{B \subseteq X \mid f^{-1}[B] \in V\}$$

If $U \leq_{RK} V$ and $V \leq_{RK} U$ then we denote this by $V \equiv_{RK} U$, and in this case there is a bijection $f : X \to Y$ such that $f_*(U) = V$. Note that if $f_*(U) = V$, then the map $g : U \to V$ defined by $g(X) = f''X$ is a cofinal map. Hence:

Proposition 1.2. *If $U \leq_{RK} V$ then $U \leq_T V$.*

It follows that cofinal types are a coarsening of the Rudin-Keisler equivalence classes of ultrafilters.

Definition 1.3. For any ultrafilter U over X and a sequence $\langle V_x \mid x \in X\rangle$ of ultrafilters over Y, define the *U-sum of the ultrafilters V_x*, to be the filter over $X \times Y$ defined as follows:

$$\sum_U V_x := \left\{ A \subseteq X \times Y :\ \left\{x \in X \mid \{y \in Y \mid \langle x, y\rangle \in B\} \in V_x\right\} \in U \right\}$$

If there is an ultrafilter V such that for every $x \in X$, $V_x = V$, we call the ultrafilter $U \cdot V := \sum_U V$ the *Fubini product of U and V*, or simply *product*.

In this paper we will also consider the Cartesian product $U \times V$, and we emphasize here the distinction between $U \cdot V$ and $U \times V$.

Proposition 1.4. *For any ultrafilters U, V, $U \times V$ is the minimal Tukey-type extension of both U and V.*

Proof. First, note that the projection maps witness that $U \times V \geq_T U, V$. These are projections since any cofinal set in $U \times V$ must be cofinal in both coordinates. Suppose that $W \geq_T U, V$ with witnessing cofinal maps $f : W \to U$ and $g : W \to V$. Then $h : W \to U \times V$ define by $h(X) = \langle f(X), g(X) \rangle$ is cofinal. □

Corollary 1.5. *For every two ultrafilters U, V, $U \times U \equiv_T U$ and $U \cdot V \geq_T U \times V$.*

Proof. It is well known that the projection to the coordinates is a Rudin-Keisler projection of $U \cdot V$ to U and V, hence $U \cdot V \geq_T U, V$ and by minimality $U \cdot V \geq_T U \times V$. □

1.2. **Some definitions of ultrafilters over a measurable cardinal.**

Definition 1.6. Let U be an ultrafilter over a regular cardinal κ.

(1) U is *uniform* if for every $X \in U$, $|X| = \kappa$.
(2) U is λ-*complete* if U is closed under intersections of less than λ many of its members.
(3) U is *normal* if U is closed under diagonal intersection, i.e. if $\langle A_i \mid i < \kappa \rangle \subseteq U$, then $\Delta_{i<\kappa} A_i := \{\nu < \kappa \mid \forall i < \nu \, (\nu \in A_i)\} \in U$.
(4) U is *Ramsey* if for any function $f : [\kappa]^2 \to 2$ there is an $X \in U$ such that $f \restriction [X]^2$ is constant.
(5) U is *selective* if for every function $f : \kappa \to \kappa$, there is an $X \in U$ such that $f \restriction X$ is either constant or one-to-one.
(6) (Kanamori [41]) U is *rapid* if for each normal function $f : \kappa \to \kappa$ (i.e. increasing and continuous), there exists an $X \in U$ such that $\mathrm{otp}(X \cap f(\alpha)) \leq \alpha$ for each $\alpha < \kappa$.
(7) U is an r-*point* if whenever f is continuous and almost injective (i.e. bounded pre-images), there is an $X \in U$ such that $|f^{-1}(\{\alpha\})| \leq \alpha$ for every $\alpha < \kappa$.
(8) U is a p-*point* if whenever $f : \kappa \to \kappa$ is not constant on a set in U, there is an $X \in U$ such that for every $\gamma < \kappa$, $|f^{-1}[\gamma] \cap X| < \kappa$.
(9) U is a q-*point* if every function $f : \kappa \to \kappa$ which is almost 1-1 (mod U) is injective (mod U).

Proposition 1.7. (1) *Normal $\Rightarrow$ Ramsey.*
(2) *Ramsey $\Leftrightarrow$ Selective.*
(3) *U is p-point iff every sequence $\langle X_i \mid i < \kappa \rangle \in [U]^\kappa$ admits a pseudo-intersection, namely, there is $X \in U$ such that for every $i < \kappa$, $X \subseteq^* X_i$.*
(4) *U is a q-point iff U is Rudin-Keisler equivalent to an ultrafilter W such that $Cub_\kappa \subseteq W$.*

For more information about these definitions and implications we refer the reader to [40] or [3].

Proposition 1.8. *Selective $\Leftrightarrow$ p-point and q-point $\Leftrightarrow$ RK-minimal among uniform ultrafilters $\Leftrightarrow$ RK-equivalent to a normal.*

Proof. See ([40] or [3] or [29]). □

Theorem 1.9. (Kanamori) [41, Theorem 1.3]

(1) *If $\kappa = \omega$ then rapid $\Leftrightarrow$ r-point.*
(2) *if $\kappa > \omega$ then rapid $\Leftrightarrow$ q-point.*

The original definition of rapid ultrafilter on ω, obviously, does not mention normality of the function. However, for measurable cardinals the normality assumption cannot be dropped:

Proposition 1.10. *Suppose that U is a normal ultrafilter over a measurable cardinal κ. Then for any increasing function $f : \kappa \to \kappa$ such that $f(\alpha) > \alpha$ for every α (in particular not continuous) and for any $X \in U$, we have that*

$$\{\alpha < \kappa \mid \mathrm{otp}(X \cap f(\alpha)) \geq \alpha + 1\} \in U$$

Proof. Let f, X be as in the proposition. Normality is equivalent to $[\mathrm{id}]_U = \kappa$. Also, since U is uniform, every $X \in U$ satisfies that $\mathrm{otp}(X) = \kappa$. Let $j_U : V \to M_U$ be the ultrapower embedding by the normal measure U. Then we have the following equality (just for normal):

$$Z \in U \Leftrightarrow \kappa \in j_U(Z)$$

In particular, since $X \in U$, we know that $\kappa \in j_U(X)$. Notice that since κ is the critical point of the embedding j_U (namely $j_U \restriction \kappa = \mathrm{id}$ and $j_U(\kappa) > \kappa$) and since $X \subseteq \kappa$ we have that $j_U(X) \cap \kappa = X$.

By elementarity, and by the assumption on f, $j_U(f) : j_U(\kappa) \to j_U(\kappa)$ and $j_U(\kappa) > \kappa$, so we my apply $j_U(f)$ to κ and by the assumption $j_U(f)(\kappa) > \kappa$. In particular,

$$j_U(X) \cap j_U(f)(\kappa) \supseteq j_U(X) \cap \kappa + 1 = X \cup \{\kappa\}$$

In particular,

$$(*) \quad \mathrm{otp}(j_U(X) \cap j_U(f)(\kappa)) \geq \kappa + 1 > \kappa$$

Consider the set $A = \{\alpha < \kappa \mid \mathrm{otp}(X \cap f(\alpha)) > \alpha\}$, we want to prove that $A \in U$, or equivalently, that $\kappa \in j_U(A)$. By elementarity, the set $j_U(A) = \{\alpha < j_U(\kappa) \mid \mathrm{otp}(j_U(X) \cap j_U(f)(\alpha)) > \alpha\}$ but this is exactly what we checked in $(*)$. □

The above also proves that p-points are not rapid in this sense.

2. The maximum Tukey class and the Galvin property

In this section, we lay forth the connections between the Galvin property and the Tukey order. Recall that a partial ordering $(P, \leq_P)$ is λ-*directed closed* if every subset of P of size less than λ has an upper bound in P.

Theorem 2.1. *Let λ and μ be any cardinals. Suppose that $(P, \leq_P)$ is λ-directed closed and that $|P| \leq \mu$. Then $(P, \leq_P) \leq_T ([\mu]^{<\lambda}, \subseteq)$. In other wards, $([\mu]^{<\lambda}, \subseteq)$ is maximal in the Tukey order among λ-directed closed posets of cardinality at most μ.*

Proof. First note that if $\mu < \lambda$ and P is a λ-directed poset of cardinality at most μ, then P has a greatest element and the theorem is trivially true. Hence, let us assume that $\lambda \leq \mu$. Fix any one-to-one function $g : P \to \mu$, and let us prove that the function $f(p) = \{g(p)\}$ is an unbounded function from P to $[\mu]^{<\lambda}$. Suppose that X is unbounded. Then $|X| \geq \lambda$ (by λ-directedness) and therefore $g''X$ has size at least λ. This implies that $\cup f''X = g''X$ has size λ. If follows that $f''X$ is unbounded in $[\mu]^{<\lambda}$. □

Any order $(P, \leq_P)$ such that $(P, \leq_P) \equiv_T ([\mu]^{<\lambda}, \subseteq)$ is called *Tukey-top*. It is important that we only restrict ourselves to λ-directed closed posets, as the following proposition suggests:

Proposition 2.2. *If $(P, \leq_P)$ is not λ directed-closed and $(Q, \leq_Q)$ is λ-directed closed then $\neg((P, \leq P) \leq_T (Q, \leq_Q))$*

Proof. Suppose otherwise that there is an unbounded function $f : P \to Q$. Since P is not λ-directed closed, there is a family $B := \{p_i \mid i < \theta\}$ for $\theta < \lambda$ which is unbounded. Thus $f''B \subseteq Q$ is unbounded. Since Q is λ-directed, and $f''B$ is a family of size less than λ, there is $q \in Q$ which bounds $f''B$, contradiction. □

So, for example, ultrafilters which are not σ-complete cannot be Tukey below ultrafilters which are κ-complete. The other direction is simply not true in general since for example $\omega \times \omega_1 \geq_T \omega_1$. More generally, we have that:

Fact 2.3. *Suppose $(P, \leq_P)$ is not λ-directed closed, then for every $(Q, \leq_Q)$, $(P, \leq_P) \times (Q, \leq_Q)$ is not λ-directed closed.*

The Galvin property (Definition 0.2) turns out to provide an equivalent condition for an ultrafilter to be Tukey-top.

Theorem 2.4. *Let $\lambda \leq \kappa$ be cardinals, and let U be a λ-complete filter over κ. Then $(U, \supseteq) \geq_T ([\kappa]^{<\lambda}, \subseteq)$ if and only if $\neg \operatorname{Gal}(U, \lambda, \kappa)$. In particular, if κ is a measurable cardinal then the κ-complete ultrafilters over κ which are Tukey-top are exactly those for which $\neg \operatorname{Gal}(U, \kappa, 2^\kappa)$ holds.*

Proof. Since $(U, \supseteq)$ is λ-directed whenever U is λ-complete, and since $|U| = 2^\kappa$, Theorem 2.1 applies (λ being itself and μ being 2^κ) and yields $(U, \supseteq) \leq_T ([2^\kappa]^{<\lambda}, \subseteq)$. So the "In particular" part of the theorem follows from the first part.

Suppose that $\langle A_i \mid i < \kappa \rangle$ witnesses that $\neg \operatorname{Gal}(U, \lambda, \kappa)$. To show that $(U, \supseteq) \geq_T ([\kappa]^{<\lambda}, \subseteq)$, define $f : [\kappa]^{<\lambda} \to U$ by $f(I) = \bigcap_{i \in I} A_i$, and let us prove that f is an unbounded map: Suppose that $X \subseteq [\kappa]^{<\lambda}$ is unbounded. Then $|\cup X| \geq \lambda$. Now

$$\bigcap f''X = \bigcap \{\cap_{i \in I} A_i \mid I \in X\} = \bigcap_{i \in \cup X} A_i.$$

By the assumption that the sequence witnesses the failure of Galvin's property, it follows that $\bigcap f''X \notin U$ and therefore $f''X$ is unbounded in U.

For the other direction, suppose $g : [\kappa]^{<\lambda} \to U$ is unbounded, and let us show that $\neg \operatorname{Gal}(U, \lambda, \kappa)$. The witnessing sequence will be $\langle X_\alpha \mid \alpha < \kappa \rangle$, where $X_\alpha = g(\{\alpha\})$. Let us prove that X_α is as wanted. Fix any $I \in [\kappa]^\lambda$. Then the set $\hat{I} = \{\{i\} \mid i \in I\} \subseteq [\kappa]^{<\lambda}$ is unbounded. By our assumption, $g''\hat{I} = \{X_i \mid i \in I\}$ is also unbounded, it follows that $\bigcap_{i \in I} X_i \notin U$. □

Galvin ultrafilters have been studied extensively in recent years. The most recent sufficient condition for an ultrafilter to be Galvin is the following [4]:

Corollary 2.5. *Let κ be a measurable cardinal and suppose that W is Rudin-Keisler equivalent to an ultrafilter of the form:*

$$\sum_{U} \sum_{U_{\langle \alpha_1 \rangle}} \cdots \sum_{\langle \alpha_1, \ldots, \alpha_{n-1} \rangle} U_{\langle \alpha_1, \ldots, \alpha_n \rangle}$$

where each of the ultrafilters $U_{\langle \alpha_1, .., \alpha_k \rangle}$ is a p-point ultrafilter. Then $\operatorname{Gal}(W, \kappa, 2^\kappa)$, and in particular, W is not Tukey-top.

Another type of ultrafilters which have the Galvin property are fine κ-complete ultrafilters over $[\lambda]^{<\kappa}$:

Definition 2.6. An ultrafilter U over $[\lambda]^{<\kappa}$ is called *fine* if for every $\alpha < \lambda$, $A_\alpha := \{X \in [\lambda]^{<\kappa} \mid \alpha \in X\} \in U$.

The assumption that there is a fine κ complete ultrafilter over λ is known to be much stronger than measurability.

Definition 2.7. A cardinal κ is called λ-*strongly-compact* if $[\lambda]^{<\kappa}$ carries a fine κ-complete ultrafilter.

It is immediate that if U is a fine ultrafilter over $[\lambda]^{<\kappa}$, then the sequence $\langle A_\alpha \mid \alpha < \lambda\rangle$ witnesses that $\neg\,\mathrm{Gal}(U, \kappa, \lambda)$. In particular, if $\lambda = 2^\kappa$, and U is a fine κ-complete ultrafilter over $[2^\kappa]^{<\kappa}$, then by Theorem 2.4, $(U, \supseteq) \equiv_T ([2^{(2^\kappa)}]^{<\kappa}, \subseteq)$. It follows that $(U, \supseteq)$ is maximal among κ-directed posets of cardinality at most $2^{(2^\kappa)}$ and in particular among κ-complete ultrafilters over κ.

