GALVIN'S PROPERTY AT LARGE CARDINALS AND AN APPLICATION TO PARTITION CALCULUS

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ABSTRACT. In the first part of this paper we produce models where κ is certain very large cardinal and every ground model κ-complete ultrafilter extends to a non-Galvin one. In the opposite direction, we also produce such models but this time every ground model κ-complete ultrafilter extends to a P-point ultrafilter, hence to a Galvin one. Finally, we apply Galvin's property to obtain consistently new instances of the classical problem in partition calculus $\lambda \to (\lambda, \omega+1)^2$ both in ZFC and ZF.

1. INTRODUCTION

Let κ be an uncountable regular cardinal and $\mathscr{F}$ a filter over it. We say that *Galvin's property* holds for $\mathscr{F}$ (in symbols, $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$) if every family[1] $\langle C_\gamma \mid \gamma < \kappa^+ \rangle \subseteq \mathscr{F}$ admits a subfamily $\langle C_{\gamma_i} \mid i < \kappa \rangle$ with the property that $\bigcap \{C_{\gamma_i} \mid i < \kappa\} \in \mathscr{F}$. It is customary to think about members of $\mathscr{F}$ as *large subsets of* κ in some prescribed sense. Bearing this in mind there is a natural interpretation of $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$ as a *saturation property*: namely, given any sufficiently long sequence of $\mathscr{F}$-large sets there is a long subsequence of it the intersection of which is still $\mathscr{F}$-large. This is, arguably, a striking feature of filters. Let us illustrate this through the *down-to-earth* example of $\mathscr{D}_{\aleph_1}$, the *club filter over* $\aleph_1$. Recall that a set $C \subseteq \aleph_1$ is a *club* if it is closed and unbounded in the order topology of $\aleph_1$. The club filter $\mathscr{D}_{\aleph_1}$ is thus the filter generated by the family of all clubs; to wit, the collection of all $X \subseteq \aleph_1$ containing some club $C \subseteq \aleph_1$. Note that the interval $(\alpha, \aleph_1)$ is a club for all $\alpha < \aleph_1$. Also, it is evident that the intersection of $\langle (\alpha, \aleph_1) \mid \alpha < \aleph_1 \rangle$ is empty, hence not a member of $\mathscr{D}_{\aleph_1}$. This shows that $\mathscr{D}_{\aleph_1}$ is not closed under uncountable intersections. However, a celebrated result of Galvin [16] establishes that, assuming the *Continuum Hypothesis*, $\mathrm{Gal}(\mathscr{D}_{\aleph_1}, \aleph_1, \aleph_2)$ holds. Morally speaking the reason why the above intersection is empty is that we did not started with enough club sets (i.e., $\aleph_2$-many).

Galvin's property connects very naturally with other well-studied properties of filters. We already ellaborated on the parallelism between Galvin's

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[1] In the absence of the *Axiom of Choice* it is important to emphasize that a family is a *sequence* in which repetitions are allowed.

property and *completeness*, but it is also deeply vinculated to *normality*.[2] So much so that Galvin's theorem establishes that, under suitable cardinal arithmetic assumptions, normality entails Galvin's property; namely, if $\kappa^{<\kappa} = \kappa$ and $\mathscr{F}$ is a normal filter over κ then $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$ holds (see [4]).

Part of the motivation of this paper is to study if, in a setting where $\kappa^{<\kappa} = \kappa$, it is possible to have non-normal (ultra)filters where Galvin's property holds. We will elaborate on this issue later on during the introduction.

Prior entering into deeper set-theoretic waters we would like to mention some of the links between Galvin's property and other natural combinatorial principles. The use of filters and ultrafilters to establish theorems both in finite and infinitary combinatorics has a long story. Paradigmatic examples of this are *The pigeonhole principle*, the partition relation $\kappa \to (\kappa)^2_2$ for measurables cardinals, *The arrow's impossibility theorem* or *Ramsey's theorem*, among many others. See [31] for an account of interesting examples.

There is a deep connection between Galvin's property and *polarized partition relations*. It is a classical result of Hajnal [23] that if κ is a measurable cardinal then $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_\theta$ for every $\theta < \kappa$.[3] A modern proof of this using $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ instead can be found in [8, Proposition 3.3]. This latter proof actually yields the enhanced polarized relation saying that the second component of the monochromatic set is a member of $\mathscr{U}$ (i.e., $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\mathscr{U}}_\theta$). Moreover, the said argument is flexible that other stronger versions of Galvin's property –such as the ones considered [7]– entail stronger forms of the polarized partition relation. But, most interestingly, the above implication is reversible. In particular the stronger the Galvin property is the better polarized partition relation is obtained, and vice-versa.

Another interesting match between Galvin's property and infinite combinatorics arises in the context of *prediction principles*; especially in the study of the *diamond-like* ones. Instances of Galvin's property are inferred from the *Devlin-Shelah weak diamond* in [18], it is compared with the *very weak diamond* over small large cardinals in [5], and it is used to show that the *club principle* is strictly weaker than *the superclub principle* in [19].

The range of applicability of Galvin's property has not been limited to the borders of infinite combinatorics. There is also a recently-opened avenue of research in *Prikry-type forcings* where Galvin's property is being used remarkably. For instance, it was instrumental to classify the intermediate models of the Prikry and Magidor generic extensions [9]. Another application is that violating Galvin's property of $\mathscr{U}$ is necessary for the corresponding Prikry forcing to project onto Cohen forcing $\mathrm{Add}(\kappa, \kappa^+)$ [10].

The first part (§2) of the paper is motivated by an open problem from [8]. Recall that Galvin's theorem says that if $\kappa^{<\kappa} = \kappa$ and $\mathscr{F}$ is a normal filter

[2] $\mathscr{F}$ is normal if $\triangle_{\alpha<\kappa} F_\alpha := \{\beta < \kappa \mid \forall \alpha < \beta\, (\beta \in F_\alpha)\}$ for all $\langle F_\alpha \mid \alpha < \kappa\rangle \subseteq \mathscr{F}$.

[3] This stands for the following statement: For every coloring $c\colon \kappa^+ \times \kappa \to \theta$ there is $A \in [\kappa^+]^\kappa$ and $B \in [\kappa]^\kappa$ such that $c \restriction (A \times B)$ is constant.

over κ then $\mathrm{Gal}(\mathscr{F},\kappa,\kappa^+)$ holds. Abraham and Shelah [1] proved that the assumption $\kappa^{<\kappa}=\kappa$ is necessary for successors of regulars and the authors [7] recently showed the same for successors of singulars. Thereby, Galvin's property is sensitive to two parameters: the cardinal-arithmetics and normality of the filter. In this part of the paper we shall be interested in the case where κ is a (very) large cardinal, hence in a context where $\kappa^{<\kappa}=\kappa$ is given from the outset.[4] In the previous situation it is meaningful to ask whether a non-normal κ-complete ultrafilter $\mathscr{U}$ over κ may violate $\mathrm{Gal}(\mathscr{U},\kappa,\kappa^+)$ - this was answered in [8]. Specifically, starting with a supercompact cardinal κ, the authors produced a generic extension where $\mathrm{Gal}(\mathscr{U},\kappa,\kappa^+)$ fails for certain κ-complete ultrafilter $\mathscr{U}$. On the other hand, it is consistent that κ is measurable and every κ-complete ultrafilter $\mathscr{U}$ satisfies $\mathrm{Gal}(\mathscr{U},\kappa,\kappa^+)$. This can be demonstrated working inside Kunen's inner model $L[\mathscr{U}]$ ([9]). Unfortunately, this analysis is not available at stronger large cardinals as inner models are limited with their tolerance to them [29]. This invites to a study of Galvin's property of non-normal ultrafilters over very large cardinals; particularly at the level of supercompact cardinals, where the current inner model techniques break down.

The following related question was raised in [8]:

Question. *Is it consistent that* $\mathrm{Gal}(\mathscr{U},\kappa,\kappa^+)$ *holds for every* κ*-complete ultrafilter* $\mathscr{U}$ *over* κ *in a context where* κ *is supercompact?*

Impelled by this problem, in §2 we will analyze how to extend (non-necessarily normal) ultrafilters into others satisfying/violating Galvin's property. We commence by showing that the method of [10] is amenable to non-normal ultrafilters. This provides a way of extending every κ-complete ultrafilter $\mathscr{U}$ in the ground model V to another κ-complete ultrafilter $\mathscr{U}^*\in V[G]$ for which $\mathrm{Gal}(\mathscr{U}^*,\kappa,\kappa^+)$ fails (Proposition 2.1). Later we argue that this construction is amenable to preserve very large cardinals; such as supercompacts, $C^{(n)}$-extendibles and *Vopěnka's Principle* (Proposition 2.3). Thus our method is flexible enough to produce models where these large cardinals co-exist with a large variety of non-Glavin ultrafilters over them.

Continuing this vein we prove a result of an opposite nature. Specifically, our first main result in this section reads as follows:

Theorem. *Assume that the* GCH *holds and that* κ *is a huge cardinal.*

Then, there is an inaccessible cardinal $\mu>\kappa$ *and a generic extension of* V_μ *where the following hold:*

(1) κ *is supercompact;*
(2) *Every* κ*-complete ultrafilter* $\mathscr{U}\in V$ *extends to a* κ*-complete ultrafilter* $\mathscr{U}^*$ *that is Rudin-Keisler equivalent to a normal ultrafilter. In particular,* $\mathrm{Gal}(\mathscr{U}^*,\kappa,\kappa^+)$ *holds.*

[4]For an extensive account on the theory of large cardinals we refer to Kanamori's monograph [29].

The above is, however, a too severe way to convert an arbitrary κ-complete ultrafilter into a Galvin one. During the rest of this section we present alternative ways to achieve the same configuration without implementing such dramatic changes. Our first attempt takes place in Theorem 2.12. There we employ iterations of *Generalized Mathias forcing* over a Mahlo cardinal κ to show that every κ-complete filter $\mathscr{U} \in V$ over κ extends to a κ-complete filter $\mathscr{U}^* \in V[G]$ for which $\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$ holds. In order for $\mathscr{U}^*$ to be an ultrafilter we need to bear on much stronger large-cardinal assumptions. This is precisely the matter of our second main theorem:

Theorem. *Assume the* GCH *holds and suppose that κ is an almost huge cardinal with measurable target λ. For unboundedly-many cardinals $\delta \in (\kappa, \lambda)$ there is a forcing extension $V[G^*_\delta]$ where $2^\kappa = \delta$ and every ground model κ-complete ultrafilter over κ extends to a P-point ultrafilter in $V[G^*_\delta]$.*

In addition, there is a truncation of $V[G^*_\delta]$ where the same configuration holds and κ is supercompact there.[5]

The previous theorem builds upon previous work of Gitik and Shelah [22], which involves a sophiticated iteration of Generalized Mathias forcing. Since every P-point ultrafilter $\mathscr{U}$ satisfies $\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$ (Proposition 2.19) the above theorem gives what we were intended for. The section is after that concluded with a brief discussion on Galvin's property in the choiceless context (see §2.2). This will be used later in §3.

In the second part of this paper (§3) we will deepen in the links between measurable cardinals, Galvin's property and polarized partition relations. Some of these connections will be investigated under the so-called *Axiom of Determinancy*, AD. This is a powerful tool in the choiceless context, with deep structural consequences in measure theory, topology and, most relevant for us, in the study of partition relations.[6] For example, under AD both $\aleph_1$ and $\aleph_2$ are measurable by an unpublished result of Solovay [29]. In particular, by the afore-mention theorem of Hajnal, AD implies the partition relation $\binom{\omega_2}{\omega_1} \to \binom{\omega_2}{\omega_1}$. Moreover, the components of the witnessing monochromatic product can be taken from the pertinent normal ultrafilters over $\aleph_1$ and $\aleph_2$, respectively. Galvin's property at $\aleph_1$, under AD, follows from this optimal polarized relation.

In §3 of this paper we combine AD and apply Galvin's property to address a classical question of [16] about possible generalizations (both in ZFC and ZF+AD) of the *Erdös-Dushnik-Miller theorem*. The first such generalization is due to Erdös and Rado, who proved that $\lambda \to (\lambda, \omega)^2$ holds for every infinite cardinal λ. Recall that this is a shorthand for the following partition

[5] By a truncation of $V[G^*_\delta]$ we mean a ZFC model of the form $V[G^*_\delta]_\theta$.

[6] The *Axiom of Determinacy* is the statement that every $A \subseteq \mathbb{R}$ is determined. It is easy to see that AD contradicts AC, the *Axiom of Choice*. For more historical and technical details about AD see [29, §6]. We also refer to Jackson's excellent handbook chapter [26] to learn more about the implications of AD in partition calculus.

relation: Every coloring $c\colon [\lambda]^2 \to \{0,1\}$ admits either a 0-monochromatic of order-type λ or a 1-monochromatic set of order-type ω.

The extension of the Erdös-Dushnik-Miller theorem considered here is the one given by the relation $\lambda \to (\lambda, \omega+1)^2$. It is well-known that this relation fails when $\mathrm{cf}(\lambda) = \omega$ and that it holds when λ is a regular uncountable cardinal [16, Theorem 11.3]. Thus the remaining case to consider is when $\omega < \mathrm{cf}(\lambda) < \lambda$. The best known result in this direction is an exquisite theorem of Shelah establishing that if $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$ and $2^\kappa < \lambda$ then $\lambda \to (\lambda, \omega+1)^2$ (see [35]). In this paper we prove (in ZF) that the same partition relation follows upon replacing the cardinal arithmetic assumption "$2^\kappa < \lambda$" by an appropriate instance of Galvin's property:

Theorem (ZF)**.** *Suppose that $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$ and that κ carries a normal measure $\mathscr{U}$. In addition, suppose that $\langle \lambda_i \mid i < \kappa\rangle$ is a cofinal sequence in λ admitting a family $\langle \mathscr{U}_i \mid i < \kappa\rangle$ such that:*

- *$\mathscr{U}_i$ is a normal filter whose dual ideal contains $[\lambda_i]^{<\lambda_i}$;*
- *$\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda_i, \lambda_i)$ holds.*

Then, $\lambda \to (\lambda, \omega+1)^2$.

The above has some few interesting corollaries. The first is in ZFC:

Corollary (ZFC)**.** *Suppose that $\mathscr{U}$ is a normal measure over κ, λ is singular with $\mathrm{cf}(\lambda) = \kappa$ and that $\mathrm{Gal}(\mathscr{U}, \lambda_i, \lambda_i)$ holds for cofinaly many regular cardinals below λ. Then, $\lambda \to (\lambda, \omega+1)^2$*

It must be said that our Galvin-like assumption is trivial whenever $2^\kappa < \lambda$ and forceable when $2^\kappa > \lambda$. Thus, we get more positive instances of $\lambda \to (\lambda, \omega+1)^2$. We believe, however, that the relation $\lambda \nrightarrow (\lambda, \omega+1)^2$ is consistent as well. Actually, our result pinpoints which instances of Galvin's property should be violated in order to force this negative partition relation.

We already mentioned that AD makes small cardinals (e.g., ω_1 or ω_2) to exhibit large cardinal properties. This is a situation of particular interest for us as under this assumption Galvin's property holds for a wide variety of cardinals. In effect, in §2.2 we show that Galvin's property holds for the so-called *boldface* GCH *cardinals* and it is known that, under AD (resp. $\mathsf{AD} + V = L(\mathbb{R})$), every cardinal $\lambda < \Theta^{L(\mathbb{R})}$ (resp. $\lambda < \Theta$) is of that kind.