Corollary 2.8. *Suppose that κ is 2^κ-strongly compact and U is a fine κ-complete ultrafilter over $[2^\kappa]^{<\kappa}$ witnessing that. Then for every κ complete ultrafilter W over κ, $(W, \supseteq) \leq_T (U, \supseteq)$*

2.1. **Generalization of Isbell's result.** It turns out that on ω, there is a ZFC construction which outright gives a Tukey-top ultrafilter:

Theorem 2.9 (Isbell [35])**.** *There is a ultrafilter U on ω such that $U \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.*

The long-standing open question in this direction is the following:

Question 2.10 (Isbell, 1965)**.** Is it consistent that for every ultrafilter U on ω, $U \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$? equivalently, that $\neg\,\mathrm{Gal}(U, \omega, \mathfrak{c})$?

On measurable cardinals, there is always a normal ultrafilter, and therefore the analog of this question to yield an ultrafilter U such that $\mathrm{Gal}(U, \kappa, 2^\kappa)$ over a measurable cardinal κ is not very interesting. However, the analog of Isbell's construction does not generalize automatically for any measurable cardinal as there are models (e.g. $L[U]$–see Theorem 7.1) where every ultrafilter satisfies $\mathrm{Gal}(U, \kappa, 2^\kappa)$.

Before entering the realm of large cardinals, let us start this subsection with a generalization of Isbell's construction to any two cardinals $\mu \leq \kappa$. As in the construction of [4] and [5], the construction uses the notion of a κ-independent family.

Definition 2.11. A κ-independent family is a family $\langle A_i \mid i < \lambda\rangle$ of subsets of κ such that for every $I, J \in [\lambda]^{<\kappa}$ such that $I \cap J = \emptyset$, we have $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j^c) \neq \emptyset$.

Hausdorff [31] proved that any regular cardinal κ satisfying $\kappa^{<\kappa} = \kappa$ admits a κ-independent family of maximal length (see also [45, Exercise (A6), Chapter VIII]). Even without the above cardinal arithmetic assumption, one can force such a family by adding mutually generic Cohen sets (see for example [4, Proposition 4.9]).

Proposition 2.12. *Fix a κ-independent family $\vec{A} = \langle A_i \mid i < \lambda\rangle$ and let $\omega \leq \mu \leq \kappa$ be any cardinal. Then the set $\{(\bigcup_{i\in I} A_i) \setminus (\bigcup_{j\in J} A_j) \mid |I| = \mu$ and $|J| < \mu\}$ has the $< \mu$-intersection property. In particular, this set generates a μ-complete filter $\mathcal{F}^\mu_{\vec{A}}$ over κ.*

Proof. Let $\langle I_\alpha, J_\alpha \mid \alpha < \rho\rangle$ for $\rho < \mu$ be such that $|I_\alpha| = \mu > |J_\alpha|$. We want to prove that $\bigcap_{\alpha<\rho}\Big((\bigcup_{i\in I_\alpha} A_i) \setminus (\bigcup_{j\in J_\alpha} A_j)\Big) \neq \emptyset$. This follows by the κ-independence of the family. □

Theorem 2.13. *If $\mathcal{F}^\mu_{\vec{A}} \subseteq U$ is an ultrafilter, then the family $\langle(\kappa \setminus A_i) \mid i < \lambda\rangle$ witnesses the fact that $\neg\,\mathrm{Gal}(U, \mu, \lambda)$.*

Proof. Clearly each $\kappa \setminus A_i$ is in $\mathcal{F}^\mu_{\vec{A}}$. To see $\neg\,\mathrm{Gal}(U, \mu, \lambda)$, let $I \in [\lambda]^\mu$, then

$$\bigcap_{i\in I}(\kappa \setminus A_i) = \kappa \setminus (\bigcup_{i\in I} A_i)$$

Since $|I| = \mu$, $\bigcup_{i\in I} A_i \in \mathcal{F}^\mu_{\vec{A}}$, and thus, also in U. It follows that $\bigcap_{i\in I}(\kappa \setminus A_i) \notin U$. □

Definition 2.14. A cardinal μ is said to have the *κ-filter extension property* if every μ-complete filter over κ can be extended to a μ-complete ultrafilter.

The axiom of choice implies that ω has the κ-filter extension property for every cardinal κ. A cardinal κ which has the κ-filter extension property is called *κ-compact*. The assumption that μ has the κ-filter extension property is a consequence of μ being a κ^+-Π^1_1-subcompact cardinal which in turn is below a supercompact cardinal [32]. For more information about the filter extension property see [33] or [32].

Corollary 2.15. *Suppose that $\kappa^{<\kappa} = \kappa$. If μ has the κ-filter extension property, then there is a μ-complete ultrafilter U over κ such that $\neg\,\mathrm{Gal}(U, \mu, 2^\kappa)$. In particular, there is a Tukey-top ultrafilter among the μ-complete ultrafilters.*

Proof. Since $\kappa = \kappa^{<\kappa}$, there is a κ-independent family $\vec{A}$ of size 2^κ, and the μ-complete filter $\mathcal{F}^\mu_{\vec{A}}$ of Proposition 2.12 exists. Applying the κ-extension property to $\mathcal{F}^\mu_{\vec{A}}$ guarantees the existence of a μ-complete ultrafilter U which extends $\mathcal{F}^\mu_{\vec{A}}$. By Theorem 2.13 we have that $\neg\,\mathrm{Gal}(U, \mu, 2^\kappa)$. □

In [4], the first author proved that if κ is 2^κ-supercompact, then there is a κ-complete ultrafilter $\mathrm{Cub}_\kappa \subseteq U$ which is non-Galvin (namely $\neg\,\mathrm{Gal}(U, \kappa, \kappa^+)$). This raises the question [4, Question 5.3] whether there is a weaker large cardinal notion which guarantees the existence of such a non-Galvin ultrafilter. Moreover, in that paper, it was conjectured that κ-compact cardinals suffice. The following corollary asserts that conjecture:

Corollary 2.16. *If κ is κ-compact, then there is a κ-complete ultrafilter U such that $\neg\,\mathrm{Gal}(U, \kappa, 2^\kappa)$.*

Corollary 2.17. *For any infinite cardinal κ, there is always an ultrafilter U over κ such that $\neg\,\mathrm{Gal}(U, \omega, 2^\kappa)$.*

Note that the ultrafilter U above does not necessarily extend the club filter (i.e. q-point). Since the club filter is only a filter on cardinals of uncountable cofinalities, henceforth we will assume that $\mathrm{cf}(\kappa) > \omega$. Let us show that a similar method as in [4], gives such an ultrafilter. For this we need the definition of a normal independent family:

Definition 2.18. A sequence $\langle A_i \mid i < \lambda \rangle$ is called a *normal* κ-independent family, if it is κ-independent and for any two disjoint subfamilies $\langle A_{\alpha_i} \mid i < \kappa \rangle, \langle A_{\beta_i} \mid i < \kappa \rangle \subseteq \langle A_i \mid i < \lambda \rangle$, the diagonal intersection $\Delta_{i<\kappa}(A_{\alpha_i} \setminus A_{\beta_i})$ is a stationary subset of κ.

Distinguishing from κ-independent families, in order to guarantee the existence of a normal κ-independent family, more is needed. Hayut [33] used the following classical prediction principle:

Definition 2.19. The *diamond at* κ principle, denoted by $\diamondsuit(\kappa)$, asserts the existence of a sequence $\langle A_i \mid i < \kappa \rangle$ such that for every set $A \subseteq \kappa$, $\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary at κ.

It is well-known that in L, $\diamondsuit(\kappa)$ holds for every regular cardinal. Moreover, Kunen and Jensen observed that if κ is measurable (or even subtle) then $\diamondsuit(\kappa)$ holds. The following proposition is due to Hayut[5]. For the proof see [4, Proposition 4.2].

Proposition 2.20. *If $\diamondsuit(\kappa)$ holds then there is a normal κ-independent family of length 2^κ.*

Next we claim that a suitable modification of the Isbell construction above will give rise to non-Galvin q-point ultrafilters.

Proposition 2.21. *Suppose that $\vec{A} := \langle A_i \mid i < \lambda \rangle$ is a normal κ-independent family, then the intersection of less than μ-many sets in $G_{\vec{A}} := \{(\bigcup_{i\in I} A_i) \setminus (\bigcup_{j\in J} A_j) \mid |I| = \mu, |J| < \mu\}$ is stationary.*

Proof. See [4, Corollary 4.4]. □

Proposition 2.22. *Let $\vec{A} := \langle A_i \mid i < \lambda \rangle$ be a normal κ-independent family. Define*

$$\mathcal{F}^{\nu,*}_{\vec{A}} = \{X \subseteq \kappa \mid \exists C \in \mathrm{Cub}_\kappa \ \exists B \in [G_{\vec{A}}]^{<\mu} \ C \cap (\bigcap B) \subseteq X\}$$

Then $\mathcal{F}^{\nu,*}_{\vec{A}}$ is a $\min(\mathrm{cf}(\kappa), \mu)$-complete filter which extends the club filter and if $\mathcal{F}^{\nu,*}_{\vec{A}} \subseteq U$ is an ultrafilter, then $\langle (\kappa \setminus A_i) \mid i < \lambda \rangle$ witnesses that $\neg \mathrm{Gal}(U, \mu, \lambda)$.

Proof. The fact that this filter is $\min(\mathrm{cf}(\kappa), \mu))$-complete, follows by the known fact that the intersection of less than $\mathrm{cf}(\kappa)$-many clubs is a club and by definition of the previous proposition. The proof that $\neg \mathrm{Gal}(U, \mu, \lambda)$ holds is identical to the proof of Theorem 2.13. □

Corollary 2.23. *If μ has the κ-extension property, $\mathrm{cf}(\kappa) \geq \mu$ and $\diamondsuit(\kappa)$ holds, then there is a μ-complete ultrafilter U which extends the club filter and $\neg \mathrm{Gal}(U, \mu, 2^\kappa)$.*

Corollary 2.24. *If $V = L$, then for every regular cardinal κ, there is an ultrafilter (which is of course not even σ-complete) such that $\neg \mathrm{Gal}(U, \omega, 2^\kappa)$ and $\mathrm{Cub}_\kappa \subseteq U$.*

The following corollary is an answer to an analogous question which is still open on ω.

Corollary 2.25. *If κ is κ-compact, then there is a Tukey-top among the κ-complete ultrafilters which is a q-point.*

[5]Hayut uses the more general framework of stationary sets over $P_\kappa(\lambda)$ due to Jech [37].

3. Bases, the Tukey order, and the $\supseteq^*$ relation

The ultrafilter number is a cardinal characteristic of the continuum which in its generalized form is defined as follows:

Definition 3.1. Let $\kappa \geq \omega$ be any cardinal.

(1) The *characteristic number* of a uniform ultrafilter U is defined by $\mathrm{Ch}(U) := \min(|\mathcal{B}| \mid \mathcal{B} \text{ is a cofinal in } (U, \supseteq^*))$.
(2) For any regular cardinal $\mu \leq \kappa$ the *μ-spectrum* of κ is denoted by
$$\mathrm{Sp}_\mu(\kappa) = \{\mathrm{Ch}(U) \mid U \text{ is a } \mu\text{-complete uniform ultrafilter over } \kappa\}$$
(3) The *μ-ultrafilter number* of κ is $\mathfrak{u}_\mu(\kappa) = \min(\mathrm{Sp}_\mu(\kappa))$.

Note that requiring $\mathcal{B}$ to be a filter base (i.e. cofinal in $(U, \supseteq)$) instead of a cofinal subset of $(U, \supseteq^*)$, yield the same cardinals. Also note that

$$\mu_1 < \mu_2 \text{ implies } \mathrm{Sp}_{\mu_1}(\kappa) \supseteq \mathrm{Sp}_{\mu_2}(\kappa) \text{ and } \mathfrak{u}_{\mu_1}(\kappa) \leq \mathfrak{u}_{\mu_2}(\kappa)$$

Definition 3.2. Let $\kappa \geq \omega$ be any cardinal.

(1) The *spectrum* of κ is $\mathrm{Sp}(\kappa) = \cup_{\mu \in (\kappa+1) \cap Reg.} \mathrm{Sp}_\mu(\kappa) = \mathrm{Sp}_\omega(\kappa)$.
(2) The *ultrafilter number* of κ is defined by $\mathfrak{u}(\kappa) = \min(\mathrm{Sp}(\kappa))$.

Note that for any $\mu \leq \kappa$ regular, $\mathfrak{u}(\kappa) \leq \mathfrak{u}_\mu(\kappa)$. Kunen proved that $\mathfrak{u}(\omega) < 2^\omega$ is consistent, and a major open problem is whether $\mathfrak{u}(\omega_1) < 2^{\omega_1}$ is consistent. For measurable cardinals, Gitik and Shelah [28] proved that $\mathfrak{u}(\kappa) < 2^\kappa$ is consistent from an extremely large cardinal—a huge cardinal. Later, the large cardinal assumption was improved by Brooke-Taylor, Fischer, Friedman, and Montoya [12] to a supercompact cardinal. Recently, a remarkable result of Raghavan and Shelah [57] established the consistency of $\mathfrak{u}(\kappa) < 2^\kappa$ for $\kappa = 2^{\aleph_0}$ starting with a much smaller large cardinal—a *measurable cardinal*. However, in their model $2^{\aleph_0}$ is still very large. They also obtained the result on the much smaller cardinal $\aleph_{\omega+1}$ but starting again from a supercompact cardinal. The ultrafilter number on small cardinals such as ω_1, or even on small measurable cardinals is not fully understood. Here we suggest some connections to the Tukey order.

The natural homomorphism to consider suitable for comparing the characteristic number of two ultrafilters U, V would be a function $f : U \to V$ such that $\mathrm{Im}(f)$ is cofinal in V. This requirement is a bit weaker than being a cofinal map. Nonetheless, if f is a monotone cofinal map, which holds in many situations, then requiring that $\mathrm{Im}(f)$ is cofinal is equivalent to being a cofinal map.

Proposition 3.3. *Suppose that U, W are any (not necessarily complete) ultrafilters such that $U \geq_T W$. Then $\mathrm{Ch}(U) \geq \mathrm{Ch}(W)$.*

Proof. Let $\{b_i \mid i < \mathrm{Ch}(U)\}$ be a filter base for U. Then $B = \{b_i \setminus \xi \mid i < \mathrm{Ch}(U), \xi < \kappa\}$ is cofinal in U. Let $f : U \to W$ be a cofinal map. Then $f''B$ is a filter base for W, and therefore $\mathrm{Ch}(W) \leq |f''B| \leq |B| = \mathrm{Ch}(U)$. □

Note that the other direction is not true in general, since for example if $U_0 \lhd U_1$, U_1 is a p-point and $2^\kappa = \kappa^+$, then $\mathrm{Ch}(U_0) = \kappa^+ = \mathrm{Ch}(U_1)$ but $U_0 \not\equiv_T U_1$ (see Theorem 7.5). As a corollary, we get that any μ-complete ultrafilter for which $\neg\,\mathrm{Gal}(U, \mu, 2^\kappa)$ will satisfy $\mathrm{Ch}(U) = \max(\mathrm{Sp}_\mu(\kappa))$. In fact, we can say more:

Proposition 3.4. *Suppose that U is an ultrafilter over κ, $\mu \leq \kappa$ and $\neg\,\mathrm{Gal}(U, \mu, 2^\kappa)$ holds. Then $\mathrm{Ch}(U) = 2^\kappa$.*

Proof. To see that $\mathrm{Ch}(U) = 2^\kappa$, let $\langle A_i \mid i < 2^\kappa\rangle$ witness that $\neg\,\mathrm{Gal}(U, \mu, 2^\kappa)$, and suppose toward a contradiction that there is cofinal set $\mathcal{B} \subseteq U$ such that $|\mathcal{B}| < 2^\kappa$. In particular, for each $\alpha < 2^\kappa$ there is $b_\alpha \in \mathcal{B}$ such that $b_\alpha \subseteq A_\alpha$. This defines a function from 2^κ to $\mathcal{B}$, and by the pigeonhole principle there is $b^* \in \mathcal{B}$ such that $I := \{\alpha < 2^\kappa \mid b_\alpha = b^*\}$ has size at least κ. It follows that $b^* \subseteq \cap_{\alpha\in I} A_\alpha$, and therefore $\cap_{\alpha\in I} A_\alpha \in U$, contradicting the fact that $\langle A_\alpha \mid \alpha < 2^\kappa\rangle$ witnesses the failure of $\mathrm{Gal}(U, \mu, 2^\kappa)$. □

It is well known that if $\langle A_i \mid i < \lambda\rangle$ is a maximal κ-independent family then $\langle A_i \mid i < \lambda\rangle$ generates an ultrafilter U such that $\mathrm{Ch}(U) = \lambda$. Since all the known constructions of non-Galvin ultrafilters rely on some independent family which witnesses the non-Galviness, the previous result is somehow not surprising. In this light, the following question seems natural:

Question 3.5. Suppose that U is a κ-complete ultrafilter over κ such that $\neg\,\mathrm{Gal}(U, \kappa, \lambda)$. Is there a λ-independent family $\langle A_i \mid i < \lambda\rangle \subseteq U$ which generates U?

Corollary 3.6. $\max(\mathrm{Sp}(\kappa)) = 2^\kappa$ *is always realized by some (any) uniform ultrafilter* U *such that* $\neg\,\mathrm{Gal}(U, \omega, 2^\kappa)$.

Proof. By Corollary 2.17, such an ultrafilter U always exists, and by the previous corollary, $2^\kappa = \mathrm{Ch}(U)$. □

We can now further exploit this connection between the Tukey order and the ultrafilter number as follows:

Corollary 3.7. *Suppose that* $|\mathrm{Sp}_\mu(\kappa)| = \theta$. *Then there are at least* θ*-many* μ*-complete ultrafilters over* κ *with distinct Tukey classes.*

There are known models due to Garti, Magidor, and Shelah where the spectrum of an infinite cardinal κ has arbitrarily many cardinals (this requires blowing up 2^κ and large cardinals) [25]. As a corollary we get:

Corollary 3.8. *Assuming large cardinals, it is consistent that there are arbitrarily many distinct Tukey-classes of ultrafilters.*

In a later section, we will present models where we have more control over how those classes are ordered.

In order to compare the characteristics it suffices to compare the cofinal types of $(U, \supseteq^*)$. The next proposition suggests that, in some sense, the converse is also true.

Proposition 3.9. *Suppose that* U, W *have generating sequences* $\langle X^U_i \mid i < \theta\rangle, \langle X^W_i \mid i < \theta\rangle$ *resp. which are* $\supseteq^*$*-increasing. Then* $(U, \supseteq^*) \equiv_T (W, \supseteq^*)$.

Proof. Let $f : U \to W$ be the map sending $f(X^U_i) = X^W_i$ and for $b \in U$, let $f(b) = f(X^U_i)$ for the minimal i such that $X^U_i \subseteq^* b$., Given a $\subseteq^*$-cofinal set B in U, then for every $b \in B$, let i_b be the minimal index such that $X^U_{i_b} \subseteq^* b$ (hence $f(g) = X^W_{i_b}$). We claim that $\{i_b \mid b \in B\}$ must be unbounded in θ. Otherwise, there is i^* which is a bound. In particular, $X^U_{i^*} \subseteq^* b$ for every $b \in B$. Take any set $Y \in U$ such that $X^U_{i^*} \setminus Y$ is unbounded in κ (just partition $X^U_{i^*}$ and take Y to be the piece which belongs to U). Now there is $b \in B$ such that $b \subseteq^* Y$ and thus

$$b \subseteq^* Y \subseteq^* X^U_{i^*} \subseteq b$$

Hence $b =^* Y =^* X^U_{i^*}$ which contradicts the fact that $X^U_{i^*} \setminus Y$ is unbounded in κ. So we conclude that $\{i_b \mid b \in B\}$ is unbounded in θ. By definition of f, we have that $f''B \subseteq \{X^W_i \mid i < \theta\}$ then is cofinal in W. It follows that f is cofinal and $(U, \supseteq^*) \geq_T (W, \supseteq^*)$. The other direction is symmetric. □

Remark 3.10. Ultrafilters with $\supseteq^*$-increasing generating sequences are forcible (see [28] or [6]).

For the rest of this section, let us focus on the poset $(U, \supseteq^*)$. The next lemma can be found in [6, Lemma 2.19]:

Lemma 3.11. *Suppose that $2^\kappa = \kappa^+$. Then U is a p-point over κ if and only if U has a strong generating sequence[6] of length κ^+. In particular, any two p-points have the same Tukey class with respect to $\supseteq^*$.*

Remark 3.12. In the previous model, where the spectrum of a cardinal is large, there are different Tukey classes of ultrafilters with respect to $\supseteq^*$.

Proposition 3.13. *If U is a p-point then $(U, \supseteq^*) <_T (U, \supseteq)$.*

Proof. Indeed, the identity map witnesses that $\leq_T$, and equality cannot hold since $(U, \supseteq^*)$ is κ^+-directed (as it is p-point) while $(U, \supseteq)$ is not κ^+-directed. □

Definition 3.14. Let U be an ultrafilter. We say that $\mathrm{Gal}^*(U, \mu, \lambda)$ for every sequence $\langle A_i \mid i < \lambda\rangle \in [U]^\lambda$ there exists $I \in [\lambda]^\mu$, such that the subsequence $\langle A_i \mid i \in I\rangle$ has a pseudointersection.

It is not hard to see that $\mathrm{Gal}(U, \mu, \lambda)$ implies $\mathrm{Gal}^*(U, \mu, \lambda)$.

Definition 3.15 (Kunen [46])**.** We say that an ultrafilter U over κ is λ-OK if for every $\langle A_\alpha \mid \alpha < \kappa\rangle \subseteq U$, there exists $\langle B_\beta \mid \beta < \lambda\rangle \subseteq U$ such that for all nonempty $\sigma \in [\lambda]^{<\kappa}$, we have $\bigcap_{\beta\in\sigma} B_\beta \subseteq^* A_{|\sigma|}$.

Proposition 3.16. *If U is a non p-point κ-complete ultrafilter which is 2^κ-OK, then $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$.*

Proof. Take any sequence $\langle A_i \mid i < \kappa\rangle$ which has no pseudo intersection. Since U is κ-complete we may assume that A_i is decreasing in $\subseteq$. This assumption guarantees that there is no pseudo intersection for $\langle A_i \mid i \in I\rangle$ where $I \in [\kappa]^\kappa$. By 2^κ-OK, there is a sequence $\langle B_i \mid i < 2^\kappa\rangle$ such that for every $a \in [2^\kappa]^{<\kappa}$, $\cap_{i\in a} B_i \subseteq^* A_{|a|}$. It follows that $\langle B_i \mid i < 2^\kappa\rangle$ witnesses $\neg\,\mathrm{Gal}(U, \kappa, 2^\kappa)$. □

Kunen proved [46] that on ω there are non-p-points which are 2^ω-OK. In the first study of the Tukey order on ultrafilters, Milovich [50] proved that if U is a non-p-point which is 2^ω-OK, then U is Tukey top. Similar to Theorem 2.13, we can characterize using $\mathrm{Gal}^*(U, \kappa, 2^\kappa)$ the ultrafilters which are Tukey top with respect to $\supseteq^*$. This improves Milovich's result (both for ω and κ).

Theorem 3.17. *Let κ be either ω or a measurable cardinal. For any κ-complete ultrafilter U over κ, $(U, \supseteq^*) \equiv_T ([2^\kappa]^{<\kappa}, \subseteq)$ if and only if $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$.*

Proof. Suppose $\langle B_\alpha \mid \alpha < 2^\kappa\rangle \in [U]^{2^\kappa}$ witnesses that $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$, and let $g : [2^\kappa]^{<\kappa} \to \{B_\alpha \mid \alpha < 2^\kappa\}$ be any enumeration. Let us prove that g is unbounded. Suppose that $\mathcal{A} \subseteq [2^\kappa]^{<\kappa}$ is unbounded. Then $|\mathcal{A}| \geq \kappa$, so $g[\mathcal{A}]$ is a subset of

[6]For the definition of a strong generating sequence see [6, Def. 2.11].

$\{B_\alpha \mid \alpha < 2^\kappa\}$ of cardinality κ and therefore has no pseudointersection; hence $g[\mathcal{A}]$ is unbounded. In the other direction, suppose that $\mathrm{Gal}^*(U, \kappa, 2^\kappa)$ holds, and let $g : [2^\kappa]^{<\kappa} \to U$. We will prove that g is not unbounded. If $|\operatorname{Im}(g)| < 2^\kappa$, then there are κ-many elements mapped to the same set in U. Any κ-many elements in $[2^\kappa]^{<\kappa}$ are unbounded (since κ is a strong limit) so g is not unbounded. If $|\operatorname{Im}(g)| = 2^\kappa$, there is $Y \subseteq \operatorname{Im}(g)$, such that $|Y| = \kappa$ which has a pseudointersection. Again $g^{-1}[Y]$ is unbounded in $[2^\kappa]^{<\kappa}$ but it is mapped to a bounded set. □

Milovich observed that a small modification in Isbell's construction gives an ultrafilter U on ω such that $\neg\,\mathrm{Gal}^*(U, \omega, 2^\omega)$. This modification, combined with the techniques of the previous section give the following:

Theorem 3.18. *Suppose that κ is κ-compact. Then there is a q-point ultrafilter U such that $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$.*

Question 3.19. It is consistent from just a measurable cardinal that there is a κ-complete ultrafilter such that $\neg\,\mathrm{Gal}^*(U, \kappa, 2^\kappa)$?

Question 3.20. Is it consistent that U is a κ-complete ultrafilter over κ such that $(U, \supseteq^*)$ is not Tukey-top but $(U, \supseteq)$ is? Namely, is it consistent that $\neg\,\mathrm{Gal}(U, \kappa, 2^\kappa)$ but $\mathrm{Gal}^*(U, \kappa, 2^\kappa)$?

4. Continuous cofinal maps with special properties

Many of the orders on ultrafilters over measurable cardinals such as the Rudin-Keisler order, the Mitchell order, the Ketonen order, and the Rudin-Frolik order are expressible in terms of the ultrapower $j_U : V \to M_U$ of a given κ-complete ultrafilter U. For example, $U \leq_{RK} W$ if and only if there is an elementary embedding $k : M_U \to M_W$ such that $k \circ j_U = j_W$. One important property which allows this characterization using ultrapowers is the fact that a Rudin-Keisler projection is just a function $f : \kappa \to \kappa$ which is seen by the ultrapower of every κ-complete ultrafilter. A major difficulty when analyzing the Tukey order on κ-complete ultrafilters is that the projection maps (namely cofinal maps) are in general functions $f : 2^\kappa \to 2^\kappa$, which are extremely large objects when compared to the closure degree of the ultrapower. However, in some situations, the information needed to compute a cofinal map can be coded into a function from a κ-sized set to a κ-sized set. This will allow us to work with cofinal maps in the ultrapower and derive some structural corollaries.

In the following, assume κ is measurable and consider 2^κ as a generalized Baire space with the topology generated by basic open sets of the form $\{x \in 2^\kappa : s \sqsubset x\}$, where $s \in 2^{<\kappa}$. We use $U \restriction X$ to denote the set $\{Y \in U : Y \subseteq X\}$.

The next theorem extends Theorem 20 of Dobrinen and Todorcevic in [16] for p-points over ω to p-points over measurable cardinals. We show that any cofinal map from a p-point U to a uniform ultrafilter on κ is actually continuous upon restricting below some suitable member of U. Our proof follows the general form of that in [16], but with some streamlining modifications utilizing the measurability of κ. For normal ultrafilters (Corollary 4.2), we obtain additional useful properties.

Theorem 4.1. *Let κ be measurable, and suppose U is a p-point on κ and V is a uniform ultrafilter on κ such that $U \geq_T V$. Then for each monotone cofinal map $f : U \to V$, there is an $X \in U$ such that the restriction of f to $U \restriction X$ is continuous and has image which is cofinal in V.*

More precisely, let $\pi : \kappa \to \kappa$ be the minimal non-constant function mod U (i.e. $[\pi]_U = \kappa$), and ρ be the function $\rho_\alpha = \sup(\pi^{-1}[\alpha+1]) + 1$. Then for any strictly increasing sequence $\langle \delta_\alpha \mid \alpha < \kappa \rangle$, we can find a set $X = X(\langle \delta_\alpha \mid \alpha < \kappa \rangle)$ such that for every $\alpha < \kappa$ and $A \in U \restriction X$, $f(A) \cap \delta_\alpha$ depends only on $A \cap \rho_\alpha$. Moreover, there is a monotone function $\hat{f} : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ such that for each $A \in U \restriction X$, there are club many $\alpha < \kappa$ such that

$$\hat{f}(A \cap \rho_\alpha) = f(A) \cap \delta_\alpha$$

and in particular

$$f(A) = \bigcup_{\alpha<\kappa} \hat{f}(A \cap \rho_\alpha)$$

Then defining $g : P(\kappa) \to P(\kappa)$ by

$$g(A) = \bigcup_{\alpha<\kappa} \hat{f}(A \cap X \cap \rho_\alpha)$$

we see that

1. *g is a continuous monotone map,*
2. *$g \restriction U : U \to V$ is a cofinal map,*
3. *$g \restriction (U \restriction X) = f \restriction (U \restriction X)$.*

Proof. Let U be a p-point on κ. Suppose V is a uniform ultrafilter such that $U \geq_T V$, and let $f : U \to V$ be a monotone cofinal map and $\langle \delta_\alpha \mid \alpha < \kappa \rangle$ be a strictly increasing sequence. Construct a $\subseteq$-decreasing sequence $\langle X_\alpha : \alpha < \kappa \rangle$ (i.e. $\alpha \leq \beta \Rightarrow X_\beta \subseteq X_\alpha$) of members of U such that for each $\alpha < \kappa$, $X_\alpha \cap \rho_\alpha = \emptyset$ and the following holds:

(∗) For each pair $s \subseteq \rho_\alpha$ and $\delta \leq \delta_\alpha$, if there is a $Y \in U$ such that $s = Y \cap \rho_\alpha$ and $\delta \notin f(Y)$, then $\delta \notin f(s \cup X_\alpha)$ (i.e. $f(s \cup X_\alpha) \cap \delta_\alpha \subseteq f(Y) \cap \delta_\alpha$).