In the above-described context we were able to answer [16, Question 11.4] by proving the following:

Corollary ($\mathsf{AD} + V = L(\mathbb{R})$)**.** *Suppose that $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$. If λ is a limit of regular cardinals then $\lambda \to (\lambda, \omega+1)^2$.*

Corollary (AD)**.** *The relation $\aleph_{\omega_1} \to (\aleph_{\omega_1}, \omega+1)^2$ holds.*

Our notation is mostly standard. If $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ then $\mathscr{D}_\kappa$ denotes the club filter over κ. If $\kappa = \mathrm{cf}(\kappa) < \lambda$ then $S^\lambda_\kappa = \{\delta \in \lambda \mid \mathrm{cf}(\delta) = \kappa\}$. The arrow symbol $\lambda \to (\alpha, \beta)^2$ is a shorthand for the following statement: for every $f : [\lambda]^2 \to 2$ either there is a 0-monochoromatic subsets of λ of

order-type α or a 1-monochromtic subset of λ of order-type β. We say that $\binom{\alpha}{\beta} \to \binom{\gamma}{\delta}$ iff for every $c : \alpha \times \beta \to \{0,1\}$ there are $A \in [\alpha]^\gamma, B \in [\beta]^\delta$ for which $c \restriction (A \times B)$ is constant. We use Θ to denote

$$\sup\{\alpha \mid \text{There exists a mapping from } {}^{\omega}\omega \text{ onto } \alpha\}.$$

For ultrafilters ultrafilters $\mathscr{U}$ and $\mathscr{V}$ over a cardinal κ, $\mathscr{U}$ is said to be *Rudin-Keisler below* $\mathscr{V}$ (notationally, $\mathscr{U} \leq_{\mathrm{RK}} \mathscr{V}$) if there is $f\colon \kappa \to \kappa$ with

$$X \in \mathscr{U} \text{ iff } f^{-1}[X] \in \mathscr{V}.$$

The ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are *Rudin-Keisler equivalent* iff

$$\mathscr{U} \leq_{\mathrm{RK}} \mathscr{V} \text{ and } \mathscr{V} \leq_{\mathrm{RK}} \mathscr{U}.$$

Regarding set-theoretic Forcing we employ the *Israel forcing convention*, thus $p \leq q$ means that q is stronger than p. For background in partition calculus we refer the reader to [16] and [38].

2. Galvin's property at very large cardinals with and without choice

2.1. **Galvin's property at large cardinals.** In [9] the following result is proved: It is consistent that κ is a measurable cardinal and every κ-complete ultrafilter $\mathscr{U}$ over κ satisfies $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$. The proof strategy is based on analyzing Kunen's inner model $L[\mathscr{U}]$, where a complete classification of the σ-complete ultrafilters over κ is available. The key observation is that in this inner model every σ-complete ultrafilter over κ is Rudin-Keisler equivalent to a finite power of the normal measure $\mathscr{U}$. Since these ultrafilters do satisfy Galvin's property one concludes that $\mathrm{Gal}(\mathscr{V}, \kappa, \kappa^+)$ holds for every κ-complete ultrafilter $\mathscr{V} \in L[\mathscr{U}]$. This phenomenon suggests the following question: How about those (very) large cardinals for which there is no available canonical inner model? The epitome of this is supercompactness.

By work of the first two authors together with S. Shelah [8] it is consistent that a supercompact cardinal κ carries a κ-complete ultrafilter $\mathscr{U}$ which extends the club filter and $\neg\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$. Shortly after, the first author together with M. Gitik [10] improved this result by showing that just a measurable cardinal suffices to obtain such a non-Galvin ultrafilter $\mathscr{U}$.

The forthcoming proposition is a spin-off of the above-mentioned result in the context of general κ-complete ultrafilters:

Proposition 2.1. *Assume that the* GCH *holds and that κ is a measurable cardinal. Then the following is true in the generic extension of* [10, Theorem 2.6]*: Every κ-complete (not necessarily normal) ultrafilter $\mathscr{U}$ of the ground model extends to a κ-complete ultrafilter $\mathscr{U}^*$ such that $\neg\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$.*

Proof. The sought model is the generic extension by the Easton support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa \rangle$ such that for $\alpha \leq \kappa$, $\mathbb{Q}_\alpha$ is trivial unless α is inaccessible, in which case it is a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\underset{\sim}{\alpha}, \alpha^+)$.

Let $\mathscr{U} \in V$ be a κ-complete ultrafilter. Let us verify that we can adjust the argument in [10] to encompass non-normal ultrafilters. We will follow the notation from the original proof, considering the elementarity embeddings

$$j_1 := j_{\mathscr{U}} : V \to M_{\mathscr{U}} =: M_1,\ j_2 := j_{\mathscr{U}^2} : V \to M_{\mathscr{U}^2} =: M_2$$

$$k : M_1 \to M_2,\ j_2 = k \circ j_1$$

where k is simply the ultrapower embedding defined in $M_{\mathscr{U}}$ using the ultrafilter $j_1(\mathscr{U})$. Denote $\kappa_1 := j_1(\kappa)$ and $\kappa_2 := j_2(\kappa)$. Let $G := G_\kappa * g_\kappa$ be V-generic for $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ and $\langle f_{\alpha,\nu} \mid \nu < \alpha^+ \rangle$ be the $V[G \restriction \alpha]$-generic functions added by G for $\mathrm{Add}(\alpha, \alpha^+)$. The argument showing that these embeddings can be lifted (within $V[G]$) does not require normality and thus remains unaltered. Thus, we form $j_1 \subseteq j_1^* : V[G] \to M_1[j_1^*(G)]$, $k \subseteq k^* : M_1[j_1^*(G)] \to M_2[j_2^*(G)]$ and $j_2 \subseteq j_2^* := k^* \circ j_1^*$ such that:

(1) $j_2^*(G) = j_2^*(G_\kappa) * j_2^*(g_\kappa)$ and $j_2^*(g_\kappa) = \langle f_{\kappa_2,\nu} \mid \nu < (\kappa_2^+)^{M_2[j_2^*(G)]} \rangle$ is $M_2[j_2^*(G_\kappa)]$-generic for $\mathrm{Add}(\kappa_2, \kappa_2^+)^{M_2[j_2^*(G_\kappa)]}$.
(2) for every $\alpha \in j_1 \text{“} \kappa^+$, $f_{\kappa_2,k(\alpha)}(\kappa_1) = 1$.
(3) for every $\alpha \in \kappa_1^+ \setminus j_1 \text{“} \kappa^+$, $f_{\kappa_2,k(\alpha)}(\kappa_1) = 0$.
(4) $f_{\kappa_2,\kappa_1}(\kappa_1) = \kappa$.

Since we might be dealing with non-normal ultrafilters we need to alter the values of the generic f_2 at $\delta^* := [\mathrm{id}]_{j_1(\mathscr{U})}$, the generator of the second ultrapower. Also, we need to eliminate the generator of the first ultrapower $\delta := [\mathrm{id}]_{\mathscr{U}}$. Note that $\kappa \leq \delta < \kappa_1 \leq \delta^* < \kappa_2$. We require that:

(1) for every $\alpha \in j_1 \text{“} \kappa^+$, $f_{\kappa_2,k(\alpha)}(\delta^*) = 1$.
(2) for every $\alpha \in \kappa_1^+ \setminus j_1 \text{“} \kappa^+$, $f_{\kappa_2,k(\alpha)}(\delta^*) = 0$
(3) $f_{\kappa_2,\delta^*}(\delta^*) = \delta$.

Notice that the amount of coordinates that were altered is small. In particular, the counting/genericity arguments of [10, Lemma 2.7] relying on ZFC still go through. Next, derive in $V[G]$ the ultrafilter generated by j_1^* and $[\mathrm{id}]_{\mathscr{U}}$,

$$\mathscr{U}^* := \{X \subseteq \kappa \mid [\mathrm{id}]_{\mathscr{U}} \in j_1^*(X)\}$$

Note that $\mathscr{U} \subseteq \mathscr{U}^*$. Finally, let

$$\mathscr{W} := \{X \subseteq \kappa \mid [\mathrm{id}]_{j_1(\mathscr{U})} \in j_2^*(X)\} \in V[G].$$

Let us prove that $\mathscr{W}$ witnesses the statement of the proposition:

Claim 2.2. *$\mathscr{W}$ is a κ-complete ultrafilter over κ such that:*

(1) $\mathscr{U} \subseteq \mathscr{W}$.
(2) $\neg\mathrm{Gal}(\mathscr{W}, \kappa, \kappa^+)$.

Proof of claim. (1): If $A \in \mathscr{U}$ then $j_1(A) \in j_1(\mathscr{U})$, hence $[\mathrm{id}]_{j_1(\mathscr{U})} \in j_2(A)$ and thus $A \in \mathscr{W}$.

(2): Let us define the witness. For each $\alpha < \kappa^+$ let

$$A_\alpha := \{\nu < \kappa \mid f_{\kappa,\alpha}(\nu) = 1\}$$

then

$$j_2^*(A_\alpha) = \{\beta < \kappa_2 \mid f_{\kappa_2, j_2(\alpha)}(\beta) = 1\}.$$

Since $j_2(\alpha) = k(j_1(\alpha))$, our modifications of the generic give

$$f_{\kappa_2, j_2(\alpha)}([\mathrm{id}]_{j_1(\mathscr{U})}) = 1,$$

hence $[\mathrm{id}]_{j_1(\mathscr{U})} \in j_2^*(A_\alpha)$. Finally $A_\alpha \in \mathscr{W}$ by definition of $\mathscr{W}$. Before proving the failure of the Galvin's property, let us denote by $j_{\mathscr{W}} : V[G] \to M_{\mathscr{W}}$ the ultrapower embedding by $\mathscr{W}$ and $k_{\mathscr{W}} : M_{\mathscr{W}} \to M_2^*$ defined by $k_{\mathscr{W}}([f]_{\mathscr{W}}) := j_2^*(f)([id]_{j_1(\mathscr{U})})$ the factor map satisfying $k_{\mathscr{W}} \circ j_{\mathscr{W}} = j_2^*$.

We show that $k_{\mathscr{W}}$ is onto, hence the identity, and thus $j_2^* = j_{\mathscr{W}}$. In effect, if $A \in M_2[j_2^*(G)]$ then there is a name $\underset{\sim}{A} \in M_2$ with $A = (\underset{\sim}{A})_{j_2^*(G)}$. Since j_2 is the second ultrapower by $\mathscr{U}$, there is $f : [\kappa]^2 \to V$ such that $j_2(f)([\mathrm{id}]_{\mathscr{U}}, [\mathrm{id}]_{j_1(\mathscr{U})}) = \underset{\sim}{A}$. By Łös theorem, we can assume that $f(\alpha, \beta)$ is a $\mathbb{P}_{\kappa+1}$-name for every $(\alpha, \beta) \in [\kappa]^2$. In $V[G]$ let $f^*(\alpha) = (f(f_{\kappa,\alpha}(\alpha), \alpha))_G$.

Then,

$$k_{\mathscr{W}}([f^*]_W) = j_2^*(f^*)([\mathrm{id}]_{j_1(\mathscr{U})}) = (j_2(f)(f_{\kappa_2, [\mathrm{id}]_{j_1(\mathscr{U})}}([\mathrm{id}]_{j_1(\mathscr{U})}), [\mathrm{id}]_{j_1(\mathscr{U})}))_{j_2^*(G)}$$

$$= j_2(f)([\mathrm{id}]_{\mathscr{U}}, [\mathrm{id}]_{j_1(\mathscr{U})})_{j_2^*(G)} = (\underset{\sim}{A})_{j_2^*(G)} = A$$

Let $\langle A_{\alpha_i} \mid i < \kappa \rangle$ be any subfamily of length κ and $\kappa \leq \eta < [\mathrm{id}]_{\mathscr{W}} = [\mathrm{id}]_{j_1(\mathscr{U})}$. Denote $j_{\mathscr{W}}(\langle A_{\alpha_i} \mid i < \kappa \rangle) := \langle A'_{\alpha'_i} \mid i < j_{\mathscr{W}}(\kappa) \rangle$.

Pick any $\kappa \leq \eta < [\mathrm{id}]_{\mathscr{U}} < j_1(\kappa)$, then $\eta \notin j_1$“κ^+ and thus $\alpha''_\eta \notin j_1$“κ^+, where $\{\alpha''_i \mid i < j_1(\kappa)\}$ is image of the $\{\alpha_i \mid i < \kappa\}$ under j_1. Moreover $k(\alpha''_\eta) = \alpha'_{k(\eta)}$ and by definition, $f_{\kappa_2, \alpha'_\eta}([\mathrm{id}]_{j_1(\mathscr{U})}) = 0$, in other words $[\mathrm{id}]_{j_1(\mathscr{U})} \notin A'_{\alpha'_\eta}$. Hence

$$[\mathrm{id}]_{j_1(\mathscr{U})} \notin \bigcap \{A'_{\alpha'_i} \mid i < \kappa_2\} = j_2^*(\bigcap_{i<\kappa} A_{\alpha_i})$$

Hence $\bigcap_{i<\kappa} A_{\alpha_i} \notin \mathscr{W}$. □

We are done with the proof of the proposition. □

Continuing with our original discussion one may ask if the conclusion of Proposition 2.1 is compatible with large cardinals stronger than supercompactness. As argued in [2, 3], the natural model-theoretic strengthening of supercompactness is $C^{(n)}$-extendibility. Fix $n < \omega$. A cardinal κ is called $C^{(n)}$-*extendible* if for every $\lambda > \kappa$ there is $\theta \in \mathrm{Ord}$ and an elementary embedding $j \colon V_\lambda \to V_\theta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \prec_{\Sigma_n} V$.[7] The classical notion of extendibility (see [29, §23]) coincides with $C^{(n)}$-extendibility whenever $n = 1$. However, when $n \geq 2$ the first $C^{(n)}$-extendible is far above, and has stronger large-cardinal-properties, than the first extendible cardinal. In addition, $C^{(n)}$-extendibility do entail a proper hierarchy of cardinals [2]. The culmination of this hierarchy is the category-theoretic axiom known as

[7]Recall that $V_\eta \prec_{\Sigma_n} V$ is a shorthand for the following statement: for every $\bar{a} \in V_\eta^{<\omega}$ and every Σ_n formula $\varphi(\bar{x})$ in the language of set theory, $V_\eta \models \varphi(\bar{a})$ iff $V \models \varphi(\bar{a})$.

Vopěnka's Principle (VP) [29, p. 335]. In effect, it was shown by Bagaria that VP is equivalent to the existence of a (proper class of) $C^{(n)}$-extendible, for all $n \geq 1$. We refer the reader to [2] for further details.

Let us come back to the argument of Proposition 2.1. If κ is an extendible cardinal performing our iteration $\mathbb{P}_{\kappa+1}$ will ruin extendibility of κ.[8] Nevertheless, if one forces with $\mathrm{Add}(\alpha, \alpha^+)$ at every inaccessible cardinal the situation changes completely. In [3] the authors develop a general theory of preservation of extendible cardinals under class-forcing iterations. Specifically, in [3, §8] it is shown that many classical class-forcing iterations (e.g., Jensen's iteration to force the GCH) do preserve extendible cardinals, as well as $C^{(n)}$-extendible cardinals and Vopěnka's Principle (VP).