Take $X_0 \in U$ such that for each $\delta \leq \delta_0$, $\delta \notin f(X_0)$. Such an X_0 exists since V is uniform. Suppose $1 \leq \alpha < \kappa$ and $\langle X_\beta \mid \beta < \alpha \rangle$ have been chosen satisfying (∗). For each $s \subseteq \rho_\alpha$ and $\delta \leq \delta_\alpha$, if there is a $Y \in U$ such that $s = Y \cap \rho_\alpha$ and $\delta \notin f(Y)$, then take $Y_{s,\delta}$ so that $Y_{s,\delta} \cap \rho_\alpha = s$ and $\delta \notin f(Y_{s,\delta})$; otherwise, let $Y_{s,\delta} = [\rho_\alpha, \kappa)$. Define

$$X_\alpha = \bigcap \{Y_{s,\delta} : s \subseteq \rho_\alpha,\ \delta \leq \delta_\alpha\} \cap \bigcap \{X_\beta : \beta < \alpha\}. \tag{1}$$

Then $X_\alpha \cap \rho_\alpha = \emptyset$ and our construction ensures that X_α satisfies property (∗): Given $s \subseteq \rho_\alpha$ and $\delta \leq \delta_\alpha$, if there is a $Y \in U$ such that $Y \cap \rho_\alpha = s$ and $\delta \notin f(Y)$ then $\delta \notin f(Y_{s,\delta})$ so by monotonicity of f, $\delta \notin f(s \cup X_\alpha)$.

By definition of the ρ_α's and the fact that U is a p-point, there is an $X \in U$ such that $X \setminus \rho_\alpha \subseteq X_\alpha$, for each $\alpha < \kappa$. We claim that f is continuous when restricted to $U \restriction X$.

Let $W \in U \restriction X$ and $\alpha < \kappa$. It follows that

$$W \setminus \rho_\alpha \subseteq X \setminus \rho_\alpha \subseteq X_\alpha \tag{2}$$

Let $s = W \cap \rho_\alpha$, and let $\delta \leq \delta_\alpha$ be given. If $\delta \notin f(W)$, then by (∗) it follows that $\delta \notin f(s \cup X_\alpha)$. Since $X \setminus \rho_\alpha \subseteq X_\alpha$, by monotonicity also $\delta \notin f(s \cup (X \setminus \rho_\alpha))$. On the other hand, if $\delta \in f(W)$, then by monotonicity $\delta \in f(s \cup X \setminus \rho_\alpha)$. Thus,

$$f(W) \cap \delta_\alpha = f((W \cap \rho_\alpha) \cup (X \setminus \rho_\alpha)) \cap \delta_\alpha \tag{3}$$

Define the function $\hat{f} : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ as follows: For $s \in [\kappa]^{<\kappa}$, letting α be the least such that $s \subseteq \rho_\alpha$, define

$$\hat{f}(s) = f(s \cup (X \setminus \rho_\alpha)) \cap \delta_\alpha \tag{4}$$

This $\hat{f}$ is monotone and uniformly continuous and for each $A \in U \restriction X$,

$$f(A) = \bigcup_{\alpha<\kappa} \hat{f}(A \cap \rho_\alpha) \tag{5}$$

since there are club many α, namely those in $\lim(A) \cap C_{\alpha \mapsto \rho_\alpha}$, for which $f(A) \cap \delta_\alpha = \hat{f}(A \cap \rho_\alpha)$.

Defining g as in the statement of the theorem, it is routine to check that g is as claimed. □

Corollary 4.2. *If U is normal, then we have that $\rho_\alpha = \alpha+1$ and thus $f(W) \cap \delta_\alpha = f((W \cap \alpha + 1) \cup (X \setminus \alpha + 1)) \cap \delta_\alpha$, for each $\alpha < \kappa$.*

Example 4.3. Suppose that U is a normal ultrafilter and $h_*(U) = V$ for some $h : \kappa \to \kappa$ (so in particular, h can be taken to be one-to-one and $h(\alpha) > \alpha$). Let $f : U \to V$ be defined by $f(X) = h''X$, noting that f is a monotone cofinal map. Pick any sequence $\langle \delta_\alpha > h(\alpha) \mid \alpha < \kappa \rangle$. In the proof of Theorem 4.1, for any $s \subseteq \alpha + 1$ and $\delta < \delta_\alpha$, if $\delta \in h''s$ then $Y_{s,\delta} = [\delta_\alpha, \kappa)$; otherwise, if $\delta \notin h''s$ then $Y_{s,\delta} = s \cup [\delta_\alpha, \kappa)$. Now $X_\alpha = [\delta_\alpha, \kappa)$ and

$$X = \Delta_{\alpha<\kappa} X_\alpha = \{\beta < \kappa \mid \forall \alpha < \beta,\ \delta_\alpha \leq \beta\}$$

Taking any $W \subseteq X$ and letting $s = W \cap \alpha + 1$, we have that $W \cap \alpha + 1$ decides that $f(W) \cap \delta_\alpha = h''(W \cap \alpha + 1)$. Since any point $h(\beta) \in h''(W \setminus \alpha + 1)$ will satisfy that $h(\alpha) < \delta_\alpha \leq \beta$, it follows that $h''W \cap (\alpha, \delta_\alpha) = \emptyset$.

We point out that $\alpha + 1$ is the best possible in the previous corollary. For if the function $f(W) \cap \delta_\alpha$ depends only on $W \cap \alpha$, then $W \cap \alpha$ decides $f(W) \cap \alpha + 1$. But this cannot be true in general since for example if f is the identity function as above, then this would mean we can decide $W \cap \alpha + 1$ using $W \cap \alpha$ below a certain set $X \in U$. But take any $\alpha \in X$, and take $W_1, W_2 \subseteq X$ such that $X \cap \alpha = W_1 \cap \alpha = W_2 \cap \alpha$, $\alpha \in W_1$ and $\alpha \notin W_2$; then we get different decisions.

If $U \neq V$, we can find $X \in U$ and $Y \subseteq X$ such that $\kappa \setminus X \in V$ and $f(Y) \subseteq \kappa \setminus X \subseteq \kappa \setminus Y$. In this situation, we know that whenever $\alpha \notin W \subseteq Y$, then $W \cap \alpha = W \cap \alpha + 1$ and we can decide $f(W) \cap \delta_\alpha$, and if $\alpha \in W$ then $\alpha \notin f(W)$.

Raghavan showed in Corollary 11 of [58] that there are no continuous cofinal maps from a Fubini product of two p-points U, W on ω to U, and the proof generalizes easily to ultrafilters on any countable base set. On the other hand, the second author showed in Theorem 4.4 of [14] that Fubini products, even countable Fubini iterates, of p-points on ω have cofinal maps which are canonized by finitary maps which behave similarly to continuous maps, but the proof is rather long. Theorem 5.6 in the next section will show that, surprisingly, for κ-complete ultrafilters, Fubini products and cartesian products have the same Tukey type.

We now extend Theorem 4.1 to cartesian products of finitely many p-points. This theorem, along with Theorem 5.6, will aid in understanding the cofinal types of Fubini products of finitely many κ-complete ultrafilters.

Theorem 4.4. *Suppose $U_1, \ldots, U_n$ are p-points over $\kappa > \omega$, $\pi_1, \ldots, \pi_n$ are functions such that $[\pi_i]_{U_i} = \kappa$, and V is a uniform ultrafilter on κ such that $U_1 \times \cdots \times$*

$U_n \geq_T V$. Then for each monotone cofinal map $f : U_1 \times \cdots \times U_n \to V$ and every sequence $\langle \delta_\alpha \mid \alpha < \kappa \rangle$, there are sets $X_i \in U_i$ such that the restriction of f to $U_1 \times \cdots \times U_n \restriction \langle X_1, \ldots, X_n \rangle$ is monotone, cofinal, and uniformly continuous, i.e. for each $\alpha < \kappa$, $f(\langle B_1, \ldots, B_n \rangle) \cap \delta_\alpha$ is determined by $\langle B_1 \cap \rho^1_\alpha, \ldots, B_n \cap \rho^n_\alpha \rangle$, where for each $1 \leq i \leq n$ and each $\alpha < \kappa$, $\rho^i_\alpha = \sup(\pi_i^{-1}[\alpha + 1]) + 1$. In particular, there is a continuous monotone cofinal map from $U_1 \times \cdots \times U_n$ to V.

Proof. As before, let us construct a $\supseteq$-decreasing sequence for each $1 \leq i \leq n$, $\langle X_{\alpha,i} : \alpha < \kappa \rangle$ of members of U_i such that for all $\alpha < \kappa$, $X_{\alpha,i} \cap \rho^i_\alpha = \emptyset$ and the following holds:

- $(*)$ For each pair $s_i \subseteq \rho^i_\alpha$ and $\delta \leq \delta_\alpha$, if there is a $\vec{Y} = \langle Y_1, \ldots, Y_n \rangle \in U_1 \times \cdots \times U_n$ such that $s_i = Y_i \cap \rho^i_\alpha$ and $\delta \notin f(\vec{Y})$, then $\delta \notin f(\langle s_1 \cup X_{\alpha,1}, \ldots, s_n \cup X_{\alpha,n} \rangle)$.

Take $X_{0,i} \in U_i$ such that $0 \notin f(\langle X_{0,1}, \ldots, X_{0,n} \rangle)$. Since V is uniform, and f is cofinal, such an $X_{0,i}$ exists. Suppose $1 \leq \alpha < \kappa$ and $\langle X_{\beta,i} : \beta < \alpha \rangle$ have been constructed. For each $s_i \subseteq \rho^i_\alpha$ and $\delta \leq \delta_\alpha$, if there are a sets $Y_i \in U_i$ such that $s_i = Y_i \cap \rho^i_\alpha$ and $\delta \notin f(\langle Y_1, \ldots, Y_n \rangle)$, then let $Y_{s_1,..,s_n,i,\delta}$ be such a Y_i; otherwise, let $Y_{s_1,..,s_n,i,\delta} = \kappa$. Define

$$X_{\alpha,i} = \bigcap \{ Y_{s_1,\ldots,s_n,i,\delta} : s_1 \subseteq \rho^1_\alpha, \ldots, s_n \subseteq \rho^n_\alpha,\ \delta \leq \delta_\alpha \} \cap \bigcap \{ X_{\beta,i} : \beta < \alpha \}. \tag{6}$$

Note that $X_{\alpha,i} \cap \rho^i_\alpha = \emptyset$. Moreover, given $s_i \subseteq \rho^i_\alpha$, if there is a $Y_i \in U_i$ such that $Y_i \cap \rho^i_\alpha = s_i$ then

$$f(\langle s_1 \cup X_{\alpha,1}, \ldots, s_n \cup X_{\alpha,n} \rangle) \cap \delta_\alpha \subseteq f(\langle Y_1, \ldots, Y_n \rangle) \cap \delta_\alpha,$$

by choice of $X_{\alpha,i}$ and monotonicity of f.

Since each U_i is a normal ultrafilter, we let $X^*_i := \Delta^*_{i<\kappa} X_{\alpha,i} \in U_i$, then $X^*_i \setminus \rho^i_\alpha \subseteq X_{\alpha,i}$. We claim that f is continuous when restricted to $U_1 \times \cdots \times U_n \restriction \langle X^*_1, \ldots, X^*_n \rangle$. Toward this, let $W_i \in U_i$ be such that $W_i \subseteq X^*_i$, let $\alpha < \kappa$ and let $s_i = W_i \cap \rho^i_\alpha$. It follows that

$$W_i \setminus \rho^i_\alpha \subseteq X^*_i \setminus \rho^i_\alpha \subseteq X_{\alpha,i}. \tag{7}$$

Thus,

$$f(\langle W_1, \ldots, W_n \rangle) \cap \delta_\alpha \subseteq f(\langle s_1 \cup X^*_1 \setminus \rho^1_\alpha, \ldots, s_n \cup X^*_n \setminus \rho^n_\alpha) \cap \delta_\alpha \subseteq$$

$$\subseteq f(s_1 \cup X_{\alpha,1}, \ldots, s_n \cup X_{\alpha,n}) \cap \delta_\alpha \subseteq f(W_1, \ldots, W_n) \cap \delta_\alpha$$

Then $f(\langle W_1, \ldots, W_n \rangle) \cap \delta_\alpha = f(\langle s_1 \cup X^*_1 \setminus \rho^1_\alpha, \ldots, s_n \cup X^*_n \setminus \rho^n_\alpha) \cap \delta_\alpha$ is continuous on $U_1 \times \cdots \times U_n \restriction \langle X^*_1, \ldots, X^*_n \rangle$. □

Let us derive similar corollaries to the ones from [16, Corollaries 21&23]. First, since there are only 2^κ-many continuous cofinal maps we get the following:

Corollary 4.5. *There are only 2^κ many ultrafilters which are Tukey below a p-point. Moreover, every $\leq_T$-chain of p-points has order-type at most $(2^\kappa)^+$.*

Next, we apply the free set lemma of Hajnal [38].

Lemma 4.6 (The free set lemma of Hajnal)**.** *If $|X| = \theta$, $\lambda < \theta$, and $F : X \to P(X)$ satisfies $x \notin F(x)$ and $|F(x)| < \lambda$, for all $x \in X$, then there is a $Y \subseteq X$, $|Y| = \theta$ with $x \notin F(y)$ and $y \notin F(x)$ for all $x, y \in Y$.*

Corollary 4.7. *If χ is a set of at least $(2^\kappa)^+$-many p-point ultrafilters on κ, then there is $Z \subseteq \chi$ of cardinality $|\chi|$ such that every distinct $U, V \in Z$ are incomparable in the Tukey order.*

Remark 4.8. In the Kunen-Paris model there are $2^{(\kappa^+)}$-many normal ultrafilters, $2^\kappa = \kappa^+$ and $2^{(\kappa^+)}$ can be made arbitrarily large. In particular, there are $2^{(2^\kappa)}$-many incomparable normal ultrafilters in the Tukey order.

5. On the Tukey class of finite powers of ultrafilters

In this section, we analyze the Tukey structure of (cartesian) products and Fubini products and limits of ultrafilters. some of these arguments are adaptations to uncountable cardinals of theorems from Section 4 of [16]. However, it turns out that the situation for measurable cardinals is very different from that of ultrafilters on ω.

Let us start with the following:

Lemma 5.1. *For any ultrafilters U, V, U_α $(\alpha < \kappa)$, we have that:*

(1) $\sum_U U_\alpha \leq_T U \times \prod_{\alpha<\kappa} U_\alpha$.
(2) $U \cdot V \leq_T U \times \prod_{\alpha<\kappa} V$.
(3) $U \cdot U \leq_T \prod_{\alpha<\kappa} U$.

Proof. The proof of (1) is a straightforward generalization of [16, Theorem 32]: Given any $A \in \sum_U U_\alpha$, define $(A)_\alpha = \{\beta < \kappa \mid \langle \alpha, \beta\rangle \in A\}$ and let $\mathcal{B}$ be the set of those $A \in \sum_U U_\alpha$ such that for each $\alpha < \kappa$, either $(A)_\alpha \in U_\alpha$ or else $(A)_\alpha = \emptyset$. Then $\mathcal{B}$ is a filter base for $\sum_U U_\alpha$. Given $A \in \mathcal{B}$ define $\pi_0(A)$ to be the set of $\alpha < \kappa$ such that $(A)_\alpha \in U_\alpha$, and for each $\alpha < \kappa$, define $u_\alpha(A) = (A)_\alpha$ if $\alpha \in \pi_0(A)$ and κ otherwise. Define $f(A)$ to be $\langle \pi_0(A)\rangle^\frown \langle u_\alpha(A) : \alpha < \kappa\rangle$. Then f is a Tukey map. (2) and (3) are immediate from (1). □

Let U be a σ-complete ultrafilter over κ, $\pi : \kappa \to \kappa$ be the least non-constant function mod U, and let $\langle A_i \mid i < \kappa\rangle$ be any sequence of sets in U, we define the *modified diagonal intersection* of the sequence by

$$\Delta^*_{i<\kappa} A_i := \{\alpha < \kappa \mid \forall i < \pi(\alpha),\ \alpha \in A_i\}$$

Fact 5.2. *For any sequence $\langle A_i \mid i < \kappa\rangle \in U$, where U is σ-complete, $\Delta^*_{i<\kappa} A_i \in U$.*

Proof. Just otherwise, the complement $B := \kappa \setminus \Delta^*_{i<\kappa} A_i$ would be in U, and for every $\beta \in B$, there is $i_\beta < \pi(\beta)$ such that $\beta \notin A_{i_\beta}$, but then $[\beta \mapsto i_\beta]_U < [\pi]_U$, and since $[\pi]_U$ is the least non-constant function, then there is a set $B' \in U \restriction B$ and i^* such that for every $\beta \in B'$, $i_\beta = i^*$. This implies that $B' \cap A_{i^*} = \emptyset$, contradicting that both B' and A_{i^*} are in U. □

Lemma 5.3. *Let $\omega < \kappa$ and U be a p-point over κ. Then there is a function $h : \kappa \to \kappa$ such that for every $\langle A_i \mid i < \kappa\rangle \in \prod_{i<\kappa} U$, and every $\alpha < \kappa$, $(\Delta^*_{i<\kappa} A_i) \setminus h(\alpha) \subseteq A_\alpha$.*

Proof. Recall that $\pi : \kappa \to \kappa$ denotes the least non-constant function. Since U is a p-point, for each $\alpha < \kappa$, $\pi^{-1}[\alpha] := \{\gamma < \kappa \mid \pi(\gamma) < \alpha\}$ is bounded in κ; let $h(\alpha) = (\sup \pi^{-1}[\alpha + 1]) + 1$. Let $\langle A_i \mid i < \kappa\rangle \in \prod_{i<\kappa} U$ be given. Then for any $\alpha < \kappa$, and any $\nu \in (\Delta^*_{i<\kappa} A_i) \setminus h(\alpha)$, $\nu \notin \pi^{-1}[\alpha + 1]$ and therefore $\pi(\nu) > \alpha$ which implies by the definition of the modified diagonal intersection that $\nu \in A_\alpha$. □

In general, if U is not a p-point we cannot prove that there is such a uniform function h which works for every sequence of sets, but instead we can do the following:

Lemma 5.4. *Let $\omega < \kappa$ and U be any κ-complete ultrafilter over κ. Then there is a sequence of sets $\langle B_\alpha \mid \alpha < \kappa\rangle$ such that $B_\alpha \notin U$ for each α, such that for every sequence $\langle A_i \mid i < \kappa\rangle \in \prod_{i<\kappa} U$, for every $\alpha < \kappa$, $(\Delta^*_{i<\kappa} A_\alpha) \setminus B_\alpha \subseteq A_\alpha$.*

Proof. Let $\pi : \kappa \to \kappa$ be as before, and define $B_\alpha = \pi^{-1}[\alpha + 1]$ for each $\alpha < \kappa$. Since π is not constant mod U and U is κ-complete, each $B_\alpha \notin U$. Let $\langle A_i \mid i < \kappa\rangle \in \prod_{i<\kappa} U$ be any sequence. To see that $(\Delta^*_{i<\kappa} A_i) \setminus B_\alpha \subseteq A_\alpha$, let $\nu \in (\Delta^*_{i<\kappa} A_i) \setminus \pi^{-1}[\alpha + 1]$. Then $\pi(\nu) \geq \alpha + 1 > \alpha$, so $\nu \in A_\alpha$ by the definition of $\Delta^*_{i<\kappa} A_i$. □

The following theorem is very different from the situation on ω:

Lemma 5.5. *Let $\omega < \kappa$ be measurable. For each κ-complete ultrafilter U over κ, $\prod_{i<\kappa} U \leq_T U$.*

Proof. Let us define the map $F : \prod_{i<\kappa} U \to U$ by

$$F(\langle A_i \mid i < \kappa\rangle) = \Delta^*_{i<\kappa} A_i$$

To see that F is unbounded, let $\{\langle A^j_i \mid i < \kappa\rangle \mid j \in I\}$ be unbounded in $\prod_{i<\kappa} U$ and suppose toward contradiction that there is $B \in U$ such that $B \subseteq \Delta^*_{i<\kappa} A^j_i$ for every $j \in I$. Let $\langle B_i \mid i < \kappa\rangle$ be the sequence from the previous lemma, and note that $\langle B \setminus B_i \mid i < \kappa\rangle \in \prod_{i<\kappa} U$. Also note that by the property of the sequence $\langle B_i \mid i < \kappa\rangle$, for each $i < \kappa$ and each $j \in I$,

$$B \setminus B_i \subseteq (\Delta^*_{i<\kappa} A^j_i) \setminus B_i \subseteq A^j_i.$$

We conclude that in the everywhere domination order of the product $\prod_{i<\kappa} U$, we have $\langle B \setminus B_i \mid i < \kappa\rangle \geq \langle A^j_i \mid i < \kappa\rangle$. Therefore, the sequence $\langle B \setminus B_i \mid i < \kappa\rangle$ bounds the set $\{\langle A^j_i \mid i < \kappa\rangle \mid j \in I\}$, contradicting our initial assumption that this set should be unbounded. □

The next several corollaries demonstrate the striking difference between ultrafilters on ω versus on a measurable cardinal. For ultrafilters on ω, Dobrinen and Todorcevic in [16] showed that Fubini products of rapid p-points are Tukey equivalent to their cartesian products, whereas if rapidness is dropped, then the equivalence fails. Here, rapidness and even p-point can be dropped: the Tukey equivalence between finite Fubini products and cartesian products holds simply under the assumption that the ultrafilters are κ-complete.

Theorem 5.6. *Let $\omega < \kappa$ be measurable. For every two κ-complete ultrafilters U, V over κ, $U \cdot V \equiv_T U \times V$. Moreover, let $U_1, \ldots, U_n$ be any κ-complete ultrafilters over κ. Then $U_1 \cdot \ldots \cdot U_n \equiv_T \prod_{i=1}^n U_i$.*

Proof. $U \times V \leq_T U \cdot V$ by Proposition 1.5. For the other direction, $U \cdot V \leq_T U \times \prod_{\alpha<\kappa} V$ by Lemma 5.1 item (2), and by Lemma 5.5, $U \times \prod_{\alpha<\kappa} V \leq_T U \times V$. The moreover part follows by a straightforward induction. □

Theorem 5.7. *For any κ-complete ultrafilter U over a measurable cardinal κ, $U^n \equiv_T U$.*

Proof. $U^n \equiv_T U \times U \times \cdots \times U \equiv_T U$. □

Corollary 5.8. *If U is a Galvin ultrafilter over a measurable cardinal κ (namely, $\mathrm{Gal}(U, \kappa, 2^\kappa)$) then for every n, U^n is Galvin.*

Proof. U is Galvin iff U is not Tukey-top, and U^n is Galvin iff U^n is not Tukey-top. Since $U \equiv_T U^n$, we see that U is Galvin iff U^n is Galvin. □

Question 5.9. Suppose that U, V_α are all Galvin. Does it follow that $\sum_U V_\alpha$ is Galvin? Is the Fubini product of two Galvin ultrafilters always Galvin?

In the next section, we will provide a partial answer for this question when we restrict ourselves to a certain class of ultrafilters called *basically generated*.

6. Basic and basically generated ultrafilters

One generalization from the theory on ω is the definition of the so-called *basic* ultrafilters, which gives another characterization of p-point. The following is the notion of convergence on the generalized Baire space 2^κ, with the topology generated by basic open sets of the form $N_s = \{f \in 2^\kappa \mid s \sqsubset f\}$, where $s \in 2^{<\kappa}$. Given $\langle A_i \mid i < \kappa\rangle \subseteq P(\kappa)$, we write $\lim_{i\to\kappa} A_i = A$ to mean that for every $\delta < \kappa$ there is $i^* < \kappa$, such that for every $i \geq i^*$, $A_i \cap \delta = A \cap \delta$, in other words, the sequence A_i converges to A.

The notion of a basic partial order first appeared in [61] in the context of separable metric spaces. The following extends the notion of basic ultrafilter on ω in [16] to uncountable cardinals.

Definition 6.1. A κ-complete ultrafilter U is called *basic* if for every sequence $\langle A_i \mid i < \kappa\rangle$ such that $\lim_{i\to\kappa} A_i = A \in U$ there is a subsequence $\langle A_{i_\alpha} \mid \alpha < \kappa\rangle$ such that $\cap_{\alpha<\kappa} A_{i_\alpha} \in U$.

Theorem 14 of [16] showed that for ultrafilters on ω, p-points are exactly basic ultrafilters. We now prove analogous results for ultrafilters on uncountable cardinals. First we present a slightly different definition of basic which will turn out to be equivalent:

Definition 6.2. A κ-complete ultrafilter U is called *uniformly basic* if for every sequence $\langle A_i \mid i < \kappa\rangle$ such that $\lim_{i\to\kappa} A_i = A \in U$ there is a function $f : \kappa \to \kappa$ such that whenever $f \leq_{bd} g$, $\cap_{\alpha<\kappa} A_{g(\alpha)} \in U$.

Clearly uniformly basic implies basic.

Proposition 6.3. *p-point implies uniformly basic.*

Proof. Suppose that U is a p-point and let π be the least non-constant function mod U. Since U is a p-point, there is $Y \in U$ such that for every $\gamma < \kappa$, $\pi^{-1}[\gamma] \cap Y$ is bounded in κ. For each $\alpha < \kappa$, pick $\rho_\alpha := \sup(\pi^{-1}[\alpha + 1] \cap Y) + 1$. To see that U is uniformly basic, let $\langle A_i \mid i < \kappa\rangle$ be a sequence converging to some set $A \in U$. By definition of convergence, there is a sequence δ_α such that for any $i \geq \delta_\alpha$, $A_i \cap \rho_\alpha = A \cap \rho_\alpha$; define $f(\alpha) = \rho_\alpha$. Let us prove that for any $f \leq_{bd} g$ we have that $\cap_{\alpha<\kappa} A_{g(\alpha)} \in U$. Towards this, let $\theta < \kappa$ be such that for every $\theta \leq \alpha < \kappa$, $\delta_\alpha \leq g(\alpha)$. It suffices to prove that $(\cap_{\beta<\theta} A_{g(\beta)}) \cap A \cap Y \cap \Delta^*_{\alpha<\kappa} A_{\delta_\alpha} \subseteq \cap_{\alpha<\kappa} A_{g(\alpha)}$, where[7]

$$\Delta^*_{\alpha<\kappa} A_{\delta_\alpha} := \{\nu < \kappa \mid \forall \alpha < \pi(\nu),\ \nu \in A_{\delta_\alpha}\}$$

[7]To see that $\Delta^*_{\alpha<\kappa} A_{\delta_\alpha} \in U$, suppose not; then for a set of ν's in U, there is $\alpha_\nu \in \pi(\nu)$ such that $\nu \notin A_{\delta_\alpha}$. Since π is minimal, then $\nu \mapsto \alpha_\nu$ is constant mod U, say on a set Z with value α^*. Then for every $\nu \in Z$, $\nu \notin A_{\delta_{\alpha^*}}$, contradiction.

Indeed, suppose $\xi \in (\cap_{\beta<\theta} A_{g(\beta)}) \cap A \cap Y \cap \Delta^*_{\alpha<\kappa} A_{\delta_\alpha}$. Let $\alpha_0 < \kappa$ and split into cases:

(1) If $\alpha_0 < \theta$, then clearly $\xi \in A_{g(\alpha_0)}$.
(2) If $\alpha_0 < \pi(\xi)$, then by the definition of $\Delta^*_{\alpha<\kappa} A_{g(\alpha)}$, $\xi \in A_{g(\alpha_0)}$.
(3) If $\alpha_0 \geq \max(\pi(\xi), \theta)$ then $\xi \in \pi^{-1}[\alpha_0 + 1] \cap Y$, which means that $\xi < \rho_{\alpha_0}$, so $\xi \in A \cap \rho_{\alpha_0}$. Also we conclude that $\delta_{\alpha_0} \leq g(\alpha_0)$, and by the choice of δ_{α_0}, it follows that $A_{g(\alpha_0)} \cap \rho_{\alpha_0} = A \cap \rho_{\alpha_0}$. Thus $\xi \in A_{g(\alpha_0)}$.