The following proposition is an easy corollary of [3, Theorem 8.4]:

Proposition 2.3. *Assume the* GCH *and let κ be either a supercompact or a $C^{(n)}$-extendible cardinal for some $n \geq 1$. Let $\mathbb{P}$ denote the Easton support class iteration forcing with $\mathrm{Add}(\alpha, \alpha^+)$ at each inaccessible cardinal.*

Then, the following hold in $V^{\mathbb{P}}$:

1. *κ is either supercompact or $C^{(n)}$-extendible;*[9]
2. *for every measurable cardinal λ every λ-complete ultrafilter $\mathscr{U} \in V$ extends to a λ-complete ultrafilter $\mathscr{U}^*$ such that $\neg\mathrm{Gal}(\mathscr{U}^*, \lambda, \lambda^+)$.*

In addition, if one assumes VP *this is preserved in $V^{\mathbb{P}}$.*

Proof. Clause (1) is an immediate consequence of [3, Theorem 8.4]. For Clause (2) we argue as follows. If λ_0 stands for the first V-inaccessible cardinal then $\mathbb{P}$ admits a gap at λ_0^{++}.[10] Thus $\mathbb{P}$ does not create new measurable cardinals in $V^{\mathbb{P}}$ [24, Corollary 2]. Let λ be a V-measurable cardinal and $\mathscr{U}$ a λ-complete ultrafilter in the ground model. By Proposition 2.1, $\mathbb{P}_{\lambda+1}$ forces that there is $\mathscr{U}^* \supseteq \mathscr{U}$ such that $\mathrm{Gal}(\mathscr{U}^*, \lambda, \lambda^+)$ fails. Clearly $\mathbb{P}/\mathbb{P}_{\lambda+1}$ is forced to be λ^{++}-directed closed (actually more), hence it preserves that $\mathscr{U}^*$ is a λ-complete ultrafilter over λ witnessing $\neg\mathrm{Gal}(\mathscr{U}^*, \lambda, \lambda^+)$. □

Let us now show how to obtain the consistency of the opposite statement "Every κ-complete ultrafilter $\mathscr{U}$ (in V) over a supercompact cardinal κ extends to a κ-complete ultrafilter $\mathscr{U}^*$ (in $V[G]$) satisfying $\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$". To this aim we will show how to turn a non (necessarily) Galvin ultrafilter $\mathscr{U}$ into another $\mathscr{U}^*$ that is *Rudin-Keisler* equivalent to a normal one. By virtue of Galvin's theorem [4] this ensures that $\mathscr{U}^*$ itself satisfies $\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$.

To show this we need a couple of preliminary observations. First, if $\mathscr{U}$ and $\mathscr{W}$ are κ-complete ultrafilters over κ and $\mathscr{U} \equiv_{\mathrm{RK}} \mathscr{W}$ then $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ holds if and only if $\mathrm{Gal}(\mathscr{W}, \kappa, \kappa^+)$ holds. Second:

Proposition 2.4. *If $\mathscr{U}$ is a κ-complete ultrafilter over κ with $|[\mathrm{id}]_{\mathscr{U}}| = \kappa$ then $\mathscr{U}$ is Rudin-Keisler equivalent to a normal κ-complete ultrafilter.*

[8] Actually, adding a single Cohen subset to κ does it.

[9] Depending, of course, of the original assumption upon κ.

[10] I.e., $\mathbb{P} \simeq \mathbb{P}_1 * \underset{\sim}{\mathbb{P}}_2$ where $|\mathbb{P}_1| < \lambda_0^{++}$ and $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{\mathbb{P}}_2$ is λ_0^{++}-distributive".

In particular, under the above conditions, $\mathrm{Gal}(\mathscr{U},\kappa,\kappa^+)$ *holds.*

Proof. Let $\mathscr{U}_0$ denote the normal measure generated from $j := j_{\mathscr{U}}$ and κ.

For each $\lambda < \kappa^+$ there is $f_\lambda\colon \kappa \to \kappa$ such that $j(f_\lambda)(\kappa) = \lambda$. We prove this by induction on λ. Suppose that $\langle f_\alpha \mid \alpha < \lambda < \kappa^+\rangle$ are defined and let $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ be cofinal in λ. Define $f_\lambda\colon \kappa \to \kappa$ as follows:

$$f_\lambda(\alpha) := \sup_{i<\alpha} f_{\lambda_i}(\alpha).$$

Note that $f_\lambda\colon \kappa \to \kappa$ due to the regularity of κ. Next, put

$$j(\langle f_\beta \mid \beta < \lambda\rangle) := \langle f'_\beta \mid \beta < j(\lambda)\rangle,\ j(\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle) := \langle \lambda'_i \mid i < j(\mathrm{cf}(\lambda))\rangle.$$

Observe that $f'_{j(\alpha)} = j(f_\alpha)$ and $\lambda'_{j(\alpha)} = j(\lambda_\alpha)$. In particular, $f'_\alpha = j(f_\alpha)$ and $\lambda'_\alpha = j(\lambda_\alpha)$ for every $\alpha < \kappa$. Hence,

$$j(f_\lambda)(\kappa) = \sup_{i<\kappa} f'_{\lambda'_i}(\kappa) = \sup_{i<\kappa} j(f)_{j(\lambda_i)}(\kappa) =$$

$$= \sup_{i<\kappa} j(f_{\lambda_i})(\kappa) = \sup_{i<\kappa} \lambda_i = \lambda$$

Thus, there is $f\colon \kappa \to \kappa$ such that $j(f)(\kappa) = [\mathrm{id}]_{\mathscr{U}}$, so that $\mathscr{U} \leq_{\mathrm{RK}} \mathscr{U}_0$. Also, it is well-known that normal filters are $\leq_{\mathrm{RK}}$-minimal (see e.g. [6, Proposition 2.6]), hence $\mathscr{U} \equiv_{\mathrm{RK}} \mathscr{U}_0$. For the in particular clause use our comments prior to the statement of the proposition. □

Theorem 2.5. *Assume that the* GCH *holds and that* κ *is a huge cardinal.*

Then, there is an inaccessible cardinal $\mu > \kappa$ *and a generic extension of* V_μ *where the following hold:*

1. κ *is supercompact;*
2. *Every* κ*-complete ultrafilter* $\mathscr{U} \in V$ *extends to a* κ*-complete ultrafilter* $\mathscr{U}^*$ *that is Rudin-Keisler equivalent to a normal ultrafilter. In particular,* $\mathrm{Gal}(\mathscr{U}^*,\kappa,\kappa^+)$ *holds.*

Proof. Let $j\colon V \to M$ be an elementary embedding witnessing that κ is huge; namely, $\mathrm{crit}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$. Fix $\langle \mathscr{U}_\alpha \mid \alpha < 2^{2^\kappa}\rangle$ an injective enumeration of the κ-complete ultrafilters over κ. For each $\alpha < 2^{2^\kappa}$ note that $j\text{“}\mathscr{U}_\alpha \in M$ and that $j\text{“}\mathscr{U}_\alpha \in [j(\mathscr{U}_\alpha)]^{<j(\kappa)}$ hence, by $j(\kappa)$-completeness of $j(\mathscr{U}_\alpha)$ in M, we can find $\epsilon_\alpha \in \bigcap j\text{“}\mathscr{U}_\alpha$. Clearly, $\epsilon_\alpha < j(\kappa)$ and

$$\mathscr{U}_\alpha \subseteq \mathscr{U}^*_{\alpha,0} := \{X \subseteq \kappa \mid \epsilon_\alpha \in j(X)\}.$$

Let λ and μ be, respectively, the first inaccessible cardinals in the intervals $(\sup_{\alpha<2^{2^\kappa}} \epsilon_\alpha, j(\kappa))$ and $(\lambda, j(\kappa))$.[11] Next, let $i\colon V \to N$ be the μ-supercompact embedding derived from j; that is, the ultrapower embedding that arises from the measure $\{X \subseteq \mathcal{P}_\kappa(\mu) \mid j\text{“}\mu \in j(X)\}$. Let $k\colon N \to M$ be the factor embedding between j and i. Usual arguments show that $\mathrm{crit}(k) > \mu$, hence $\mathscr{U}^*_{\alpha,0} = \{X \subseteq \kappa \mid \epsilon_\alpha \in i(X)\}$ for each $\alpha < 2^{2^\kappa}$.

Now, force over V with *Woodin's fast function forcing* $\mathbb{F}_\kappa$. Loosely speaking, forcing with $\mathbb{F}_\kappa$ adds a generic function $f\colon \kappa \to \kappa$ that can be used

[11] This choice is possible as $j(\kappa)$ is a limit of inaccessibles.

to *guess* any prescribed ordinal $\chi < i(\kappa)$. More precisely, by virtue of Lemma 1.10 in [25], we know that i lifts to a μ-supercompact embedding $i\colon V[f] \to N[i(f)]$ such that $i(f)(\kappa) = \mu$. Notice that using the fast function $f\colon \kappa \to \kappa$ we can easily represent λ as well: let $f^*\colon \kappa \to \kappa$ be defined as $\alpha \mapsto \sup\{\beta < f(\alpha) \mid \beta \text{ is inaccessible}\}$ and note that $i(f^*)(\kappa) = \lambda$.[12]

Next, over $V[f]$, force with the two-step iteration $\mathbb{C} := \mathbb{C}_\kappa * \underset{\sim}{\mathrm{Col}}(\kappa, <\lambda)$ where $\mathbb{C}_\kappa$ is the Easton-supported iteration defined as follows: for $\alpha < \kappa$, the α^{th}-stage of the iteration is trivial unless α is inaccessible, $f``\alpha \subseteq \alpha$ and $\alpha < f^*(\alpha)$, in which case it forces with $\underset{\sim}{\mathrm{Col}}(\alpha, <f^*(\alpha))$.

Claim 2.6. *After forcing with $\mathbb{C}$ the embedding $i\colon V[f] \to N[i(f)]$ lifts to a μ-supercompact embedding $i^*\colon V[f*C] \to N[i(f*C)]$ in $V[f*C]$.*

Proof of claim. Denote $\bar{V} := V[f]$ and $\bar{N} := N[i(f)]$. Let $C := C_\kappa * c \subseteq \mathbb{C}$ generic over $V[f]$. We can lift the embedding after forcing with $\mathbb{C}_\kappa$ to another $i\colon \bar{V}[C_\kappa] \to \bar{N}[C_\kappa * c * h] \subseteq \bar{V}[C]$. There are two points here: first, the κ^{th}-stage of the iteration from the perspective of $\bar{N}$ is $\mathrm{Col}(\kappa, <\lambda)^{\bar{V}}$;[13] second, the tail forcing $i(\mathbb{C}_\kappa)/\mathbb{C}$ is trivial in the interval (κ, μ) because $i(f)(\kappa) = \mu$ and so the next closure point of $i(f)$ past κ is $\geq (\mu^+)^{\bar{V}}$.

Finally, one can lift i after forcing with $\mathrm{Col}(\kappa, <\lambda)_{\bar{V}[C_\kappa]}$ to another embedding $i\colon \bar{V}[C] \to \bar{N}[C_\kappa * c * h * h']$. For this one uses the fact that $i``c \subseteq \mathrm{Col}(i(\kappa), <i(\lambda))_{\bar{N}[C_\kappa * c * h]}$ is a directed set of conditions in $N[C_\kappa * c * h]$, $|i``c| < i(\kappa)$ and $\mathrm{Col}(i(\kappa), <i(\lambda))_{\bar{N}[C_\kappa * c * h]}$ is μ-directed-closed in the model $N[C_\kappa * c * h]$. Standard arguments show that the resulting embedding witnesses μ-supercompactness of κ. □

Working in $V[f*C]$, for each $\alpha < (2^{2^\kappa})^V$ define

$$\mathscr{U}^*_\alpha := \{X \subseteq \kappa \mid \epsilon_\alpha \in i^*(X)\}.$$

Clearly, $\mathscr{U}^*_\alpha$ is a κ-complete ultrafilter satisfying $\mathscr{U}_\alpha \subseteq \mathscr{U}^*_\alpha$. The point now is that $|[\mathrm{id}]_{\mathscr{U}^*_\alpha}|^{V[f*C]} \leq |\epsilon_\alpha|^{V[f*C]} = \kappa$ hence $\mathrm{Gal}(\mathscr{U}^*_\alpha, \kappa, \kappa^+)$ holds in $V[f*C]$.

Let $M := V[f*C]_\mu$. This is certainly a model of ZFC because μ remains inaccessible. Also, M satisfies that κ is supercompact. Finally, note that $M = V_\mu[f*C]$. Since every κ-complete ultrafilter $\mathscr{U}$ over κ from the ground model actually comes from V_μ we obtain Clause (2) of the theorem. □

In the model of Theorem 2.5 our target cardinal κ cannot be extendible. To make it so, one should perform a class-forcing iteration that is nice enough to carry the previous arguments. This suggests the following question:

Question 2.7. Is the statement of the previous theorem compatible with κ being extendible or, more generally, $C^{(n)}$-extendible?

[12] Here we are implicitly assuming that $N[i(f)]$ is μ-closed, hence it thinks that μ is inaccessible and that λ is the first inaccessible below it.

[13] Because $i(f)``\kappa \subseteq \kappa$ and $i(f^*)(\kappa) = \lambda > \kappa$.

In the proof of Theorem 2.5 we showed how to *correct* ultrafilters that do not satisfy Galvin's property. The technique used for this purpose consisted of collapsing the generators of every V-ultrafilter to yet another generator of cardinality κ; namely, the normal generator. In that manner we accomplished our *correction* of non-Galvin ultrafilters by making them essentially *minimal* (to wit, normal) from the Rudin-Keisler-perspective. This is, certainly, a too harsh way to ensure Galvin's property in the final model.

In the light of this one may wonder whether a similar Galvin-like configuration is possible without *trivializing* the relevant ultrafilters. In what is left we show that this is indeed possible. As a warm up exercise we begin describing how to turn a general κ-complete ultrafilter into a Galvin one by using a generalization of *Mathias forcing*. The classical Mathias forcing dealing with subsets of ω appeared in [33], while the version that we will use here follows the template of [20, Definition 3.1]:

Definition 2.8 (Generalized Mathias forcing)**.** Let κ be a regular cardinal and $\mathscr{U}$ a non-principal κ-complete filter over κ.

The forcing notion $\mathbb{M}_{\mathscr{U}}$ consists of pairs (a, A) such that $a \in [\kappa]^{<\kappa}, A \in \mathscr{U}$ and $\sup(a) < \min(A)$. For the order, one writes $(a_0, A_0) \leq (a_1, A_1)$ if and only if $a_0 \subseteq a_1, A_0 \supseteq A_1$ and $a_1 \setminus a_0 \subseteq A_0$. Also, given a condition $p = (a, A)$ we denote $a^p = a$ and $A^p = A$.

The next is a brief account of the main properties of $\mathbb{M}_{\mathscr{U}}$:

Proposition 2.9 (Properties of $\mathbb{M}_{\mathscr{U}}$)**.**

(1) $\mathbb{M}_{\mathscr{U}}$ *is* κ^+*-centered, provided* $\kappa^{<\kappa} = \kappa$*;*
(2) $\mathbb{M}_{\mathscr{U}}$ *is* κ*-directed-closed;*
(3) $\mathbb{M}_{\mathscr{U}}$ *is countably parallel closed;*[14]
(4) *If* $G \subseteq \mathbb{M}_{\mathscr{U}}$ *is a* V*-generic filter then* $V[G] = V[a_G]$*, where*

$$a_G := \bigcup \{a \mid \exists A \in \mathscr{U}\ ((a, A) \in G)\};$$

(5) *The set* a_G diagonalizes $\mathscr{U}$*: i.e.,* $a_G \subseteq^* A$ *for every* $A \in \mathscr{U}$*.*

In particular, if $\mathscr{U}$ *is an ultrafilter then either* $a_G \subseteq^* A$ *or* $a_G \subseteq^* \kappa \setminus A$ *for all* $A \in \mathcal{P}(\kappa)^V$*.*

The next proposition describes how to turn a κ-complete filter into one satisfying Galvin's property by means of $\mathbb{M}_{\mathscr{U}}$:

Proposition 2.10. *Let* κ *be a regular cardinal,* $\mathscr{U}$ *a* κ*-complete filter over* κ *and* $G \subseteq \mathbb{M}_{\mathscr{U}}$ *a generic filter. Then the following hold in* $V[G]$*:*

(1) $\mathscr{U}^* := \{A \subseteq \kappa \mid a_G \subseteq^* A\}$ *is a* κ*-complete filter;*
(2) $\mathscr{U} \subseteq \mathscr{U}^*$*;*
(3) $\mathrm{Gal}(\mathscr{U}^*, \kappa, \kappa^+)$ *holds.*

[14] I.e., every two decreasing sequences of conditions $\langle p_n \mid n < \omega\rangle$, $\langle q_n \mid n < \omega\rangle$ with $p_n \parallel q_n$ admit an upper bound. See [14].

Proof. $\mathscr{U}^*$ is clearly a filter and $\mathscr{U} \subseteq \mathscr{U}^*$ by virtue of Proposition 2.8(4). The argument for κ-completeness of $\mathscr{U}^*$ is essentially the same as the one for Clause (3): Let $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \subseteq \mathscr{U}^*$ and find $I \in [\kappa^+]^{\kappa^+}$ and $\alpha^* < \kappa$ such that $a_G \setminus \alpha^* \subseteq A_\alpha$ for all $\alpha \in I$. Thus, $\bigcap_{\alpha \in I} A_\alpha \in \mathscr{U}^*$. □

The above argument repeats the one from [7, Proposition 4.5]: the point is that $\mathbb{M}_{\mathscr{U}}$ creates a *generating sequence* of length 1.

Definition 2.11. A family $\mathcal{A} = \langle x_\alpha \mid \alpha < \lambda \rangle \subseteq \mathscr{U}$ is a *generating sequence for* $\mathscr{U}$ if for every $A \in \mathscr{U}$ there is $\alpha < \lambda$ such that $x_\alpha \subseteq^* A$. In addition, $\mathcal{A}$ is called *strong generating* if it is $\subseteq^*$-decreasing.

As demonstrated in [7, §4] the analysis of (strong) generating sequences provides an effective way to produce certain Galvin-like configurations. The main obstacle, however, is to ensure that the departing κ-complete ultrafilter $\mathscr{U}$ extends to yet another κ-complete ultrafilter. This will be eventually addressed in Theorem 2.20.

Our next goal will be to iterate Mathias forcing over a given filter (and its extensions along the way) so that it will generate a κ-complete ultrafilter with a strong generating sequence of arbitrary length. This idea traces back to Kunen who employed it to separate the ultrafilter number $\mathfrak{u}$ from 2^ω (see [32, Ch. VII Question (A10)]). A similar argument, yet involving a more complex iteration, was considered in [12]. There the authors separate $\mathfrak{u}(\kappa)$ and 2^κ in a context where κ is supercompact.

The *naïve* approach would consist of iterating Mathias forcing over and over with κ-complete filters. Unfortunately, this strategy is doomed to failure and so an additional structure on the forcing is required. Let us illustrate where the problem arises. Suppose that x_0 is a Mathias set for a κ-complete filter $\mathscr{U}$. Working in the generic extension $V[x_0]$ let $\mathscr{U}_0$ be a κ-complete filter extending $\{x_0\} \cup \mathscr{U}$ (e.g., by Proposition 2.10 we can take $\{X \in \mathcal{P}(\kappa)^{V[x_0]} \mid x_0 \subseteq^* X\}$). Next, over $V[x_0]$, force a Mathias set x_1 through $\mathscr{U}_0$ and working over the resulting extension $V[x_0, x_1]$ let $\mathscr{U}_1$ a κ-complete filter extending $\{x_1\} \cup \mathscr{U}_0$. One can proceed in this fashion ω-many times. Formally speaking, this is forced by the following full-support iteration $\langle \mathbb{P}_n, \mathbb{Q}_n \mid n < \omega \rangle$: for each $n \geq 1$, $\mathbb{Q}_n$ is a $\mathbb{P}_n$-name for $\mathbb{M}_{\mathscr{U}_n}$ where $\mathscr{U}_n$ is a $\mathbb{P}_n$-name for a κ-complete ultrafilter extending $\{\underset{\sim}{x}_n\} \cup \mathscr{U}_{n-1}$.

An essential obstacle arises at stage $\omega + 1$. Here one needs to find a κ-complete filter which includes all the Mathias sets $\langle x_n \mid n < \omega \rangle$ constructed so far. However, notice that $\mathscr{W} := \{X \subseteq \kappa \mid \exists n < \omega \, (x_n \subseteq^* X)\}$ (i.e., the filter generated by the Mathias sets) is not σ-complete: for if $\bigcap_{n<\omega} x_n \in \mathscr{W}$ then there would be some $n^* < \omega$ such that $x_{n^*} \subseteq^* \bigcap_{n<\omega} x_n$, hence x_{n^*} would be $\subseteq^*$-included in x_{n^*+1}. This latter is certainly impossible in that x_{n^*+1} is a Mathias set for a filter including x_{n^*}. What we will end up doing in ω (or other limit stages of low cofinality) is looking at the filter generated by $\bigcap_{n<\omega} x_n$. The main issue will be to verify that the set is unbounded.

For the moment, and as a warm up for Theorem 2.20, we show how to produce κ-complete filters with arbitrarily long strong generating sequences using Mathias forcing.

Theorem 2.12. *Let $\mathscr{U}$ be a κ-complete filter over a Mahlo cardinal κ.*

Then, for every $\lambda \in \mathrm{Ord}$ there is a κ-directed-closed and κ^+-cc poset $\mathbb{P}(\lambda)$ forcing that $\mathscr{U}$ can be extended to a κ-complete filter $\mathscr{U}^*$ with a strong generating sequence $\langle x_\alpha \mid \alpha < \lambda\rangle$.

Proof. Let $\mathscr{U}$ and λ be as above. Define a $<\kappa$-supported iteration $\mathbb{P}(\lambda)$, $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \beta < \alpha \leq \lambda\rangle$, as follows. Suppose that $\mathbb{P}_\alpha$ is defined for $\alpha < \lambda$. In $V^{\mathbb{P}_\alpha}$ we will define a filter $\mathscr{U}^*_\alpha$ over κ which we will prove to be κ-complete. Bearing this in mind, we shall let $\underset{\sim}{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{M}_{\mathscr{U}^*_\alpha}$ and denote by $x_\alpha := a_{G_{\mathbb{Q}_\alpha}}$ the generic Mathias set added after forcing with $\underset{\sim}{\mathbb{Q}}_\alpha$. In addition, we shall inductively maintain that the x_α's are $\subseteq^*$-increasing.

For $\alpha = 0$, we let $\mathscr{U}_0 := \mathscr{U}$. At successor $\alpha + 1$, in $V^{\mathbb{P}_{\alpha+1}}$ we have x_α and we let $\underset{\sim}{\mathscr{U}}_{\alpha+1}$ be a $\mathbb{P}_{\alpha+1}$-name for the filter generated by $\underset{\sim}{x}_\alpha$; to wit, the canonical $\mathbb{P}_{\alpha+1}$-name for the set $\{X \subseteq \kappa \mid x_\alpha \subseteq^* X\}$. By proposition 2.10, we have that $0_{\mathbb{P}_{\alpha+1}} \Vdash \underset{\sim}{\mathscr{U}}_\alpha \subseteq \underset{\sim}{\mathscr{U}}_{\alpha+1}$ and $\underset{\sim}{\mathscr{U}}_{\alpha+1}$ is κ-complete.

Regarding the limit stages, let us split the discussion into cases

Claim 2.13. *Suppose that $\alpha < \lambda$ is such that $\mathrm{cf}(\alpha) \geq \kappa$. Then $\langle x_\beta \mid \beta < \alpha\rangle$ generates a κ-complete filter $\mathscr{U}^*_\alpha$ in $V^{\mathbb{P}_\alpha}$ that extends $\mathscr{U}^*_\beta$ for every $\beta < \alpha$.*

Proof. Working over $V^{\mathbb{P}_\alpha}$ let us consider

$$\mathscr{U}^*_\alpha := \{X \subseteq \kappa \mid \exists \beta < \alpha\, (x_\beta \subseteq^* X)\}.$$

Clearly this is a filter over κ in $V^{\mathbb{P}_\alpha}$. Let $\{X_i \mid i < \mu < \kappa\} \subseteq \mathscr{U}^*_\alpha$ in $V^{\mathbb{P}_\alpha}$. For each $i < \mu$ there is $\beta_i < \alpha$ such that $x_{\beta_i} \subseteq^* X_i$. Since the cofinality of α is at least κ, $\sup_{i<\mu} \beta_i =: \beta^* < \alpha$. Hence, $x_{\beta^*} \subseteq^* x_{\beta_i} \subseteq^* X_i$ for every $i < \mu$. It follows that for some $\epsilon_i < \kappa$, $x_{\beta^*} \setminus \epsilon_i \subseteq X_i$. Take $\epsilon^* = \sup_{i<\mu} \epsilon_i < \kappa$, then $x_{\beta^*} \setminus \epsilon^* \subseteq \cap_{i<\mu} X_i$, by definition, $\cap_{i<\mu} X_i \in \mathscr{U}^*_\alpha$. Also, for every $\beta < \alpha$, $\mathscr{U}^*_\beta \subseteq \mathscr{U}^*_{\beta+1}$ and $\mathscr{U}^*_{\beta+1}$ is by definition the filter generated by x_β, which is clearly a subset of $\mathscr{U}^*_\alpha$. □

Claim 2.14. *Suppose that $\alpha < \lambda$ is limit such that $\mathrm{cf}(\alpha) < \kappa$, and let*

$$\alpha = \kappa^{\delta_1}\gamma_1 + ... + \kappa^{\delta_n}\gamma_n$$

be the Cantor normal form of α. Consider the following cofinal subset of α: $I_\alpha := \{\kappa^{\delta_1}\gamma_1 + ... + \kappa^{\delta_n}\gamma \mid \gamma < \gamma_n\}$. Then $x := \cap_{i\in I_\alpha} x_i \in V^{\mathbb{P}_\alpha}$ is unbounded in κ. In particular, $x \subseteq^* x_\beta$ for every $\beta < \alpha$, and the filter $\mathscr{U}^*_\alpha$ generated by x, is a κ-complete filter in $V^{\mathbb{P}_\alpha}$ which extends $\mathscr{U}^*_\beta$ for every $\beta < \alpha$.

Proof. Let $p \in \mathbb{P}_\alpha$ and $\alpha_0 < \kappa$. We shall now proceed with a density argument to prove that there are $\alpha_0 < \gamma^* < \kappa$ and $p \leq p_{fin}$ such that $p_{fin} \Vdash \gamma^* \in \cap_{i\in I_\alpha} \underset{\sim}{x}_i$. Construct two sequences $\langle M_\rho \mid \rho < \kappa\rangle$ and $\langle q_\rho \mid \rho < \kappa\rangle$ such that:

(1) $M_\rho \prec H(\theta)$ for θ regular and sufficiently large.

(2) M_ρ is increasing and continuous.
(3) $|M_\rho| := \gamma_\rho < \kappa$ and $\mathrm{cf}(\gamma_\rho) = \gamma_\rho$ if $\rho = 0$ or if ρ is successor.
(4) If γ_ρ is regular then $M_\rho^{<\gamma_\rho} \subseteq M_\rho$.
(5) $\alpha, p, \mathbb{P}_\alpha, \kappa \in M_\rho$.
(6) If $\beta \in M_\rho \cap \alpha + 1$ has cofinality less than κ, then $\mathrm{cf}(\beta) \cup I_\beta \subseteq M_{\rho+1}$.

For the conditions q_ρ we require that:

(1) $\langle q_\rho \mid \rho < \mu^* \rangle$ is increasing and continuous;[15]
(2) q_ρ is M_ρ-generic for $\mathbb{P}_\alpha$: namely, for every dense open $D \subseteq \mathbb{P}_\alpha$, $D \in M_\rho$, $q_\rho \in D$;
(3) $supp(q_\rho) = M_\rho \cap \alpha$;
(4) $q_\rho \in M_{\rho+1}$.

For this construction, we need κ to be Mahlo. Such conditions exist by κ-closure of $\mathbb{P}_\alpha$ and by standard construction of an increasing sequence of conditions in M. Let $\mu^* < \kappa$ be regular such that $|M_{\mu^*}| = \mu^*$. Denote $M^* = M_{\mu^*}$ and $p^* = q_{\mu^*} = \sup_{i<\mu^*} q_i$.

Claim 2.15. *For every $\beta \in Supp(p^*)$ the following hold:*[16]

(1) *There is δ_β such that $p^* \restriction \beta \Vdash \underset{\sim}{a}_\beta^{p^*} \subseteq \delta_\beta$.*
(2) *If $\beta = \beta_0 + 1$ is successor, then there is ϵ_β such that $p^* \restriction \beta \Vdash \underset{\sim}{x}_{\beta_0} \setminus \epsilon_\beta \subseteq \underset{\sim}{A}_\beta^{p^*}$.*
(3) *If $\beta \in Supp(p^*)$ is of cofinality less than κ then there is ϵ_β such that $p^* \restriction \beta \Vdash \cap_{i \in I_\beta} \underset{\sim}{x}_i \setminus \epsilon_\beta \subseteq \underset{\sim}{A}_\beta^{p^*}$.*
(4) *If $\beta \in Supp(p^*)$ is of cofinality at least κ, then there $i_\beta < \beta$ and ϵ_β such that $p^* \restriction \beta \Vdash \underset{\sim}{x}_{i_\beta} \setminus \epsilon_\beta \subseteq \underset{\sim}{A}_\beta^{p^*}$.*

Proof. To see (1), since $\beta \in Supp(p^*) = M^* \cap \alpha$, find $\xi_0 < \mu^*$ such that $\beta \in M_{\xi_0} \cap \alpha$. In M_{ξ_0+1} we can define the dense open set

$$D_{\xi_0} := \{q \in \mathbb{P}_\alpha \mid \exists \delta < \kappa . q \restriction \beta \Vdash \underset{\sim}{a}_\beta^{q_{\xi_0}} \subseteq \delta\}$$

Since q_{ξ_0+1} is $M_{\xi_0}-$generic, $q_{\xi_0+1} \in D_{\xi_0}$. For every $\xi_0 \leq i < \mu^*$ we have that $q_i \in M_{i+1}$, hence we can define in M_{i+1} the dense open set

$$D_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists \delta < \kappa . q \restriction \beta \Vdash \underset{\sim}{a}_\beta^{q_i} \subseteq \delta\}$$

By genericity, $q_{i+1} \in D_{i+1}$. For every such i, pick $\delta^{(i)}$ witnessing $q_{i+1} \in D_{i+1}$. Let $\delta_\beta = \sup_{i<\mu^*} \delta^{(i)} < \kappa$, by continuity, $p^* \restriction \beta \Vdash \underset{\sim}{a}_\beta^{p^*} \subseteq \delta_\beta$.

Let us now prove (2). First, note that (by definition) $\mathscr{U}_\beta$ is the filter generated by $\underset{\sim}{x}_{\beta_0}$. Let us now replace the dense D_{i+1} by

$$E_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists \epsilon . q \restriction \beta \Vdash \underset{\sim}{x}_{\beta_0} \setminus \epsilon \subseteq \underset{\sim}{A}_\beta^{q_i}\},$$

[15] I.e. for limit ρ, we let $q_\rho(\gamma) = \langle \cup_{\rho'<\rho} \underset{\sim}{a}_\gamma^{q_{\rho'}}, \cap_{\rho'<\rho} \underset{\sim}{A}_\gamma^{q_{\rho'}} \rangle$.