□

The following proposition implies that basic, uniformly basic, and p-points are all equivalent:

Proposition 6.4. *Basic implies p-point.*

Proof. Suppose that U is basic and uniform. Let $\langle X_i \mid i < \kappa \rangle \in [U]^\kappa$ be a sequence of sets in U. We want to prove that this sequence has a pseudointersection. By κ-completeness we may assume that the sequence of X_i is decreasing. Define the sets $Y_i = X_i \cup i$. Clearly, $Y_i \supseteq X_i$ and $\lim_{i\to\kappa} Y_i = \kappa$. Hence, since U is basic, there is a subsequence $\langle i_\alpha \mid \alpha < \kappa \rangle$ such that $X^* := \cap_{\alpha<\kappa} Y_{i_\alpha} \in U$. Note that for each $i < \kappa$, there is $\alpha < \kappa$ such that $i < i_\alpha$ and therefore $X_i \supseteq X_{i_\alpha} =^* Y_{i_\alpha} \supseteq X^*$. Hence U is a p-point. □

Proposition 6.5. *Basic implies Galvin.*

Although this proposition follows from Theorem 6.8, we will provide a short proof for this proposition separately as we believe that it may be of interest to the reader who is familiar with the proof of Galvin's Theorem for normal filters, to see the similarities with the parallel theory for ultrafilters on ω.

Proof. Let $\langle A_i \mid i < \kappa^+ \rangle$ be any sequence of sets in U, and for each $\alpha < \kappa$, $i < \kappa^+$ define

$$H_{\alpha,i} := \{j < \kappa^+ \mid A_i \cap \alpha = A_j \cap \alpha\}$$

Galvin observed that:

Lemma 6.6. *There is an i^* such that for every $\alpha < \kappa$, $|H_{i^*,\alpha}| = \kappa^+$.*

The proof can be found in [9, Proposition 5.13]. Then we can find a sequence of distinct i_α's such that $A_{i_\alpha} \cap \alpha = A_{i^*} \cap \alpha$. In particular, $\lim_{\alpha\to\kappa} A_{i_\alpha} = A_{i^*} \in U$, and since U is basic, there is a subsequence $\langle \alpha_j \mid j < \kappa \rangle$ such that $\cap_{j<\kappa} A_{i_{\alpha_j}} \in U$. □

The next notion of basically generated ultrafilter on ω first appeared in [16] as a means to abstract properties of p-points responsible for being strictly below the Tukey top. In [16], Dobrinen and Todorcevic proved that the collection of basically generated ultrafilters on ω contains all p-points and is closed under countable iterates of Fubini limits, and that all basically generated ultrafilters are not Tukey-top [16, Theorems 16, 18]. We prove the analogues for κ-complete ultrafilters on κ (Theorems 6.8 and 6.10) building on those proofs but then utilizing some different techniques. This is another improvement of Galvin's Theorem, and the theorem due to the first author from [4] which asserts that a p-point sum of p-points has the Galvin property.

Definition 6.7. An ultrafilter U on κ is called *basically generated* if it is uniform, κ-complete, and has a κ-complete filter base[8] $B \subseteq U$ with the property that to each sequence $\{A_i \mid i < \kappa\} \subseteq B$ converging to an element of B, there corresponds a function f such that for every $f \leq_{bd} g$, we have $\cap_{\alpha<\kappa} A_{g(\alpha)} \in U$.

Note that this definition differs from the one in [16].

Theorem 6.8. *If U is a basically generated ultrafilter on κ, then U is not Tukey-top.*

Proof. Let $B \subseteq U$ be a filter base witnessing that U is basically generated. Fix a subset $C \subseteq U$ of cardinality 2^κ; for each $X \in C$ fix one $Y_X \in B$ such that $Y_X \subseteq X$. If there is a subset $D \subseteq C$ of cardinality κ for which all Y_X, $X \in D$, are equal, then there is a subset of C of size κ with intersection in U.

Otherwise, $|\{Y_X \mid X \in C\}| \geq \kappa^+$. Without loss of generality, assume that for each $X \neq X'$ in C, $Y_X \neq Y_{X'}$. For each $s \in 2^{<\kappa}$, let $C_s = \{X \in C \mid Y_X \cap \mathrm{dom}(s) = s^{-1}[\{1\}]\}$. Note that for each $\alpha < \kappa$, there is at least one $s \in 2^\alpha$ for which $|C_s| \geq \kappa^+$.

Claim 6.9. *There is an $X \in C$ such that for every $\alpha < \kappa$, $|C_{s_\alpha}| \geq \kappa^+$, where s_α is the characteristic function of $Y_X \cap \alpha$.*

Proof of claim. Suppose otherwise. Then for each $X \in C$, there is a minimal $\alpha_X < \kappa$ such that C_{s_X} has size at most κ, where s_X is the characteristic function of $Y_X \cap \alpha_X$. Let $F = \{s_X \mid X \in C\}$. Then $F \subseteq 2^{<\kappa}$ so $|F| \leq \kappa$. But then $C = \bigcup_{s \in F} C_s$ has size at most κ, a contradiction. □

Thus, fix $X \in C$ and the sequence $\langle s_\alpha \mid \alpha < \kappa\rangle$ from the claim such that each $|C_{s_\alpha}| \geq \kappa^+$. Now we build a convergent subsequence of $\{Y_X \mid X \in C\}$ quite simply: For $\alpha < \kappa$, take any $X_\alpha \in C_{s_\alpha}$. Then $\langle Y_{X_\alpha} \mid \alpha < \kappa\rangle$ is a sequence of members of B which converges to Y_X which is a member of B. Since B witnesses that U is basically generated, there is a subsequence $\langle \alpha_i \mid i < \kappa\rangle$ such that $Y^* := \bigcap_{i<\kappa} Y_{X_{\alpha_i}}$ is in U. Finally, let $D = \{X \in C \mid Y^* \subseteq X\}$, and note that $\{X_{\alpha_i} \mid i < \kappa\} \subseteq D$; in particular, $|D| \geq \kappa$. Then $\bigcap D \supseteq Y^*$ which is in U. Hence, U is not Tukey-top. □

Theorem 6.10. *Suppose that U and V_α, $\alpha < \kappa$, are basically generated ultrafilters on κ. Then $W := \sum_U V_\alpha$ is basically generated (with respect to the product topology on $\kappa \times \kappa$).*

Proof. Let B and $(B_\alpha)_{\alpha<\kappa}$ be κ-complete filter bases for $U, (V_\alpha)_{\alpha<\kappa}$ witnessing that $U, (V_\alpha)_{\alpha<\kappa}$ are basically generated, respectively. Let $\pi_0 : \kappa \times \kappa \to \kappa$ be the projection map to the first coordinate: $\pi_0(\alpha, \beta) = \alpha$. For $X \subseteq \kappa \times \kappa$ and $\alpha < \kappa$, let $(X)_\alpha$ denote $\{\beta < \kappa \mid (\alpha, \beta) \in X\}$, the α-th fiber of X. Define

$$C = \{X \in W \mid \pi_0[X] \in B \wedge \forall \alpha < \kappa\, ((X)_\alpha = \emptyset \vee (X)_\alpha \in B_\alpha)\}$$

Then C is a filter base for W which is κ-complete.

Consider a converging sequence $X_\alpha \to X$ in C. Note that $\pi_0[X_\alpha]$ converges to some member of U (possibly not in B) containing $\pi_0[X]$. Since $\pi_0[X]$ is in B and since $X \in C$, we make the following modification: Let $Y_\alpha = X_\alpha \cap (\pi_0[X] \times \kappa)$, note that Y_α is in C as $\pi_0[Y_\alpha] = \pi_0[X_\alpha] \cap \pi_0[X] \in B$, and for every $\xi < \kappa$, if $\xi \notin \pi_0[Y_\alpha]$ then $(Y_\alpha)_\xi = \emptyset$ and if $\xi \in \pi_0[Y_\alpha]$, then $(Y_\alpha)_\xi = (X_\alpha)_\xi$. Let A_α denote $\pi_0[Y_\alpha]$ and A denote $\pi_0[X]$. Then we have that $\lim_{\alpha\to\kappa} A_\alpha = A$, $\lim_{\alpha\to\kappa} Y_\alpha = X$, and for each

[8]Namely, B is cofinal in $(U, \supseteq)$, and closed under intersecting less than κ-many of its elements.

$\xi < \kappa$, $\lim_{\alpha\to\kappa}(Y_\alpha)_\xi = (X)_\xi$[9]. Since B witnesses that U is basically generated, there is a function f_0 such that for every $f_0 \leq_{bd} g$, $\bigcap_{\alpha<\kappa} A_{g(\alpha)} \in U$.

For each $\xi \in A$, consider the sequence

$$Z^\xi_\alpha = \begin{cases} (Y_\alpha)_\xi & \xi \in A_\alpha \\ Z^\xi_* & \text{else} \end{cases}$$

where Z^ξ_* is some (any) fixed element of B_ξ. Note that since A_α converges to A, then there is $\theta_\xi < \kappa$ such that for every $\alpha \in (\theta_\xi, \kappa)$, $\xi \in A_\alpha$ and therefore $Z^\xi_\alpha = (Y_\alpha)_\xi$. Also note that A^ξ_α converges to $(Y)_\xi$ and since V_ξ is basically generated, there is some f_ξ such that for every $f_\xi \leq_{bd} g$, $\cap_{\alpha<\kappa} A^\xi_{g(\alpha)} \in V_\xi$. Find a function f^* such that for every $i \in \{0\} \cup A$, $f_i \leq_{bd} f^*$ and also $\mathrm{id} \leq_{bd} f^*$.

Let $f^* \leq_{bd} g$. Since $f_0 \leq_{bd} g$, $A_g := \cap_{\alpha<\kappa} A_{g(\alpha)} \in U$ and $A_g \subseteq A$. Fix $\xi \in A_g$. We have that $f_\xi <_{bd} g$ and therefore $\cap_{\alpha<\kappa} Z^\xi_{g(\alpha)} \in V_\xi$. Since $\mathrm{id} \leq_{bd} g$, there is some $\alpha^* < \kappa$ such that $\theta_\xi < g(\alpha)$ whenever $\alpha^* \leq \alpha$. It follows that for every $\alpha^* \leq \alpha$, $Z^\xi_{g(\alpha)} = (Y_{g(\alpha)})_\xi$ and thus

$$(\cap_{\beta<\alpha^*}(Y_{g(\alpha)})_\xi) \cap (\cap_{\alpha<\kappa} Z^\xi_{g(\alpha)}) \subseteq \cap_{\alpha<\kappa}(Y_{g(\alpha)})_\xi =: W_\xi$$

It follows that $W_\xi \in V_\xi$. Letting $X^* = \cup_{\xi\in A_g}\{\xi\} \times W_\xi$, we conclude that $X^* \in \sum_U V_\xi$ and also $X^* \subseteq \cap_{\alpha<\kappa} Y_{g(\alpha)} \subseteq \cap_{\alpha<\kappa} X_{g(\alpha)}$, concluding that $\cap_{\alpha<\kappa} X_{g(\alpha)} \in \sum_U V_\xi$.

□

Remark 6.11. Note that if $\phi : \kappa \times \kappa \to \kappa$ is any bijection and $(A_i)_{i<\kappa}$ is a sequence of sets such $\lim_{\alpha\to\kappa} A_\alpha = A$, then $\lim_{\alpha\to\kappa} \phi'' A_\alpha = \phi'' A$, as there are many points $\mu < \kappa$ such that $\phi \restriction \mu \times \mu : \mu \times \mu \to \mu$. In particular, transferring a basically generated ultrafilter on $\kappa \times \kappa$ (such as in the previous theorem) to an ultrafilter on κ will remain basically generated and thus non-Galvin.

We point out that there are ultrafilters on ω which are not basically generated and yet are not Tukey-top. The first example of this was seen in work of Blass, Dobrinen and Raghavan [11]. This particular ultrafilter was then shown by the second author to have Tukey type which is the immediate successor of the Ramsey ultrafilter below it; further, hierarchies of non-basically generated ultrafilters whose cofinal types form downwards closed chains of any finite length were also constructed [13]. Thus, it seems unlikely that basically generated ultrafilters on measurable cardinals exactly capture non-Tukey topness. However, no such examples are known at this time, so we ask:

Question 6.12. Is it consistent that there are ultrafilters over a measurable cardinal which are not Tukey-top and also not basically generated?

7. Models of the Tukey-order

In this section, we study the Tukey classes of some well-known models of measurable cardinals. We start with perhaps the best known such model, $L[U]$ which was studied extensively by K. Kunen in [44].

[9]Note that it is possible that $(X)_\xi$ is empty but then all the $(Y_\alpha)_\xi$'s will be empty from a certain point.

Theorem 7.1. *In $L[U]$ there is no ultrafilter among the σ-complete ultrafilters over κ which is Tukey-top.*

Proof. In [9], Gitik and the first author observed that in $L[U]$ every κ (or even σ) complete ultrafilter satisfies the Galvin property so in particular, by Theorem 2.4, none of them are Tukey-top. □

Recall that in Section 5, we showed that for normal ultrafilters U, W we have $U \cdot W \equiv_T U \times W$ (and also $U^n \equiv_T U$). This can be used to say more about the model of $L[U]$:

Theorem 7.2. *Let U be a normal κ-complete ultrafilter over κ. In $L[U]$, the σ-complete ultrafilters over κ form a single Tukey class which is the union of ω-many Rudin Keisler equivalence classes.*

Proof. Since κ is the unique measurable in $L[U]$, every σ-complete ultrafilter is a κ-complete ultrafilter. It is well known that every κ-complete ultrafilter $W \in L[U]$ is RK isomorphic to U^n for some $n < \omega$. In particular, by Theorem 5.7, $W \equiv_T U^n \equiv_T U$. Hence the Tukey equivalence class of U is composed of the ω-many RK equivalence classes with representatives U^n. □

Remark 7.3. In contrast, we point out that it is still unknown whether there is exactly one Tukey type for ultrafilters over ω in $L(\mathbb{R})[U]$, the Solovay model extended by a forced Ramsey ultrafilter U on ω.

Although we do not have a Tukey-Top ultrafilter in $L[U]$, we do have a maximal class among the classes of κ-complete ultrafilters. As we will see in a moment, from larger cardinals we can obtain more complicated structures in the Tukey-order. In order to do that, let us establish a connection between the Mitchell order and the Tukey order.

Definition 7.4. Let U, W be σ-complete ultrafilters. We say that U is *Mitchell below* W, and denote it by $U \triangleleft W$, if $U \in M_W$, where M_W is the (transitive collapse of the) ultrapower V^κ/W.