[16] Recall that a condition $p \in \mathbb{P}_\alpha$ is a sequence $p = \langle p_\beta \mid \beta < \alpha \rangle$ such that $p \restriction \beta \Vdash p_\beta \in \underset{\sim}{\mathbb{Q}}_\beta$ and therefore we might assume that $p_\beta = \langle \underset{\sim}{a}, \underset{\sim}{A} \rangle$, where $\underset{\sim}{a}$ is just a $\mathbb{P}_\beta$-name for bounded subset of κ and $\underset{\sim}{A}$ is a name for a set in $\underset{\sim}{U}_\beta$. Denote $\underset{\sim}{a} := \underset{\sim}{a}_\beta^p$ and $\underset{\sim}{A} := \underset{\sim}{A}_\beta^p$.

which is definable from $q_i \in M_{i+1}$. By genericity, $q_{i+1} \in E_{i+1}$ and we let $\epsilon^{(i)}$ witness that. Let $\epsilon_\beta = \sup_{i<\mu^*} \epsilon^{(i)}$, we have that $p^* \restriction \beta \Vdash \forall i < \mu^*, \underset{\sim}{x}_\beta \setminus \epsilon_\beta \subseteq \underset{\sim}{A}^{q_i}_\beta$. It follows then (again by continuity) that $p^* \restriction \beta \Vdash \underset{\sim}{x}_\beta \setminus \epsilon_\beta \subseteq \underset{\sim}{A}^{p^*}_\beta$.

To see (4) we follow a similar argument. First, define

$$F_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists j_\beta < \beta . \exists \epsilon .\ q \restriction \beta \Vdash \underset{\sim}{x}_{j_\beta} \setminus \epsilon \subseteq \underset{\sim}{A}^{q_i}_\beta\}$$

Note that F_{i+1} is dense open for every $i < \mu^*$: this is by definition of $\underset{\sim}{\mathscr{U}}_\beta$ whenever $\mathrm{cf}(\beta) \geq \kappa$. By genericity of q_{i+1}, we can pick for every $i < \mu^*$ and index $j_{\beta,i} \in M_{\rho^*} \cap \beta$ witnessing $q_{i+1} \in F_{i+1}$. Since the cofinality of β is at least κ we can let $i_\beta := \sup_{i<\mu^*} j_{\beta,i} < \beta$. Note that i_β has the property that for every $i < \mu^*$, there is $\epsilon^{(i)}$ such that for $q_{i+1} \Vdash \underset{\sim}{x}_{i_\beta} \setminus \epsilon^{(i)} \subseteq \underset{\sim}{A}^{q_{i+1}}_\beta$. As before, letting $\epsilon_\beta := \sup_{i<\mu^*} \epsilon^{(i)}$ we have that $p^* \restriction \beta \Vdash \underset{\sim}{x}_{i_\beta} \setminus \epsilon_\beta \subseteq \underset{\sim}{A}^{p^*}_\beta$.

Finally, to see (3), let

$$H_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists j_\beta < \beta . \exists \epsilon . q \restriction \beta \Vdash \cap_{j\in I_\beta} \underset{\sim}{x}_j \setminus \epsilon \subseteq \underset{\sim}{A}^{q_i}_\beta\}$$

By the definition of $\underset{\sim}{U}_\beta$ in the case that $cf(\beta) < \kappa$, H_{i+1} is dense open, however, we need the parameters $I_\beta, q^i \in M_{i+1}$ in order to conclude that $H_{i+1} \in M_{i+1}$. As before, $q_i \in M_{i+1}$. As for I_β, note that $\beta \in M_{\xi_0}$, so $I_\beta \subseteq M_{\xi_0+1}$ and also $I_\beta \in M_{\xi+2}$. Hence for every $\xi_0 < i < \mu^*$, we have that $H_{i+1} \in M_{i+1}$ is dense open and so $q_{i+1} \in H_{i+1}$. Let $\epsilon^{(i)} \in M_{\rho^*} \cap \beta$ be that $q_{i+1} \restriction \beta \Vdash \cap_{i\in I_\beta} \underset{\sim}{x}_i \setminus \epsilon^{(i)} \subseteq \underset{\sim}{A}^{q_i}_\beta$. Once again, define $\epsilon_\beta = \sup_{i<\mu^*} \epsilon^{(i)}$, then by continuity one last time we have that $p^* \restriction \beta \Vdash \cap_{i\in I_\beta} \underset{\sim}{x}_i \setminus \epsilon \subseteq \underset{\sim}{A}^{p^*}_\beta$, as wanted. □

By (1), for every $\beta \in Supp(p^*)$ we have $\delta_\beta < \kappa$, pick $\delta^* = \sup \delta_\beta$. By (2), (3) and (4) we choose ϵ_β and set $\epsilon^* = \sup \epsilon_\beta < \kappa$. Pick $\gamma^* \in A^{p^*}_0$ above $\epsilon^*, \delta^*, \alpha_0$ and define p_{fin}. Define the support of p_{fin} to be $Supp(p^*) \cup \{i_\beta \mid \beta \in Supp(p^*), cf(\beta) \geq \kappa\}$. Define

$$p_{fin}(\gamma) := \begin{cases} \langle \underset{\sim}{a}^{p^*}_\gamma \cup \{\gamma^*\}, \underset{\sim}{A}^{p^*}_\gamma \setminus \gamma^* \rangle, & \gamma \in Supp(p^*), \\ \langle \{\gamma^*\}, \kappa \setminus \gamma^* + 1 \rangle, & otherwise. \end{cases}$$

Clearly $p_{fin} \Vdash \gamma^* \in \cap_{i\in I_\alpha} \underset{\sim}{x}_i$. It remains to argue that p_{fin} is an extension of p^*: Indeed $p_{fin}(0) \geq p^*_0$ since $\gamma^* \in A_0$ and $\gamma^* > \delta_0 > \sup(a^{p^*}_0)$. Suppose that $\beta \in Supp(p^*)$ and that $p_{fin} \restriction \beta \geq p^* \restriction \beta$. If $\beta = \beta_0 + 1$ is successor then $p_{fin} \restriction \beta \Vdash \gamma^* \in \underset{\sim}{x}_{\beta_0} \setminus \epsilon^* \subseteq \underset{\sim}{A}^{p^*}_\beta$, hence $p_{fin} \restriction \beta \Vdash p_{fin}(\beta) \geq p^*(\beta)$. If β is limit of cofinality less than κ, then since $\beta \in M^*$, we have that $I_\beta \subseteq M^* \cap \beta$, hence by induction $p_{fin} \restriction \beta \Vdash \gamma^* \in \cap_{i\in I_\beta} \underset{\sim}{x}_\beta \setminus \epsilon^* \subseteq \underset{\sim}{A}^{p^*}_\beta$. Finally, if the cofinality of β is at least κ, then there is $i_\beta < \kappa$ such that $p^* \restriction \beta \Vdash \gamma^* \in x^*_{i_\beta} \setminus \epsilon^* \subseteq \underset{\sim}{A}^{p^*}_\beta$. □

This completes the proof of the theorem. □

Let us briefly describe a natural, yet unfruitful, strategy to make $\mathscr{U}^*$ become a κ-complete ultrafilter. Given a Laver-indestructible supercompact

cardinal κ and $\alpha < \lambda$ force over $V^{\mathbb{P}_\alpha}$ with $\mathbb{M}_{\mathscr{V}_\alpha}$, where $\mathscr{V}_\alpha$ is an extension of $\mathscr{U}^*_\alpha$ (in $V^{\mathbb{P}_\alpha}$) to a κ-complete ultrafilter. Note that since $\mathbb{P}_\alpha$ is κ-directed-closed, κ is still supercompact in the corresponding extension and thus the choice of $\mathscr{V}_\alpha$ is available. In addition, if the length of the iteration is $\lambda = \kappa^+$ then the union of the (tower of) κ-complete ultrafilters generated along the way will be also an ultrafilter, $\mathscr{U}_\infty$. The problem with this approach is that we lose control upon κ-completeness of $\mathscr{U}_\infty$. Indeed, even the union of the first ω-many ultrafilters generated might not be κ-complete, as we argued in the discussion preceding Theorem 2.12.

The approach of Theorem 2.20 is to iterate $\mathbb{M}_{\mathscr{U}}$ more carefully so that we have complete control on the completeness of the ultrafilter $\mathscr{U}_\infty$.

Definition 2.16. For $f\colon \kappa \to \kappa$ and an ultrafilter $\mathscr{U}$ over κ we say that f *is constant* $\mathrm{mod}(\mathscr{U})$ if there is $\gamma < \kappa$ such that $f^{-1}\{\gamma\} \in \mathscr{U}$. Similarly, f *is* 1-1$\mathrm{mod}(\mathscr{U})$ if there is $X \in \mathscr{U}$ such that $|f^{-1}\{\gamma\} \cap X| < \kappa$ for every $\gamma < \kappa$.

Definition 2.17. A κ-complete ultrafilter $\mathscr{U}$ is called a *P-point* if every function $f : \kappa \to \kappa$ that is not constant $\mathrm{mod}(\mathscr{U})$ is 1-1$\mathrm{mod}(\mathscr{U})$.

Remark 2.18. $\mathscr{U}$ is a P-point if and only if every sequence $\langle X_\alpha \mid \alpha < \kappa\rangle$ of elements in $\mathscr{U}$ has a pseudo intersection in $\mathscr{U}$; namely, there is $X \in \mathscr{U}$ such that $X \subseteq^* X_\alpha$ for every $\alpha < \kappa$.

Lemma 2.19. *Every κ-complete ultrafilter $\mathscr{U}$ carrying a strong generating sequence*[17] *of size κ^+ is a P-point. In particular,* $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$.

Proof. Let $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa^+\rangle$ be a generating sequence of $\mathscr{U}$. To see it is P-point suppose that $\langle X_\alpha \mid \alpha < \kappa\rangle$ is any $\kappa-$sequence of members of $\mathscr{U}$. By definition of generating sequence, for each $\alpha < \kappa$ there is $\beta_\alpha < \kappa^+$ such that $A_{\beta_\alpha} \subseteq^* X_\alpha$. Consider $\beta^* = \sup_{\alpha<\kappa} \beta_\alpha < \kappa^+$. Then A_{β^*} is a pseudo intersection of the sequence $\langle X_\alpha \mid \alpha < \kappa\rangle$. For the in particular claim use the fact that every P-point ultrafilter $\mathscr{U}$ satisfies $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ (see [9, Proposition 5.13]). □

In analogy to Theorem 2.5, next we show that every ground model κ-complete ultrafilter can be extended to a *well-behaved* one: to wit, to a P-point. By virtue of the above lemma, this gives an alternative (and less severe way) to trasnmute an arbitrary κ-complete ultrafilter into a Galvin one. Unlike Theorem 2.5, the models we produce this time have the extra feature that 2^κ can be made arbitrarily large. Our construction owes much to previous work of Gitik and Shelah [22]. Recall that κ is *almost huge with target λ* if there is $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $M^{<\lambda} \subseteq M$.

Theorem 2.20. *Assume the* GCH *holds and suppose that κ is an almost huge cardinal with measurable target λ. For unboundedly-many cardinals $\delta \in (\kappa, \lambda)$ there is a forcing extension $V[G^*_\delta]$ where $2^\kappa = \delta$ and every ground model κ-complete ultrafilter over κ extends to a P-point ultrafilter in $V[G^*_\delta]$.*

[17] Recall Definition 2.11.

In addition, there is a truncation of $V[G^*_\delta]$ where the same configuration holds and κ is supercompact there.[18]

Proof. Let $j\colon V \to M$ be such that $\mathrm{crit}(j) = \kappa$, $M^{<\lambda} \subseteq M$ and $j(\kappa) = \lambda$. Since λ is assumed to be measurable in the ground model we can let $\mathcal{U}$ be a normal measure over it. Let us define an Easton supported iteration

$$\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \beta < \kappa,\, \alpha \leq \kappa \rangle$$

as follows: First, $\mathbb{Q}_\alpha$ is trivial unless α is measurable in $V^{\mathbb{P}_\alpha}$ in which case $\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for the two-step iteration $\mathbb{Q}_{\alpha,0} * \mathbb{Q}_{\alpha,1}$. Here $\mathbb{Q}_{\alpha,0}$ stands for the atomic forcing choosing an ordinal $F(\alpha) < \kappa$ and $\mathbb{Q}_{\alpha,1}$ for an $<\alpha$-supported iteration $\langle \mathbb{R}^\alpha_\beta, \mathbb{S}^\alpha_\gamma \mid \beta \leq F(\alpha),\, \gamma < F(\alpha) \rangle$ defined in the following manner: for each $\beta < F(\alpha)$, $\mathbb{S}^\alpha_\beta$ is trivial unless $\Vdash_{\mathbb{P}_\alpha * \mathbb{R}^\alpha_\beta}$ "α is measurable", in which case $\mathbb{S}^\alpha_\beta$ is forced to be the $<\alpha$-supported product $\prod_{\mathscr{U}} \mathbb{M}_{\mathscr{U}}$, where $\mathscr{U}$ ranges over all $\mathbb{P}_\alpha * \mathbb{R}^\alpha_\beta$-names for an α-complete ultrafilter over α.

Note that this iteration is coherent in the following sense: for each $\alpha < \kappa$, if $G \subseteq \mathbb{P}_\alpha$ is V-generic and $\delta < \delta' < \kappa$ then $(\mathbb{Q}_{\alpha,1})_{G*\{\delta\}} = (\mathbb{Q}_{\alpha,1})_{G*\{\delta'\}} \restriction \delta$.

Since $\mathbb{Q}_{\alpha,1}$ is an $<\alpha$-supported iteration of α^+-stationary c.c., $<\alpha$-closed and countably-parallel closed forcing (cf. Proposition 2.9) it follows from [14, Theorem 1.2] that $\mathbb{Q}_{\alpha,1}$ is α^+-cc. Also, $\mathbb{P}_\kappa$ is κ-cc because κ is Mahlo and $\mathbb{P}_\kappa$ is Easton-supported. Hereafter $G_\kappa \subseteq \mathbb{P}_\kappa$ will be a V-generic filter.

Let us factor $j(\mathbb{P}_\kappa)$ as $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}}_{\kappa,0} * \underset{\sim}{\mathbb{Q}}_{\kappa,1} * \underset{\sim}{\mathbb{P}}'_{(\kappa,j(\kappa))}$.

Claim 2.21. *Let $\kappa \leq \rho < \lambda$. Then κ is $\theta(\rho)$-supercompact in $V[G_\kappa * \{\rho\}]^{\mathbb{Q}_{\kappa,1}}$ and $M[G_\kappa * \{\rho\}]^{\mathbb{Q}_{\kappa,1}}$, where $\theta(\rho)$ is the first inaccessible cardinal above ρ.*

In particular, κ is fully supercompact in $(V[G_\kappa * \{\rho\}]^{\mathbb{Q}_{\kappa,1}})_{\theta(\rho)}$.

Proof. To ease the notation let us put $\theta := \theta(\rho)$. Let W be a fine normal measure over $\mathcal{P}_\kappa(\theta)$ and let $j_W\colon V \to N$ be the corresponding elementary embedding. Let G^ρ_κ a $V[G_\kappa * \{\rho\}]$-generic for $\mathbb{Q}_{\kappa,1}$, and let us lift j_W to the model $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$. Note that in $N[G_\kappa * \{\rho\} * G^\rho_\kappa]$ we have $2^\kappa \geq \rho$, for at each step of the iteration $\mathbb{Q}_{\kappa,1}$ we add a new subset to κ. Also, $N[G_\kappa * \{\rho\} * G^\rho_\kappa]$ is closed under θ-sequences in $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$, and so there are no $N[G_\kappa * \{\rho\} * G^\rho_\kappa]$-measurable cardinals in (κ, θ).