Similar to the Rudin-Keisler order, in the case of normal ultrafilters, the Tukey order is in some sense orthogonal to the Mitchell order:

Theorem 7.5. *Suppose that $U \triangleleft W$ are two κ-complete ultrafilters over κ and W is a p-point. Then $\neg(W \leq_T U)$.*

Proof. Since W is p-point, by Theorem 4.1, there is $f : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ a function which determines a continuous cofinal map from U to W. Note that $f \in M_W$ (since M_W is closed under κ-sequences and $\mathrm{crit}(j_W) = \kappa$). Also since $U \triangleleft W$, $U \in M_W$ and therefore from f and U we can reproduce $W \in M_W$. This is known to be impossible (see for example [36, Lemma 17.9.ii]). □

Corollary 7.6. *Suppose that $U_1 \triangleleft U_2 \triangleleft \cdots \triangleleft U_n$ are any normal ultrafilters, then $\neg(U_n \leq_T U_1 \times \cdots \times U_{n-1})$.*

Proof. In M_{U_n} we have the ultrafilters $U_1, \ldots, U_{n-1}$. If $U_n \leq_T U_1 \times \cdots \times U_{n-1}$, then this would be witnessed by a continuous map (Theorem 4.4) and the previous argument applies. □

Corollary 7.7. *Suppose that $o(\kappa) = \alpha$ for $\alpha \leq \omega$, then there is a Tukey-chain of κ-complete ultrafilters of order type α.*

Proof. Indeed, let $\langle U_n \mid n < \alpha\rangle$ be an increasing chain in the Mitchell order. Then $U_1 \leq_T U_1 \cdot U_2 \leq_T \ldots$ is a $\leq_T$-chain. To see that this is strictly increasing, suppose toward a contradiction that $U_1 \cdot \ldots \cdot U_n \equiv_T U_1 \cdot \ldots \cdot U_{n+1}$. Then $U_1 \cdot \ldots \cdot U_n \equiv_T U_1 \times \cdots \times U_n \geq_T U_{n+1}$, contradicting the previous corollary. □

Corollary 7.8. *Let $L[\vec{U}]$ be the Mitchell model for $o(\kappa) = \omega$ and let $\langle U_n \mid n < \omega\rangle$ be the $\lhd$-increasing sequence of ultrafilter on κ. Then the sequence $\langle [U_1 \times \cdots \times U_n]_T \mid n < \omega\rangle$ is strictly increasing, cofinal and unbounded in the Tukey order. In particular, there is no maximal Tukey class.*

In order to get a chain of order type $\omega + 1$, we can start with the large cardinal assumption that κ is a κ-compact cardinal (and in particular $o(\kappa) \geq \omega$). By Theorem 2.4, there is a κ-complete ultrafilter U such that $\neg \mathrm{Gal}(U, \kappa, 2^\kappa)$ and such an ultrafilter is Tukey-top. Hence we can extend the chain from the previous corollary by one more ultrafilter.

Actually, we can get such a chain from a much weaker assumption. Assume that the Rudin-Keisler order is σ-closed, i.e., given a Rudin-Keisler increasing sequence $\langle W_n \mid n < \omega\rangle$ of κ-complete ultrafilters on κ, there exists an ultrafilter W such that for each n, $W_n \leq_{RK} W$.

Proposition 7.9. *If $o(\kappa) \geq \omega$ and the Rudin-Keisler order is σ-closed on κ then there is a Tukey chain of length $\omega + 1$.*

Proof. Similar to the previous construction, but the top ultrafilter will be any Rudin-Keisler bound for the sequence $\langle U_1 \cdot \ldots \cdot U_n \mid n < \omega\rangle$. Since Rudin-Keisler reduction implies Tukey reduction, we are done. □

The assumption that the Rudin-Keisler order is σ-closed follows from the ω-*gluing property*, due to Poveda and Hayut [34, Lemma 4.1]. In this paper, Poveda and Hayut proved that the ω-gluing property is equiconsistent with $o(\kappa) = \omega_1$ which is much less than κ being κ-compact.

Finally, let us move back to the optimal assumption of a measurable cardinal. Gitik and the first author [10] constructed a model with a κ-complete ultrafilter over κ such that $\neg \mathrm{Gal}(U, \kappa, 2^\kappa)$ from the minimal large cardinal assumption (i.e. a measurable cardinal). Thus in this model, we have a chain of length 2. Finally, let us prove that using the usual Kunen-Paris construction [47] of a model with many distinct normal ultrafilters we have a chain or ordertype $\omega + 1$:

Theorem 7.10. *Assume GCH and that κ is a measurable cardinal. Then there is a generic extension where GCH holds and there is a Tukey-chain of order type $\omega + 1$.*

Proof. Consider the forcing from [10], namely, let $\mathbb{P} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa$ such that $\mathbb{P}_\kappa$ is just an Easton support iteration of $\mathrm{Cohen}(\alpha, \alpha^+)$ for inaccessible α's and trivial otherwise. Also, $\mathbb{Q}_\kappa$ is a $\mathbb{P}_\kappa$-name for $\mathrm{Cohen}(\kappa, \kappa^+)$. Let $G = G_\kappa * g$ be a V-generic filter for $\mathbb{P}$. Then by [10, Theorem 2.6], $2^\kappa = \kappa^+$ and in $V[G]$ there is a κ-complete ultrafilter W over κ such that $\neg \mathrm{Gal}(W, \kappa, 2^\kappa)$. Over $V[G]$ let us force with $\mathrm{Cohen}(\kappa^+, \kappa^{+3})$, let H be $V[G]$-generic for this forcing. Then in $V[G, H]$, we note by the closure of $\mathrm{Cohen}(\kappa^+, \kappa^{+3})$, W is still an ultrafilter in $V[G, H]$ and $\neg \mathrm{Gal}(W, \kappa, 2^\kappa)$ holds. Also, the usual Kunen-Paris argument shows that in $V[G, H]$ there are $2^{(2^\kappa)} = 2^{\kappa^+} = \kappa^{+3}$-many normal ultrafilters over κ. Now let us apply Corollary 4.7 to conclude that there are $(2^\kappa)^+$-many non-normal ultrafilters which are Tukey incomparable. Take any ω-many of those ultrafilters; the same chain

of products as before produces a Tukey strictly increasing chain and the top one which is the non-Galvin one. □

Kuratowski and Sierpiński [48] (See also [65, Thm. 46.1]) proved the following:

Lemma 7.11. *For any cardinal λ and $k < \omega$, given any function $f : [\lambda^{+k}]^k \to [\lambda^{+k}]^{<\lambda}$ such that for any $x_1, \dots, x_k$, $\{x_1, \dots, x_k\} \cap f(x_1, \dots, x_k) = \emptyset$, there is a free set Y of size $k+1$, namely, there are distinct $y_1, \dots, y_{k+1} \in \lambda^{+k}$ such that for every $x_1, \dots, x_k \in [\{y_1, \dots, y_{k+1}\}]^k$, $f(x_1, \dots, x_k) \cap \{y_1, \dots, y_{k+1}\} = \emptyset$.*

Corollary 7.12. *For every $0 < k < \omega$ it is consistent that $(P(k) \setminus \{\emptyset\}, \subseteq)$ can be embedded into the Tukey classes.*

Proof. Assume $k > 0$. Consider the Kunen-Paris model M where $2^{2^\kappa} = \kappa^{+k+2}$ and $2^\kappa = \kappa^+$. In particular, the set of normal ultrafilters $\beta_{\rm nor}(\kappa)$ has size κ^{+k+2} in M. Consider the function: $F : [\beta_{\rm nor}(\kappa)]^k \to P(\beta_{\rm nor}(\kappa))$ defined by

$$(*) \quad F(U_1, \dots, U_k) = \{W \in \beta_{\rm nor}(\kappa) \mid W \leq_T U_1 \times \cdots \times U_k\} \setminus \{U_1, \dots, U_k\}$$

By Theorem 4.4 $|F(U_1, \dots, U_k)| \leq 2^\kappa = \kappa^+$. Apply lemma 7.11 for $\lambda = (2^\kappa)^+ = \kappa^{+2}$ to find a free set of ultrafilters $\{U_1, \dots, U_{k+1}\}$ for the function F of size $k+1$. It follows that for each $1 \leq i \leq k+1$,

$$\neg(U_i \leq_T U_1 \times \cdots \times U_{i-1} \times U_{i+1} \times \cdots \times U_{k+1}).$$

It follows that if $a \subsetneq b \subseteq \{1, \dots, k+1\}$, then $\prod_{i\in a} U_i <_T \prod_{j\in b} U_j$. Just otherwise, take any $j \in b \setminus a$, then $U_j \leq_T \prod_{i\in a} U_i \leq_T \prod_{i\neq j} U_i$ contradicting $(*)$. For the same reason, it follows that if $a \neq b$ then $\prod_{i\in a} U_i \not\equiv_T \prod_{j\in b} U_j$. We conclude that the map g defined on $P(\{1, \dots, k+1\})$ by $g(a) = [\prod_{i\in a} U_i]_T$ is an order embedding. □

Remark 7.13. We point out some similar results on ω and the contrast between embedding and exact Tukey structures. Dobrinen, Mijares, and Trujillo [15] showed that it is consistent with ZFC that $([\omega]^{<\omega}, \subseteq)$ appears as an *initial* (downwards closed in the Tukey order) structure among the cofinal types of p-points on ω. Raghavan and Shelah [56] later showed that it is consistent that $\mathcal{P}(\omega)/\text{fin}$ embeds into the Tukey types of p-points. Except for Theorem 7.2, the above results in this section so far are of this latter sort, showing that certain partial orders embed into the Tukey types of a given structure. The Ultrafilter Axiom and the minimality conjecture below will ensure more examples where the Tukey structure is *exactly* known.

By a result of Todorcevic in [58], every Ramsey ultrafilter on ω is Tukey minimal. The proof utilizes a theorem of Pudlák and Rödl [55] canonizing equivalence relations on barriers; this theorem does not generalize to uncountable cardinals. Hence a crucial open problem is the following:

Question 7.14. Are normal ultrafilters on a measurable cardinal κ Tukey minimal among κ complete ultrafilters?

We conjecture a positive answer for this question, and call this the *minimality conjecture*. The *general minimality conjecture* is the assertion that if $U_1, .., U_n, W$ are normal ultrafilters over κ and $W \notin \{U_1, \dots, U_n\}$ then $\neg(W \leq_T \prod_{i=1}^n U_i)$

There is good reason to believe that the general minimality conjecture holds. Indeed, it is well known that normal ultrafilters are RK-minimal and the only

ultrafilters which are RK below a product of normals are the ultrafilters participating in the product. Also, by Theorem 4.4, if $f : \prod_{i=1}^{n} U_i \to W$ witness that $U_1 \cdot \ldots \cdot U_n \geq_T W$ then f is actually determined by a function from $([\kappa]^{<\kappa})^n$ to $[\kappa]^{<\kappa}$. On ω it is known how to make such maps into Rudin-Keisler projections under various hypotheses on the U_i and W (see [58] and [14]).

Example 7.15. As we have seen above, if $U_0 \lhd U_1$ then $\neg(U_1 \lhd U_0)$. A natural approach for trying to produce a counter-example for the minimality conjecture is to try to prove that $U_0 \lhd U_1$ implies $U_0 \leq_T U_1$ (where U_0, U_1 are assumed to be normal). There is a natural map for this. Since $U_0 \in M_{U_1}$ there is a function $f : \kappa \to V_\kappa$ such that $[f]_{U_1} = U_0$. Recall that U_1 is normal and therefore $\kappa = [id]_{U_1}$. By Loś Theorem, since

$$M_{U_1} \models [f]_{U_1} \text{ is a normal ultrafilter over } [id]_{U_1},$$

we have that $E = \{\alpha < \kappa \mid f(\alpha) \text{ is a normal ultrafilter over } \alpha\} \in U_1$. For every $\alpha \in E$, denote $f(\alpha) = U_{0,\alpha}$.

Let us define a function $f : U_0 \to U_1$ by

$$f(X) = \{\alpha < \kappa \mid X \cap \alpha \in U_{0,\alpha}\}$$

This is well defined since for every $X \subseteq \kappa$, $X \in U_0$ iff $\{\alpha < \kappa \mid X \cap \alpha \in U_{0,\alpha}\} \in U_1$. We claim that this f is not unbounded. Indeed, let $A_\alpha = \kappa \setminus \{\alpha\}$, then $\{A_\alpha \mid \alpha < \kappa\}$ is unbounded in U_0 since $\cap_{\alpha<\kappa} A_\alpha = \emptyset$. On the other hand, for every α, $f(A_\alpha) = \{\alpha < \kappa \mid A_\alpha \cap \alpha \in U_{0,\alpha}\} = E$ and therefore $f''\{A_\alpha \mid \alpha < \kappa\}$ is bounded.

The minimality conjecture can be used to completely characterize the Tukey order in the canonical inner models for relatively small cardinals (and possibly further). For example, we can classify the Tukey classes in the Mitchell models $L[\vec{U}]$ where $\vec{U}$ is a coherent sequence of normal measures [53] of length which is not too long (below κ^{++}). For this we will need the following easy corollary from the general minimality conjecture.