Thus, by closure under θ-sequences, the closure of

$$j_W(\mathbb{P}_\kappa)/(G_\kappa * \{\rho\} * G^\rho_\kappa)$$

is at least θ^+, even in $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$. Also note that every dense set in this quotient is represented by a function $f : \mathcal{P}_\kappa(\theta) \to \mathcal{P}(\mathbb{P}_\kappa)$. Since there are in total $(2^\kappa)^\theta = \theta^+$ of such functions we can construct a master sequence which induces an $N[G_\kappa * \{\rho\} * G^\rho_\kappa]$-generic filter, $G_{(\kappa, j_W(\kappa))}$. Next, the top-most forcing $\mathbb{Q}_{\kappa,0} * \mathbb{Q}_{\kappa,1}$ is κ^+-cc and when the length of $\mathbb{Q}_{\kappa,1}$ is restricted to ρ its size becomes $\rho \cdot 2^{2^\kappa} < \theta$. Since every maximal antichain in $N[G_\kappa * \{\rho\} * G^\rho_\kappa * G_{(\kappa, j_W(\kappa))} * \{j_W(\rho)\}]$ for $j_W(\mathbb{Q}_{\kappa,1})$ is represented by a

[18] By a truncation of $V[G^*_\delta]$ we mean a ZFC model of the form $V[G^*_\delta]_\theta$.

function $F\colon \mathcal{P}_\kappa(\theta)\to\mathcal{P}_\kappa((\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}})\in V[G_\kappa*\{\rho\}]$ and $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ is of size $<\theta$ there are at most θ^+ many such functions. What remains in order to argue that κ is θ-supercompact in $V[G_\kappa*\{\rho\}*G^\rho_\kappa]$ is standard.

Finally, κ is also supercompact in $M[G_\kappa*\{\rho\}*G^\rho_\kappa]$ as this model is closed under $<\lambda$-sequences in $V[G_\kappa*\{\rho\}*G^\rho_\kappa]$. Also, θ remains inaccessible in all of these generic extensions, which entails our in particular assertion. □

For each $\rho<\lambda$, the poset $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ is κ^+-cc and of size less than λ. Thus, there are less than λ-many nice names for subsets of κ. Denote these names by $\langle \underset{\sim}{A}^\rho_\tau\mid\tau<\theta_\rho\rangle$. We can find in $V[G_\kappa]$ an enumeration $\langle \underset{\sim}{A}_\tau\mid\tau<\lambda\rangle$ of subsets of κ such that for every $\tau_1\leq\tau_2$, there are δ_1,δ_2 such that

$$\underset{\sim}{A}_{\tau_1}=\underset{\sim}{A}^{\rho(\tau_1)}_{\delta_1},\ \underset{\sim}{A}_{\tau_2}=\underset{\sim}{A}^{\rho(\tau_2)}_{\delta_2}\text{ and }\rho(\tau_1)\leq\rho(\tau_2).$$

Let C' be the club of closure points of the map φ assigning each $\tau<\lambda$ to its $\rho(\tau)<\lambda$. Since λ is measurable there is $S\in\mathcal{U}$ consisting of inaccessibles.[19] In particular, $\varphi``\rho\subseteq\rho$ for all $\rho\in C:=C'\cap S$. Thus, for each $\rho\in C$, the sequence $\langle \underset{\sim}{A}_\tau\mid\tau<\rho\rangle$ codes all the $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-names for subsets of κ.[20]

Let $\epsilon<\lambda$ be an ordinal above the generators of all κ-complete ultrafilters over κ lying in V. More precisely, for each such κ-complete ultrafilter $U\in V$ let some $\epsilon_U\in\bigcap j``U$ and define $\epsilon:=\sup\{\epsilon_U\mid U\in V\text{ measure over }\kappa\}$. Note that $\epsilon<\lambda$ as κ is an almost huge cardinal with (inaccessible) target λ.

For every $\rho\in C$ and every $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-name $\underset{\sim}{\mathscr{U}}$ for a κ-complete ultrafilter over κ let us define a condition $r_{\underset{\sim}{\mathscr{U}}}$ as follows:

$$r_{\underset{\sim}{\mathscr{U}}}:=\langle a_{\underset{\sim}{\mathscr{U}}}\cup(A_{\underset{\sim}{\mathscr{U}}}\cap\epsilon),A_{\underset{\sim}{\mathscr{U}}}\setminus(\epsilon+1)\rangle,$$

where here we used the following notations:

(1) $a_{\underset{\sim}{\mathscr{U}}}$ is the standard $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-name for the $\mathbb{M}_{\underset{\sim}{\mathscr{U}}}$-generic;
(2) $A_{\underset{\sim}{\mathscr{U}}}$ is the standard $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-name for the set $\bigcap j``\underset{\sim}{\mathscr{U}}$.

Notice that $r_{\underset{\sim}{\mathscr{U}}}$ is forced by the trivial condition of $(\mathbb{Q}_{j(\kappa),1})_{j(G_\kappa)*\{j(\rho)\}}$ to be a member of $\mathbb{M}_{j(\underset{\sim}{\mathscr{U}})}$: First, this latter condition forces "$\min A_{\underset{\sim}{\mathscr{U}}}\geq\kappa$", and thus $a_{\underset{\sim}{\mathscr{U}}}\cup(A_{\underset{\sim}{\mathscr{U}}}\cap\epsilon)$ is a legitimate stem. Second, the same condition also forces $j(\underset{\sim}{\mathscr{U}})$ to be $j(\kappa)$-complete. Besides, $M[j(G_\kappa)*\{j(\rho)\}]$ is closed enough (within $V[j(G_\kappa)*\{j(\rho)\}]$) to grant that the standard name for $j``\underset{\sim}{\mathscr{U}}$ belongs to it. Therefore, $0\Vdash_{(\mathbb{Q}_{j(\kappa),1})_{j(G_\kappa)*\{j(\rho')\}}}\bigcap j``\underset{\sim}{\mathscr{U}}\in j(\underset{\sim}{\mathscr{U}})$.

The key feature of $r_{\underset{\sim}{\mathscr{U}}}$ is that it forces $a_{j(\underset{\sim}{\mathscr{U}})}$ to contain all the generators ϵ_W for measures $W\in V$ that are forced inside $A_{\underset{\sim}{\mathscr{U}}}$.

For every $\rho\in C$ let q_ρ be the condition in $(\mathbb{Q}_{j(\kappa),1})_{j(G_\kappa)*\{j(\rho)\}}$ with support $j``\rho$ (hence of cardinality $<j(\kappa)$) defined as follows: For every $\rho'<\rho$ and a $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho'\}}$-name for a κ-complete ultrafilter $\underset{\sim}{\mathscr{U}}$ over κ,

$$q_\rho\restriction j(\rho')\Vdash q_\rho(j(\rho'))(j(\underset{\sim}{\mathscr{U}}))=r_{\underset{\sim}{\mathscr{U}}}.$$

[19]Recall that $\mathcal{U}$ is the normal measure on λ we fixed at the beginning of the proof.

[20]Specifically, if σ is a $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-name for a subset of κ then there is $\tau<\rho$ such that $0\Vdash_{(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}}\sigma=\underset{\sim}{A}_\tau$.

Observe that in this definition we are implicitly using the coherency of the iteration: more specifically that $\underset{\sim}{\mathscr{U}}$ is a $(\mathbb{Q}_{\kappa,1})_{G_\kappa * \{\rho\}} \restriction \rho'$-name.

A moment's reflection makes clear that q_ρ is a *master condition* for the $V[G_\kappa * \{\rho\}]$-generic filter G^ρ_κ: i.e., $j(p) \leq q_\rho$ for all $p \in G^\rho_\kappa$. In addition, $\langle q_\rho \mid \rho \in C\rangle$ is coherent: namely, $q_\rho \restriction \rho' = q_{\rho'}$ for all $\rho' \leq \rho$ in C.

Fix $\rho \in C \setminus (\epsilon+1)$ and work inside $V[G_\kappa]$. For each $\tau < \rho$, $\zeta < \epsilon$ and $t \in (\mathbb{Q}_{\kappa,1})_{G_\kappa * \{\rho\}}$ let $\underset{\sim}{D}_{\tau,\zeta,t} \in M[G_\kappa]$ be a $(\mathbb{Q}_{\kappa,1})_{G_\kappa * \{\rho\}}$-name for the dense open set

$$\{p \in \mathbb{P}'_{(\kappa,j(\kappa)]} \mid \exists s_\kappa \geq t\ \exists i \in 2\ ((s_\kappa, p) \Vdash^i_{(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}} * \mathbb{P}'_{(\kappa,j(\kappa)]}} \zeta \in j(\underset{\sim}{A}_\tau))\}.^{21}$$

Since the trivial condition of $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ forces that $\mathbb{P}'_{(\kappa,j(\kappa)]}$ is ρ^+-closed, the intersection $\bigcap_{\tau,\zeta,t} \underset{\sim}{D}_{\tau,\zeta,t}$ is still dense open. In particular, the trivial condition of $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ forces the existence of $p_\rho \in \bigcap_{\tau,\zeta,t} \underset{\sim}{D}_{\tau,\zeta,t}$ such that

$$p_\rho \restriction j(\kappa) \Vdash p_\rho(j(\kappa)) \geq \{j(\rho)\} * q_\rho.^{22}$$

Notice that p_ρ has the property that for every $\tau < \rho$, $\zeta < \epsilon$,

$$(\otimes) \qquad \{s_\kappa \in (\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}} \mid \exists i \in 2\ (s_\kappa, p_\rho) \Vdash^i_{(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}} * \mathbb{P}_{(\kappa,j(\kappa)]}} \zeta \in j(\underset{\sim}{A}_\tau)\}$$

is dense open in $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$.

Therefore if G^ρ_κ is a $M[G_\kappa * \{\rho\}]$-generic filter for $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ and $t \in G^\rho_\kappa$ then there is $\langle s, s_\kappa\rangle \in (G_\kappa * \{\rho\}) * G^\rho_\kappa$ with $s_\kappa \geq t$ such that, for some $i \in 2$,

$$\langle s, s_\kappa, p_\rho\rangle \Vdash^i \zeta \in j(\underset{\sim}{A}_\tau).$$

Working in $M[G_\kappa]$ define for each $\langle s, s_\kappa\rangle \in (G_\kappa * \{\rho\}) * (\underset{\sim}{\mathbb{Q}}_{\kappa,1})_{G_\kappa*\{\rho\}}$, ordinals $\zeta < \epsilon$ and $\tau < \lambda$, and $i < 2$, the set

$$A^i_{(s,s_\kappa,\zeta,\tau)} := \{\rho < \lambda \mid \langle s, s_\kappa, p_\rho\rangle \Vdash^i \zeta \in j(\underset{\sim}{A}_\tau)\}.$$

Denote by $A^2_{(s,s_\kappa,\zeta,\tau)}$ the complement of the union of the above two sets.

Note that the $A^i_{(s,s_\kappa,\zeta,\tau)}$'s belong to $V[G_\kappa]$, while we would like to decide which of them are $\mathcal{U}$-large. Since $\mathcal{U}$ just measures V-subsets of κ this is, in principle, troublesome. However, the forcing $\mathbb{P}_\kappa$ is small (i.e., of cardinality $<\lambda$). Hence, by Lévy-Solovay, $\mathcal{U}$ naturally extends to a normal measure over λ in $V[G_\kappa]$. For simplicity, let us keep denoting this measure by $\mathcal{U}$.

For each quadruple $(s, s_\kappa, \zeta, \tau)$ as before let $i_{(s,s_\kappa,\zeta,\tau)} < 3$ be the unique i for which $A^i_{(s,s_\kappa,\zeta,\tau)} \in \mathcal{U}$. Let us now amalgamate these sets by taking a sort of diagonal intersection. More precisely, let us define a set A as follows:

$$\{\rho < \lambda \mid (\langle s, s_\kappa\rangle \in (G_\kappa * \{\rho\}) * (\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}} \wedge \max(\zeta,\tau) < \rho) \Rightarrow \rho \in A^{i_{(s,s_\kappa,\zeta,\tau)}}_{(s,s_\kappa,\zeta,\tau)}\}.$$

Claim 2.22. *$A \in \mathcal{U}$.*

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[21] Here we are adopting the usual convention that $p \Vdash^1 \varphi$ (resp. $p \Vdash^0 \varphi$) stands for $p \Vdash \varphi$ (resp. $p \Vdash \neg\varphi$). Also, here $\mathbb{P}'_{(\kappa,j(\kappa)]}$ denotes the iteration $\mathbb{P}'_{(\kappa,j(\kappa))} * \{j(\rho)\} * j(\mathbb{Q}_{\kappa,1})$.

[22] Note that this implies that the 0-component of $p_\rho(j(\kappa))$ is $j(\rho)$.

Proof of claim. Let $j_{\mathcal{U}}\colon V \to N$ denote the ultrapower embedding by the normal measure $\mathcal{U}$. As we indicated before, the usual Lévi-Solovay argument allows to lift $j_{\mathcal{U}}$ to $j_{\mathcal{U}}\colon V[G_\kappa] \to N[G_\kappa]$. Let us now work in $V[G_\kappa]$. Since $\mathbb{Q}_{\kappa,0}$ is the atomic forcing for choosing an ordinal less than $j(\kappa) = \lambda$, by elementarity, $j_{\mathcal{U}}(\mathbb{Q}_{\kappa,0})$ randomly chooses any ordinal $< j_{\mathcal{U}}(\lambda)$. In particular, it might choose λ. Thus, after this choice, $j_{\mathcal{U}}(\mathbb{Q}_{\kappa,1})$ becomes a λ-length and $<\lambda$-supported iteration such that $j_{\mathcal{U}}(\mathbb{Q}_{\kappa,1}) \restriction \rho = (\mathbb{Q}_{\kappa,1})_{G*\{\rho\}}$ for all $\rho < \lambda$.

Let $\langle s, s_\kappa\rangle \in (G_\kappa * \{\lambda\}) * j_{\mathcal{U}}(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\lambda\}}$ and ordinals $\zeta, \tau < \lambda$. Since s_κ has support $<\lambda$ there is $\rho < \lambda$ such that $s_\kappa \restriction \mathrm{supp}(s_\kappa) \in (\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$, hence $\langle s, s_\kappa \restriction \mathrm{supp}(s_\kappa)\rangle$ is a pair among the ones we considered before. Therefore,

$$\lambda \in j_{\mathcal{U}}(A^{i_{\langle s, s_\kappa\restriction \mathrm{supp}(s_\kappa),\zeta,\tau\rangle}}_{\langle s, s_\kappa\restriction \mathrm{supp}(s_\kappa),\zeta,\tau\rangle}).$$

Set $B := j_{\mathcal{U}}(A^{i_{\langle s,s_\kappa,\zeta,\tau\rangle}}_{\langle s,s_\kappa,\zeta,\tau\rangle})$ and let us denote by B' the right-hand-side set above. It is not hard to check that $B = B'$. We thus conclude that $\lambda \in j_{\mathcal{U}}(A)$. $\square$

Put $C^* := A \cap C$. Let $\rho, \rho' \in C^*$ be with $\rho < \rho'$. For each pair $\langle s, s_\kappa\rangle \in (G_\kappa * \{\rho\}) * (\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ and $\zeta, \tau < \rho$, we have $\rho, \rho' \in A^{i_{(s,s_\kappa,\zeta,\tau)}}_{(s,s_\kappa,\zeta,\tau)}$.[23] Hence, given these conditions and ordinals, it follows for each $i < 2$ that

$$(\star) \qquad \langle s, s_\kappa, p_\rho\rangle \Vdash^i \zeta \in j(\underset{\sim}{A}_\tau) \text{ iff } \langle s, s_\kappa, p_{\rho'}\rangle \Vdash^i \zeta \in j(\underset{\sim}{A}_\tau),$$

Next, for all $\rho \in C^*$ and $\zeta < \epsilon$ consider

$$\mathscr{U}_{\rho,\zeta} := \{(\underset{\sim}{A}_\tau)_{G_\kappa*\{\rho\}*G^\rho_\kappa} \subseteq \kappa \mid \exists \langle s, s_\kappa\rangle \in G_\kappa*\{\rho\}*G^\rho_\kappa\, (\langle s, s_\kappa, p_\rho\rangle \Vdash \zeta \in j(\underset{\sim}{A}_\tau))\}.$$

Claim 2.23. *$\mathscr{U}_{\rho,\zeta}$ is a κ-complete ultrafilter in $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$.*

Proof of claim. Clearly $\mathscr{U}_{\rho,\zeta}$ is a filter and it belongs to $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$. Working inside $V[G_\kappa * \{\rho\} * G^\rho_\kappa]$ let $A \subseteq \kappa$. Let $\underset{\sim}{A}$ be a $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$-name representing A and $\langle s, s_\kappa\rangle \in G_\kappa*\{\rho\}*G^\rho_\kappa$ forcing "$\underset{\sim}{A} \subseteq \kappa$". Due to our initial preparation (see Footnote 20) there is $\tau < \rho$ such that $s_\kappa \Vdash^{V[G_\kappa*\{\rho\}]} \underset{\sim}{A} = \underset{\sim}{A}_\tau$. Since $\rho \in C^* \subseteq C$, the above give us a 4-tuple $\vec{\sigma} := \langle s, s_\kappa, \tau, \zeta\rangle$ of the form we considered above. By extending $\langle s, s_\kappa\rangle$ if necessary we can ensure that s_κ belongs to the set displayed in equation $(\otimes)$ above. Note that this move does not ruin that $\langle s, s_\kappa\rangle \in G_\kappa * \{\rho\} * G^\rho_\kappa$. Hence, there is $i \in 2$ such that $\langle s, s_\kappa, p_\rho\rangle \Vdash^i \zeta \in j(\underset{\sim}{A}_\tau)$. This means that

$$\text{either } A \in \mathscr{U}_{\rho,\zeta} \text{ or } \kappa \setminus A \in \mathscr{U}_{\rho,\zeta}.$$

Let $\langle A_\alpha \mid \alpha < \theta < \kappa\rangle \in V[G_\kappa * \{\rho\} * G^\rho_\kappa]$ be subsets of κ. By our special coding there are names $\underset{\sim}{A}_{\tau_\alpha}$ representing each A_α such that each τ_α's is $<\rho$. Let H be some $\mathbb{P}'_{(\kappa,j(\kappa))}$-generic over $M[G_\kappa * \{\rho\} * G^\rho_\kappa]$ such that $p_\rho \in H$. Remember that $p_\rho \restriction j(\kappa) \Vdash p_\rho(j(\kappa)) \geq \{j(\rho)\} * q_\rho$ and that q_ρ was a master condition for G^ρ_κ. Hence, our almost huge embedding j lifts to

$$j\colon V[G_\kappa * \{\rho\} * G^\rho_\kappa] \to M[G_\kappa * \{\rho\} * G^\rho_\kappa * H].$$

[23]Note that this requires invoking the natural identification of conditions in $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho\}}$ as conditions in $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\rho'\}}$.

Clearly, $j(\langle A_{\tau_\alpha} \mid \alpha<\theta\rangle)=\langle j(A_{\tau_\alpha})\mid\alpha<\theta\rangle$. Also, $\zeta\in\bigcap_{\alpha<\theta}j(A_{\tau_\alpha})$.

Work for a moment in $M[G_\kappa*\{\rho\}]$. For each $\alpha<\theta$ recall that

$$E_{\alpha,\rho}:=\{s_\kappa\in\mathbb{Q}_{\kappa,1}\mid\exists i<2\,(\langle s_\kappa,p_\rho\rangle\Vdash^i\zeta\in j(\underset{\sim}{A}_{\tau_\alpha}))\}$$

was dense open. Since $\mathbb{Q}_{\kappa,1}$ is κ-closed and $\alpha<\theta$, $E_\rho:=\bigcap_{\alpha<\theta}E_{\alpha,\rho}$ is also dense open, and thus there is $s_\kappa\in G^\rho_\kappa\cap E_\rho$. Note that

$$\langle s_\kappa,p_\rho\rangle\Vdash^i\zeta\in\bigcap_{\alpha<\theta}j(\underset{\sim}{A}_{\tau_\alpha})$$

for some $i\in 2$. Since $\langle s_\kappa,p_\rho\rangle\in G^\rho_\kappa*H$ and $\zeta\in\bigcap_{\alpha<\theta}j(A_{\tau_\alpha})$ it cannot be that $i=0$, so $\langle s_\kappa,p_\rho\rangle\Vdash\zeta\in\bigcap_{\alpha<\theta}j(\underset{\sim}{A}_{\tau_\alpha})$. Now let $s\in G_\kappa*\{\rho\}$ be such that $\langle s,s_\kappa,p_\rho\rangle\Vdash\zeta\in\bigcap_{\alpha<\theta}j(\underset{\sim}{A}_{\tau_\alpha})$. This shows that $\bigcap_{\alpha<\theta}A_\alpha\in\mathscr{U}_{\rho,\zeta}$. □

Claim 2.24. *For each $\zeta<\epsilon$, $\langle\mathscr{U}_{\rho,\zeta}\mid\rho\in C^*\rangle$ defines a tower of ultrafilters.*

Proof of claim. Suppose $\rho<\rho'\in C^*$ and let $A\in\mathscr{U}_{\rho,\zeta}$. Then, there is a pair $\langle s,s_\kappa\rangle$ such that $\langle s,s_\kappa,p_\rho\rangle\Vdash\zeta\in j(\underset{\sim}{A}_\tau)$. By equation $(\star)$ above this is also true when replacing p_ρ by $p_{\rho'}$. Thereby, $A\in\mathscr{U}_{\rho',\zeta}$. □

Let $\langle\rho_\alpha\mid\alpha<\kappa^+\rangle\subseteq C^*$ and $\delta:=\sup_{\alpha<\kappa^+}\rho_\alpha$. By the previous claim $\langle\mathscr{U}_{\rho_\alpha,\zeta}\mid\alpha<\kappa^+\rangle$ is a tower of measures. Working in $V[G_\kappa*\{\delta\}*G^\delta_\kappa]$ define

$$\mathscr{V}_{\delta,\zeta}:=\bigcup_{\alpha<\kappa^+}\mathscr{U}_{\rho_\alpha,\zeta}.$$

Since $V[G_\kappa*\{\rho_\alpha\}*(G^\delta_\kappa\restriction\rho_\alpha)]$ is a submodel of $V[G_\kappa*\{\delta\}*G^\delta_\kappa]$ and $(\mathbb{Q}_{\kappa,1})_{G_\kappa*\{\delta\}}$ is κ^+-cc it is immediate that $\mathscr{V}_{\delta,\zeta}$ is a κ-complete ultrafilter.

Claim 2.25. *$\mathscr{V}_{\delta,\zeta}$ admits a strong generating sequence of size κ^+.*
In particular, $\mathscr{V}_{\delta,\zeta}$ is a P-point (see Lemma 2.19).

Proof. For each $\alpha<\kappa^+$, $\mathscr{U}_{\rho_\alpha,\zeta}\in V[G_\kappa*\{\delta\}*(G^\delta_\kappa\restriction\rho_\alpha)]$ hence the iteration $\mathbb{Q}_{\kappa,1}$ at stage ρ_α shoots a Mathias set $x_\alpha\subseteq\kappa$ (over $V[G_\kappa*\{\delta\}*(G^\delta_\kappa\restriction\rho_\alpha)]$) for the measure $\mathscr{U}_{\rho_\alpha,\zeta}$. Namely, x_α is almost included in every $A\in\mathscr{U}_{\rho_\alpha,\zeta}$.

We claim that $\langle x_\alpha\mid\alpha<\kappa^+\rangle$ is the sought strong generating sequence. First, for each $A\in\mathscr{V}_{\delta,\zeta}$ there is $\alpha<\kappa^+$ such that $A\in\mathscr{U}_{\rho_\alpha,\zeta}$ and so $x_\alpha\subseteq^*A$. Second, $\langle x_\alpha\mid\alpha<\kappa^+\rangle$ is $\subseteq^*$-decreasing: Fix $\alpha<\beta<\kappa^+$. We would like to show that $x_\alpha\in\mathscr{U}_{\rho_\beta,\zeta}$. Recall that $x_\alpha=(a_{\mathscr{U}_{\rho_\alpha,\zeta}})_{G_\kappa*\{\rho_\beta\}*G^{\rho_\alpha}_\kappa}$ (see item (1) in p.19). Hence we need to check that

$$\exists\langle s,s_\kappa\rangle\in(G_\kappa*\{\rho_\beta\}*G^{\rho_\beta}_\kappa)\ \langle s,s_\kappa,p_{\rho_\beta}\rangle\Vdash\zeta\in j(a_{\mathscr{U}_{\rho_\alpha,\zeta}}).$$

By elementarity of j, $j(a_{\mathscr{U}_{\rho_\alpha,\zeta}})=a_{j(\mathscr{U}_{\rho_\alpha,\zeta})}$ is the canonical $j(\mathbb{P}_\kappa*\{\rho_\alpha\}*\mathbb{Q}_{\kappa,1})$-name for the Mathias generic of $\mathbb{M}_{j(\mathscr{U}_{\rho_\alpha,\zeta})}$. Since $p_{\rho_\beta}\geq q_{\rho_\beta}$, the definition of the condition $r_{\mathscr{U}_{\rho_\alpha,\zeta}}$ (see p.19) entails the following:

$$\langle 0,p_{\rho_\beta}\rangle\Vdash a_{j(\mathscr{U}_{\rho_\alpha,\zeta})}\cap(\epsilon+1)\supseteq(\bigcap j\text{“}\underset{\sim}{\mathscr{U}}_{\rho_\alpha,\zeta})\cap(\epsilon+1).$$

Working in $M[G_\kappa * \{\rho_\alpha\} * G_\kappa^{\rho_\alpha}]$, we have that for every $A \in \mathscr{U}_{\rho_\alpha,\zeta}$, there is a name A_τ for A such that $p_{\rho_\alpha} \Vdash \zeta \in j(\underset{\sim}{A}_\tau)$. Hence $p_{\rho_\alpha} \Vdash \zeta \in \bigcap j\text{``}\underset{\sim}{\mathscr{U}}_{\rho_\alpha,\zeta}$. Hence there is $\langle s, s_\kappa\rangle \in G_\kappa * \{\rho_\alpha\} * G_\kappa^{\rho_\alpha}$ such that

$$\langle s, s_\kappa, p_{\rho_\alpha}\rangle \Vdash \zeta \in j(a_{\mathscr{U}_{\rho_\alpha,\zeta}}).$$

Since $\rho_\alpha < \rho_\beta$ are in C^* equation $(\star)$ implies that $\langle s, s_\kappa, p_{\rho_\beta}\rangle \Vdash \zeta \in j(a_{\mathscr{U}_{\rho_\alpha,\zeta}})$. This, by definition, yields $x_\alpha \in \mathscr{U}_{\rho_\beta,\zeta}$. □

Claim 2.26. *Every κ-complete ultrafilter $\mathscr{U}$ from the ground model extends to $\mathscr{V}_{\delta,\zeta}$, for some $\zeta < \epsilon$.*

Proof of claim. Let $\zeta < \epsilon$ be such that $\mathscr{U} = \{X \subseteq \kappa \mid \zeta \in j(X)\}$. Clearly, $\mathscr{U} \subseteq \mathscr{U}_{\rho_\alpha,\zeta}$ for all $\alpha < \kappa^+$, hence $\mathscr{U} \subseteq \mathscr{V}_{\delta,\zeta}$. □

Letting $G^*_\delta := G_\kappa * \{\delta\} * G^\delta_\kappa$ the above arguments show that the generic extension $V[G^*_\delta]$ models the desired configuration: On one hand, every ground model κ-complete ultrafilter extends to a P-point. On the other hand, $2^\kappa = \delta$ in $V[G^*_\delta]$ due to GCH in V and the usual considerations about nice names. Finally, for the in particular claim of the theorem let $W := V[G^*_\delta]_{\theta(\delta)}$. By virtue of Claim 2.21 this truncation is as needed. □

Remark 2.27. Note that every measurable cardinal κ always carries a κ-complete ultrafilter which is not a P-point. To see this, take any κ-complete ultrafilter $\mathscr{U}$ over κ, and a bijection $\phi : [\kappa]^2 \to \kappa$ and define $\mathscr{W} := \phi_*(\mathscr{U} \times \mathscr{U})$. One can check that $\mathscr{W}$ is a κ-complete non-P-point ultrafilter. As witnessed by $L[\mathscr{U}]$, it is consistent that every ultrafilter is a finite power of a normal one (hence of a P-point), and such ultrafilters are always Galvin (see [9, Corollary 5.29]). If κ is κ-compact[24] then there is a κ-complete ultrafilter which is not a finite power of a P-point (see, e.g., [28, §3.9]).

Question 2.28. Is it consistent to have a measurable cardinal carrying a κ-complete ultrafilter $\mathscr{U}$ such that $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ but it is not Rudin-Keisler equivalent to a finite power of P-points?

Question 2.29. Is it ZFC-provable that a supercompact cardinal always admits a κ-complete non Galvin ultrafilter?

2.2. **Galvin's property in the choiceless context.** Another way to examine Galvin's property at very large cardinals is to consider relatively small cardinals in ZF. A typical example is $\aleph_1$ under AD. Indeed, Solovay proved that both $\aleph_1$ and $\aleph_2$ are measurable under AD and that $\mathscr{D}_{\aleph_1}$ is a normal ultrafilter (see [27, Theorem 33.12]). In fact, $\aleph_1$ is $\aleph_2$-supercompact under AD, by a result of Martin (see [29, p. 401]). Moreover, under $\mathrm{AD}_{\mathbb{R}}$, $\aleph_1$ is γ-supercompact for all $\gamma < \Theta$ [29, Theorem 28.22]. However, the classical proof of Galvin's theorem employs the Axiom of Choice in a crucial way.

[24] A cardinal $\kappa \geq \aleph_1$ is called κ-*compact* if every κ-complete filter over κ extends to a κ-complete ultrafilter.

In private communication [13], W. Chan informed us that it is possible to get Galvin's property at many cardinals even in the absence of AC. The argument involves Steel's notion of *boldface* GCH *cardinal* [37, p. 1678]:

Definition 2.30. A cardinal κ is called *boldface* GCH if there is no injection $F\colon \kappa^+ \to \mathcal{P}(\kappa)$.

In [37, Theorem 8.26] Steel shows that under "$V = L(\mathbb{R}) + \mathsf{AD}$" every cardinal $\kappa < \Theta$ is boldface GCH. Moreover, Chan indicated that using *Moschovakis coding lemma* one can show that AD alone implies that every $\kappa < \Theta^{L(\mathbb{R})}$ is boldface GCH.

Proposition 2.31. *If κ is boldface* GCH *and κ^+ is regular then* $\mathrm{Gal}(\mathscr{F}, \kappa^+, \kappa^+)$ *holds for every κ-complete filter $\mathscr{F}$ over κ.*[25]

Proof. Note that if κ is boldface GCH and κ^+ is regular then every function $F\colon \kappa^+ \to \mathcal{P}(\kappa)$ admits some $B \in \mathrm{Im}(F)$ such that $|(F^{-1})\text{“}B| = \kappa^+$.

Suppose $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of elements of a κ-complete filter $\mathscr{F}$ over κ. By the previous observation there is a κ^+-length subsequence which is constant. Hence the corresponding intersection is this constant value, which belongs to $\mathscr{F}$. Thereby, $\mathrm{Gal}(\mathscr{F}, \kappa^+, \kappa^+)$ holds. □

Remark 2.32. Even ditching the regularity assumption on κ^+ one can show that $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$ holds for every κ-complete ultrafilter $\mathscr{F}$ over κ. This is because every $F\colon \kappa^+ \to \mathcal{P}(\kappa)$ admits a $B \in \mathrm{Im}(F)$ such that $|(F^{-1})\text{“}B| \geq \kappa$.