Corollary 7.16. *Let $U_1, U_2, \ldots, U_n$ be distinct normal ultrafilters. Let $1 \leq i \leq n$ and $\emptyset \neq a \subseteq \{1, \ldots, n\}$.*

1. *If $i \in a$ then $U_i \leq_{RK} \prod_{j\in a} U_j$.*
2. *If the general minimality conjecture holds then for $i \notin a$, U_i and $\prod_{j\in a} U_j$ are Tukey incomparable.*

Proof. (1) is clear. For (2), we note that by the minimality assumption $\neg(U_i \leq_T \prod_{j\in a} U_j)$ and also $\neg(U_i \geq_T \prod_{j\in a} U_j)$. Otherwise since $a \neq \emptyset$, there is $j_0 \in a$ and $j_0 \neq i$, we get that $U_i \geq_T \prod_{j\in a} U_j \geq_T U_{j_0}$. But this contradicts the minimality conjecture since $U_i \neq U_{j_0}$. □

Next, let us state some known results which generalize Kunen's classification of ultrafilters in $L[U]$. The framework in which we are going to analyze the structure of the Tukey classes is due to G. Goldberg [29], who discovered the *Ultrapower Axiom*, which holds in all known canonical inner models, and in particular in the models $L[\vec{U}]$. Goldberg showed that in the context of UA, the structure of κ-complete ultrafilters is quite rigid and well understood. The first result which is relevant for our needs is the generalization of Kunen's classification of ultrafilters in the model $L[U]$. This is a direct corollary of [29, Lemma 5.3.20]:

Lemma 7.17 (Goldberg)**.** *Assume* UA *and that* κ *is a measurable cardinal with* $o(\kappa) < 2^{(2^\kappa)}$*. Then every* κ*-complete ultrafilter is Rudin-Keisler equivalent to a finite product of normal ultrafilters.*

The following remarkable theorem of Goldberg is also relevant for us:

Theorem 7.18 (Goldberg)**.** *Under* UA*, the Mitchell order is linear (when restricted to normal measures).*

Let us prove that the above results suffice to completely characterize the Tukey classes under UA and small enough large cardinal assumption:

Theorem 7.19. *Assume* UA *and the general minimality conjecture. Suppose that* κ *is a measurable cardinal with* $o(\kappa) < 2^{(2^\kappa)}$*. Then the Tukey classes of ultrafilters is isomorphic to* $([o(\kappa)]^{<\omega}, \subseteq)$*.*

Proof. By theorem 7.18 let $\langle U_\alpha \mid \alpha < \gamma\rangle$ be the $\lhd$-increasing sequence of all the normal ultrafilters, where $\gamma = o(\kappa)$. Next, by Lemma 7.17, any κ-compete ultrafilter is Rudin-Keisler isomorphic to a finite product of normal ultrafilters which are listed in the sequence $\langle U_\alpha \mid \alpha < \gamma\rangle$. As in Theorem 7.2, we have that for any ultrafilter W on κ, $[W]_T = [\prod_{i\in a} U_i]_T$ for some finite set $a \subseteq \gamma$. Let us check that the function $f : [\gamma]^{<\kappa} \to \{[U]_T : U \in \mathcal{U}\}$, where $\mathcal{U}$ is the set of κ-complete ultrafilters on κ, defined by $f(a) = [\prod_{i\in a} U_i]_T$ is an order isomorphism. Clearly if $a \subseteq b$ then $\prod_{i\in a} U_i \leq_{RK} \prod_{i\in b} U_i$. If $a \not\subseteq b$ then we cannot have $\prod_{i\in a} U_i \leq_T \prod_{i\in b} U_i$ since this will contradict Corollary 7.16. Also if $a \subsetneq b$, then again Corollary 7.16 implies that $\prod_{i\in a} U_i <_T \prod_{i\in b} U_i$. □

Since the models $L[\vec{U}]$ are models of UA, we obtain the following corollary:

Corollary 7.20. *Assume the general minimality conjecture and that* $\vec{U}$ *is a coherent sequence for* $o(\kappa) = \gamma < 2^{2^\kappa}$*. Then in* $L[\vec{U}]$*, the Tukey order on equivalence classes of* κ*-complete ultrafilters is isomorphic to* $([\gamma]^{<\omega}, \subseteq)$*.*

Example 7.21. Suppose that $\vec{U}$ is a coherent sequence for $o(\kappa) = 2$. Then in $L[\vec{U}]$ there are 2 minimal Tukey classes for κ corresponding to $U(\kappa, 0), U(\kappa, 1)$, and there is a third class corresponding to $U(\kappa, 0)\cdot U(\kappa, 1)$ strictly above $U(\kappa, 0), U(\kappa, 1)$ (and that's it!)

Remark 7.22. In the case of $(U, \supseteq^*)$, in the models $L[\vec{U}]$ there is a single class of ultrafilters since any two normal ultrafilters are equivalent (by 3.11) and therefore products are are also equivalent.

8. Open problems and further directions

In an attempt to obtain a model with longer Tukey-chains, Gitik's construction from [26] turning a Mitchell increasing sequence of ultrafilters into a Rudin-Keisler increasing sequence seems like a promising direction to generate longer chains of non-Tukey equivalent ultrafilters:

Question 8.1. In the model of [26], is there a long Tukey-chain?

Note that the extensions of the ultrafilters are not going to be q-point nor p-point and therefore our results about such ultrafilters are not applicable.

Todorcevic classified the ultrafilters which are Tukey below a Ramsey ultrafilter U on ω to consist exactly of the Rudin-Keisler classes of countable Fubini iterates

of U (Theorem 24, [58]). As all countable Fubini iterates of a Ramsey ultrafilter are Tukey equivalent, it follows that Ramsey ultrafilters on ω are Tukey minimal. We do not know whether any of these statements generalize to measurable cardinals:

Question 8.2. Suppose that U is normal. Is it true that whenever $V \leq_T U$ then $V \equiv_{RK} U^n$ for some $n < \omega$?

This is certainly the case under UA, but is it true in general? Note also that the minimality conjecture from Question 7.14. The problem with trying to directly extend Theorem 24 in [58] to normal ultrafilters, is that Todorcevic's proof heavily relies on combinatorial features of fronts and barriers on ω such as the Nash-Williams [54] and Pudlák-Rödl [55] Theorems. These Theorems are not available on measurable cardinals, as their relevant generalizations would in particular imply the existence of large homogeneous sets for colorings $f : [\kappa]^\omega \to 2$. This is known to be impossible by the classical example of Erdős and Rado [19] (see also [42, Proposition 7.1])

In the Kunen-Paris model we used the Kuratowski-Sierpiński characterization of λ^{+k}. It is known that this is best possible if we consider any possible map. However, we apply this characterization to the specific map

$$F(U_1, \ldots, U_k) = \{W \mid W \leq_T U_1 \times \cdots \times U_k\} \setminus \{U_1, \ldots, U_k\}.$$

This map has further properties, for example, if $x_1, \ldots, x_k \in F(U_1, \ldots, U_k)$ then $F(x_1, \ldots, x_k) \subseteq F(U_1, \ldots, U_k)$.

Question 8.3. How large of a free set can we guarantee for the function F? Can we guarantee an infinite set? What about the case where $F : [\beta_{\mathrm{nor}}(\kappa)]^{<\omega} \to P(\beta_{\mathrm{nor}}(\kappa))$?

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References

1. Uri Abraham and Saharon Shelah, *On the intersection of closed unbounded sets*, The Journal of Symbolic Logic **51** (1986), no. 1, 180–189.
2. Omer Ben-Neria, Shimon Garti, and Yair Hayut, *Weak prediction principles*, Fundamenta Mathematicae **245** (2019), no. 2, 109–125.
3. Tom Benhamou, *Prikry forcing and tree Prikry forcing of various filters*, Archive if Mathematical Logic **58** (2019), 787–817.
4. ———, *Saturation properties in canonical inner models*, submitted (2023), arXiv.
5. Tom Benhamou, Shimon Garti, Moti Gitik, and Alejandro Poveda, *Non-Galvin filters*, Journal of Mahtematical Logic (2022), to appear.
6. Tom Benhamou, Shimon Garti, and Alejandro Poveda, *Galvin's property at large cardinals and an application to partition calculus*, Israel Journal of Mathematics (2022), to appear.
7. ———, *Negating the Galvin property*, Journal of the London Mathematical Society **108** (2023), no. 1, 190–237.
8. Tom Benhamou, Shimon Garti, and Saharon Sehlah, *Kurepa trees and the failure of the Galvin property*, Proceedings of the American Mathematical Society **151** (2023), 1301–1309.
9. Tom Benhamou and Moti Gitik, *Intermediate models of Magidor-Radin forcing-II*, Annals of Pure and Applied Logic **173** (2022), 103107.
10. ———, *On Cohen and Prikry forcing notions*, The Journal of Symbolic Logic (2023), 1–47.

11. Andreas Blass, Natasha Dobrinen, and Dilip Raghavan, *The next best thing to a p-point*, Journal of Symbolic Logic **80** (15), no. 3, 866–900.
12. A.D. Brooke-Taylor, V. Fischer, S.D. Friedman, and D.C. Montoya, *Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model*, Annals of Pure and Applied Logic **168** (2017), no. 1, 37–49.
13. Natasha Dobrinen, *High dimensional Ellentuck spaces and initial chains in the Tukey structure of non-p-points*, Journal of Symbolic Logic **81** (2016), no. 1, 237–263.
14. ———, *Continuous and other finitely generated canonical cofinal maps on ultrafilters*, Fundamenta Mathematicae **249** (2020), no. 2, 111–147.
15. Natasha Dobrinen, José G. Mijares, and Timothy Trujillo, *Topological Ramsey spaces from Fraïssé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points*, Archive for Mathematical Logic, special issue in honor of James E. Baumgartner **56** (2017), no. 7-8, 733–782, (Invited submission).
16. Natasha Dobrinen and Stevo Todorcevic, *Tukey types of ultrafilters*, Illinois Journal of Mathematics **55** (2011), no. 3, 907–951.
17. ———, *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 1*, Transactions of the American Mathematical Society **366** (2014), no. 3, 1659–1684.
18. ———, *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 2*, Transactions of the American Mathematical Society **367** (2015), no. 7, 4627–4659.
19. P. Erdős and R. Rado, *Combinatorial theorems on classifications of subsets of a given set*, Proceedings of the London Mathematical Society **s3-2** (1952), no. 1, 417–439.
20. Paul Erdős, András Hajnal, Attila Máté, and Richard Rado, *Combinatorial set theory: partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, vol. 106, North-Holland Publishing Co., Amsterdam, 1984. MR 795592
21. David Fremlin, *The partially ordered sets of measure theory and Tukey's ordering*, Note di Matematica **XI** (1991), 177–214.
22. Shimon Garti, *Weak diamond and Galvin's property*, Periodica Mathematica Hungarica **74** (2017), no. 1, 128–136. MR 3604115
23. ———, *Tiltan*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 4, 351–359. MR 3787522
24. Shimon Garti, Yair Hayut, Haim Horowitz, and Menachem Magidor, *Forcing axioms and the Galvin number*, Periodica Mathematica Hungarica **84** (2022), 250–258.
25. Shimon Garti, Menachem Magidor, and Saharon Shelah, *On the spectrum of characters of ultrafilters*, Notre Dame Journal of Formal Logic **59** (2018), no. 3, 371 – 379.
26. Moti Gitik, *Changing Cofinalities and the Nonstationary Ideal*, Israel Journal of Mathematics **56** (1986), no. 3, 280–314.
27. ———, *On density of old sets in Prikry type extensions*, Proceedings of the American Mathematical Society **145** (2017), no. 2, 881–887.
28. Moti Gitik and Saharon Shelah, *On densities of box products*, Topology and its Applications **88** (1996), 219–237.
29. Gabriel Goldberg, *The ultrapower axiom*, Berlin, Boston:De Gruyter, 2022.
30. András Hajnal, *On some combinatorial problems involving large cardinals*, Fundamenta Mathematicae **69** (1970), no. 1, 39–53.
31. Felix Hausdorff, *Über zwei sätze von g. fichtenholz und l. kantorovitch*, Studia Mathematica **6** (1936), no. 1, 18–19 (ger).
32. Yair Hayut, *Partial strong compactness and squares*, Fundamenta Mathematicae **246** (2019), 193–204.
33. ———, *A note on the filter extension property*, Procceedings of the Amererical Mathematical Society **148** (2020), 3129–3133.
34. Yair Hayut and Alejandro Poveda, *The gluing property*, preprint (2022), arXiv:2212.03333.
35. John R. Isbell, *The category of cofinal types. II*, Transactions of the American Mathematical Society **116** (1965), 394–416.
36. Thomas Jech, *Set Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513
37. ———, *Stationary Sets*, Handbook of set theory, Springer, 2010, pp. 93–128.
38. I. Juhász, A. Verbeek, and N.S. Kroonenberg, *Cardinal functions in topology*, Mathematical Centre tracts, Mathematisch Centrum, 1979.

39. Istvan Juhász, *Remarks on a theorem of B. Pospíšil*, General Topology and its Relations to Modern Analysis and Algebra, Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967, pp. 205–206.
40. Akihiro Kanamori, *Ultrafilters over a measurable cardinal*, Annals of Mathematical Logic **10** (1976), 315–356.
41. ———, *On p-points over a measurable cardinal*, The Journal of Symbolic Logic **46** (1981), no. 1, 59–66.
42. ———, *The Higher Infinite*, Springer, 1994.
43. Péter Komjáth and Vilmos Totik, *Ultrafilters*, The American Mathematical Monthly **115** (2008), no. 1, 33–44.
44. Kenneth Kunen, *Some applications of iterated ultrapowers in set theory*, Annals of Mathematical Logic **1** (1970), 179–227.
45. ———, *Introduction to independence proofs*, North-Holand, 1980.
46. ———, *Weak p-points in n**, pp. 100–108, Birkhäuser Basel, Basel, 1993.
47. Kenneth Kunen and J.B. Paris, *Boolean extensions and measurable cardinals*, Annals of Mathematical Logic **2** (1971), no. 4, 359–377.
48. Casimir Kuratowski, *Sur une caractérisation des alephs*, Fundamenta Mathematicae **38** (1951), 14–17.
49. Borisa Kuzeljevic and Dilip Raghavan, *A long chain of p-points*, Journal of Mathematical Logic **19** (18), no. 1, 1850004, 38 pp.
50. David Milovich, *Tukey classes of ultrafilters on ω*, Topology Proceedings **32** (2008), 351–362.
51. ———, *Forbidden rectangles in compacta*, Topology and Its Applications **159** (2012), no. 14, 3180–3189.
52. ———, *Between homeomorphism type and Tukey type*, Topology and Its Applications **271** (2020), 106985, 12 pp.
53. William J. Mitchell, *Sets constructible from sequences of ultrafilters*, The Journal of Symbolic Logic **39** (1974), no. 1, 57–66.
54. C. St. J. A. Nash-Williams, *On well-quasi-ordering transfinite sequences*, Proceedings of the Cambridge Philosophical Society **61** (1965), 33–39.
55. Pavel Pudlák and Vojtech Rödl, *Partition theorems for systems of finite subsets of integers*, Discrete Mathematics **39** (1982), 67–73.
56. Dilip Raghavan and Saharon Shelah, *Embedding partial orders into the p-points under Rudin-Keisler and Tukey reducibility*, Transactions of the American Mathematical Society **369** (17), no. 6, 4433–4455.
57. ———, *A small ultrafilter number at smaller cardinals*, Archive for Mathematical Logic **59** (2020), 325–334.
58. Dilip Raghavan and Stevo Todorcevic, *Cofinal types of ultrafilters*, Annals of Pure and Applied Logic **163** (2012), no. 3, 185–199.
59. Dilip Raghavan and Jonathan L. Verner, *Chains of p-points*, Canadian Mathematics Bulletin **62** (2019), no. 4, 856–868.
60. Jürgen Schmidt, *Konfinalität*, Zeitschrift für Mathematische Logik und Grundlagen der Matematik **1** (1955), 271–303.
61. Slawomir Solecki and Stevo Todorcevic, *Cofinal types of topological directed orders*, Annales de L'Institut Fourier **54** (2004), no. 6, 1877–1911.
62. Stevo Todorcevic, *Directed sets and cofinal types*, Transactions of the American Mathematical Society **290** (1985), no. 2, 711–723.
63. ———, *A classification of transitive relations on ω_1*, Proceedings of the London Mathematical Society **(3) 73** (1996), no. 3, 501–533.
64. John W. Tukey, *Convergence and uniformity in topology*, Princeton University Press, 1940.
65. Neil H. Williams, Paul Erdös, András Hajnal, Attila Máté, and Richard Rado, *Combinatorial set theory: Partition relations for cardinals*, Combinatorial Set Theory: Partition Relations for Cardinals, 2012.

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