Anticipating the results in the next section we consider a natural generalization of Proposition 2.31. The crucial point is that both the family of sets with large intersection and the corresponding large set contained in all of them are explicitly constructed. We will apply the claim to many pairs simultaneously, and since these objects are explicitly definable we do not need the axiom of choice in order to pick them.

Definition 2.33. We say that $\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda, \lambda)$ holds if for every sequence $\mathcal{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$, there is a subsequence $\langle C_{\rho_j} \mid j < \lambda \rangle$ definable from $\mathcal{C}$ such that $\bigcap_{j<\lambda} C_{\rho_j} \in \mathscr{U}$.

Proposition 2.34. *Suppose that κ is measurable, $\kappa < \lambda$ is regular and there is no injection $f\colon \lambda \to \mathcal{P}(\kappa)$. Then,* $\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda, \lambda)$ *holds for every κ-complete ultrafilter $\mathscr{U}$ over κ.*

If one seeks for a parallel of the above in ZFC then one may consider real-valued measurable cardinals, which can be described as measurables without the cardinal arithmetic. A cardinal κ is real-valued measurable if there exists a non-trivial κ-additive measure over κ. The size of such cardinals, if exist at all, is at most $2^{\aleph_0}$. Solovay proved that if there is a

[25] Recall that $\mathrm{Gal}(\mathscr{F}, \kappa^+, \kappa^+)$ stands for every sequence $\langle F_\gamma \mid \gamma < \kappa^+ \rangle$ of members of $\mathscr{F}$ admits a subfamily $\langle F_{\gamma_\delta} \mid \delta < \kappa^+ \rangle$ with intersection in $\mathscr{F}$. Observe that this form of Galvin's property is slightly different compared to the ones previously considered.

measurable cardinal κ and one forces a κ-product of random reals then one obtains $2^{\aleph_0} = \kappa$ in the generic extension and κ is real-valued measurable. It is tempting to try an amalgamation between this theorem and the forcing construction of Abraham-Shelah [1]. The framework will be similar, but rather than Cohen reals (added in the Abraham-Shelah model) one can try random reals. The most difficult part is to verify that the *Main lemma* [1, Lemma 1.7] still holds true when replacing the Cohen's by the Random reals. This is in principle not evident for the original argument of [1] relied upon some specific properties of Cohen reals. Once this is accomplished, the failure of Galvin's property at $\aleph_1$ follows. However, there is the additional caveat of ensuring that the Abraham-Shelah poset preserves the real-valued measurability of κ. All of this suggests the following question:

Question 2.35. Is it consistent that κ is a real-valued measurable cardinal and $\mathrm{Gal}(\mathscr{D}_{\aleph_1}, \aleph_1, \kappa)$ fails?

3. An application to ordinary partition relations

Ramsey's theorem from [34] says that $\omega \to (\omega, \omega)^2$. That is, for every $c : [\omega]^2 \to 2$ there exists an infinite monochromatic subset $A \subseteq \omega$. A natural generalization is obtained by replacing ω with some cardinal $\kappa > \aleph_0$. The resulting relation is $\kappa \to (\kappa, \kappa)^2$ and implies that κ is weakly compact.

There is yet another possible way to generalize Ramsey's theorem to uncountable cardinals and in this way one obtains a positive relation at small cardinals as well. Recall that $\lambda \to (\kappa, \theta)^2$ means that for every $c : [\lambda]^2 \to 2$, either there is $A \subseteq \lambda, |A| = \kappa$ such that $c``[A^2] = \{0\}$ or $B \subseteq \lambda, |B| = \theta$ such that $c``[B]^2 = \{1\}$. Ramsey's theorem generalizes to the statement $\kappa \to (\kappa, \omega)^2$ in which we increase the first component only. A theorem of Erdős, Dushnik and Miller says, indeed, that this relation holds at every infinite cardinal κ, see [15] and [16]. In terms of graph theory this means that if G is a complete graph of size κ then either G contains an edge-free subset of size κ or an infinite clique.

Can one improve the positive relation $\lambda \to (\lambda, \omega)^2$? The lightest possibility would be $\lambda \to (\lambda, \omega+1)^2$, but this relation does not hold at every infinite cardinal λ anymore. Suppose that $\lambda > \mathrm{cf}(\lambda) = \omega$ and let $\lambda = \bigcup_{n<\omega} \Delta_n$ where $m \neq n \Rightarrow \Delta_m \cap \Delta_n = \emptyset$ and $|\Delta_n| < \lambda$ for every $n < \omega$. Define $c : [\lambda]^2 \to 2$ by letting $c(\alpha, \beta) = 0$ iff there exists $n < \omega$ for which $\{\alpha, \beta\} \subseteq \Delta_n$. A 0-monochromatic set A must be contained in some Δ_n, so there is no such a set of size λ. A 1-monochromatic set B satisfies $|B \cap \Delta_n| \leq 1$ for every $n < \omega$, so there is no such a set of order-type $\omega + 1$. Hence there is a class of cardinals which fail to satisfy $\lambda \to (\lambda, \omega + 1)^2$. On the other hand, if $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ then $\lambda \to (\lambda, \omega + 1)^2$, see [16, Theorem 11.3]. In fact, one can prove a slightly stronger statement.

Proposition 3.1. *Suppose that $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. For every $c : [\lambda]^2 \to 2$, either there is a stationary set $T \subseteq \lambda$ such that $c \restriction [T]^2$ is 0-monochromatic or there is $B \subseteq \lambda, \mathrm{otp}(B) = \omega + 1$ such that $c \restriction [B]^2$ is 1-monochromatic.*

Proof. Suppose that $c\colon [\lambda]^2 \to 2$. If there exists $B \subseteq \lambda$ of order type $\omega + 1$ such that $c``[B]^2 = \{1\}$ then we are done. Suppose that there is no such B, and let S be the set of limit ordinals of λ. For every $\delta \in S$ choose a sequence $\bar{\alpha}_\delta = \langle \alpha^\delta_0, \dots, \alpha^\delta_{n-1} \rangle$ such that $\bar{\alpha}_\delta \frown \langle \delta \rangle$ is 1-monochromatic and if $\max(\bar{\alpha}_\delta) < \xi < \delta$ then $\bar{\alpha}_\delta \frown \langle \xi, \delta \rangle$ is not 1-monochromatic. The choice is possible by our assumption that there is no 1-monochromatic sequence of length $\omega + 1$.

By shrinking S if needed, we may assume that $|\bar{\alpha}_\delta| = n$ for some fixed $n < \omega$ and every $\delta \in S$. We remark that in this shrinking process we retain the fact that S is stationary. Let ξ_δ be the top-element of $\bar{\alpha}_\delta$ for every $\delta \in S$. The function $h(\delta) = \xi_\delta$ is regressive on S, so by Fodor's lemma there is a stationary $T_0 \subseteq S$ and a fixed ordinal $\xi < \lambda$ such that $\delta \in T_0 \Rightarrow \xi_\delta = \xi$. By repeating this process n-many times we obtain a stationary set T_n and a fixed sequence $\bar{\alpha}$ such that $\bar{\alpha} \frown \langle \delta \rangle$ is 1-monochromatic and $\bar{\alpha} \frown \langle \zeta, \delta \rangle$ is not 1-monochromatic whenever $\delta \in T_n$ and $\max(\bar{\alpha}) < \zeta < \delta$.

In particular, if $T = T_n \setminus (\max(\bar{\alpha}) + 1)$ then T is a stationary subset of λ and if $\zeta, \delta \in T, \zeta < \delta$ then necessarily $c(\zeta, \delta) = 0$. Otherwise, $c(\zeta, \delta) = 1$ and since $\zeta, \delta \in T$ we have that $\bar{\alpha}_\zeta = \bar{\alpha}_\delta = \bar{\alpha}$. Hence $\bar{\alpha} \frown \langle \zeta \rangle = \bar{\alpha}_\zeta \frown \langle \zeta \rangle$ is 1-monochromatic and so is $\bar{\alpha} \frown \langle \delta \rangle$ for the same reason. Finally, since by our assumption $c(\zeta, \delta) = 1$ we get that $\bar{\alpha} \frown \langle \zeta \rangle \frown \langle \delta \rangle$ is 1-monochromatic, which contradicts maximality of $\bar{\alpha} = \bar{\alpha}_\delta$. All in all $c``[T]^2 = \{0\}$, as needed. □

As mentioned above, the statement of the proposition gives a bit more than just $\lambda \to (\lambda, \omega + 1)^2$ since it yields a *stationary* 0-monochromatic set. Notice that the argument applies to singular cardinals of uncountable cofinality for which the concepts of club and stationary subsets are sound. However, if T is a stationary subset of a singular cardinal λ then it is possible that $|T| < \lambda$. Therefore, one may wonder about $\lambda \to (\lambda, \omega + 1)^2$ in such cardinals. The following is [16, Question 11.4]:

Question 3.2. Does the relation $\lambda \to (\lambda, \omega + 1)^2$ hold for $\lambda > \mathrm{cf}(\lambda) > \omega$?

Actually, the question is phrased in [16] with respect to $\lambda = \aleph_{\omega_1}$, the first relevant instance. We indicate that in [16] there appears a partial answer, based on canonization theorems of Shelah, which apply to strong limit singular cardinals. Namely, if $\lambda > \mathrm{cf}(\lambda) > \omega$ and λ is a strong limit cardinal then $\lambda \to (\lambda, \omega + 1)^2$. In some sense, this result gives many ZFC instances since for every $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ there is a class of singular cardinals which are strong limit of cofinality κ. On the other hand, for every specific $\lambda > \mathrm{cf}(\lambda) = \kappa$ one can choose $\theta < \lambda$ and force $2^\theta > \lambda$, thus locally it is not a theorem of ZFC.

A substantial progress with regard to the above question was made by Shelah in [35]. Using methods of pcf theory, Shelah proved that if $\lambda >$

$\mathrm{cf}(\lambda) = \kappa > \aleph_0$ and $2^\kappa < \lambda$ then $\lambda \to (\lambda, \omega+1)^2$. The assumption $2^\kappa < \lambda$ is a weakening of the assumption that λ is a strong limit cardinal, but the overall question remains open if one wishes to eliminate any further assumption.

In this section we would like to replace $2^\kappa < \lambda$ by a version of Galvin's property. Our assumption on the Galvin's property is also easily forced in the ZFC context if $\kappa = \mathrm{cf}(\lambda)$ is measurable. This gives a slight improvement to Shelah's result, but it seems that the real importance of the ZFC result is that it guides us towards the (tentative) direction of forcing the negative relation $\lambda \nrightarrow (\lambda, \omega+1)^2$. In effect, our approach indicates that one must kill all the pertinent instances of the Galvin's property to obtain $\lambda \nrightarrow (\lambda, \omega+1)^2$. As observed in the previous section, AD yields many instances of Galvin's property; consequently, it also gives several instances of $\lambda \to (\lambda, \omega+1)^2$.

To this end, we must render the proof of [35] by removing any use of choice apart from AC_ω. Let us begin with models of ZF. Our first mission is to prove that $\kappa \to (\kappa, \omega+1)^2$ holds for many regular cardinals. The proof of Theorem 3.1 seems to make use of the axiom of choice in two places. Firstly, when one chooses the maximal green sequence $\bar{\alpha}_\delta$ below δ for every $\delta \in S$. Secondly, when one employs Fodor's lemma (finitely many times). The first issue is not a real obstacle, since finite sequences are well ordered even without choice. The second issue is more substantial, but can be settled if one works with normal filters. In this respect, we note that the club filter might not be normal, as proved by A. Karagila [30]. Measurability does not suffice, either. In effect, in [11] Bilinsky and Gitik constructed a model of ZF with a measurable cardinal which does not carry any normal ultrafilter.

Proposition 3.3 (ZF)**.** *Suppose that $\mathscr{F}$ is a normal filter over κ whose dual ideal contains $[\kappa]^{<\kappa}$. Then, for every $c\colon [\kappa]^2 \to 2$ one can find either $A \in \mathscr{F}^+$, such that $c``[A]^2 = \{0\}$ or $B \subseteq \kappa$ with $\mathrm{otp}(B) = \omega+1$ such that $c``[B]^2 = \{1\}$. In particular, $\kappa \to (\kappa, \omega+1)^2$.*

Proof. Let $c\colon [\kappa]^2 \to 2$ be a coloring. We refer to the first color as red and to the second color as green. Repeat the arguments in the proof of Proposition 3.1 with the following changes. Set $S := \{\alpha < \kappa \mid \alpha \text{ is a limit ordinal}\}$. Clearly, $S \in \mathscr{F}^+$. For every $\delta \in S$ let $\bar{\alpha}_\delta$ be the first $<_{\mathrm{lex}}$-finite sequence which is green with δ and maximal with respect to this property. Now use the normality and apply Fodor's lemma to $\mathscr{F}$ finitely many times to obtain a fixed maximal green sequence $\bar{\alpha}$ with respect to some set $A \in \mathscr{F}^+$. It follows that $c``[A]^2 = \{0\}$. Also, by our assumption on $\mathscr{F}$, $|A| = \kappa$. □

Now we come to the main result of this section.

Theorem 3.4 (ZF)**.** *Suppose that $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$ and that κ carries a normal measure $\mathscr{U}$. In addition, suppose that $\langle \lambda_i \mid i < \kappa\rangle$ is a cofinal sequence in λ admitting a family $\langle \mathscr{U}_i \mid i < \kappa\rangle$ such that:*

- *$\mathscr{U}_i$ is a normal filter whose dual ideal contains $[\lambda_i]^{<\lambda_i}$;*
- *$\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda_i, \lambda_i)$ holds.*

Then, $\lambda \to (\lambda, \omega+1)^2$.

Proof. We may assume that for every $i < \kappa$, $\lambda_i > \kappa$. Given a function $f\colon \kappa \to \mathrm{Ord}$ we define a rank function $\neg(f)$ as follows. If $f =_{\mathscr{U}} 0$ (i.e., $\{\alpha < \kappa \mid f(\alpha) = 0\} \in \mathscr{U}$) then we set $\neg(f) = 0$. Otherwise, for $\alpha > 0$, set $\neg(f) = \alpha$ iff $\neg(f) \neq \beta$ for any $\beta < \alpha$ and for every $g\colon \kappa \to \mathrm{Ord}$ which satisfies $g <_{\mathscr{U}} f$ (i.e., $\{\alpha < \kappa \mid g(\alpha) < f(\alpha)\} \in \mathscr{U}$) there is $\beta < \alpha$ such that $\neg(g) = \beta$.

Let $c\colon [\lambda]^2 \to 2$ be a coloring and suppose that there is no 1-monochromatic subset of order type $\omega+1$. Define $\Delta_0 := \lambda_0$ and $\Delta_{1+i} := [\lambda_i, \lambda_{i+1})$ for every $i < \kappa$. By our assumption, there is a full-sized 0-monochromatic subset of Δ_i for every $i < \kappa$. Moreover, this set is explicitly constructible by the proof of Proposition 3.3 (Indeed, for every i we take the $<_{lex}$-minimal sequence $\vec{\alpha}_i^*$ such that the set of $\delta \in \Delta_i$ such that $\bar{\alpha}_\delta = \vec{\alpha}_i^*$ is in $\mathscr{U}_i$). Hence we may assume without loss of generality that $c``[\Delta_i]^2 = \{0\}$ for every $i < \kappa$.

For each $\alpha < \kappa$ let $\eta(\alpha)$ be the unique $i < \kappa$ for which $\alpha \in \Delta_i$. For every $0 < i < \kappa$ let Seq_i be the set $\{\langle \alpha_0, \ldots, \alpha_{n-1}\rangle \mid \eta(\alpha_0) < \cdots < \eta(\alpha_{n-1}) < i\}$. For $i < \kappa$ and $\zeta \in \Delta_i$ we define a tree $\mathscr{T}_\zeta$ as follows. For every $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle \in Seq_i$ and every $\zeta \in \Delta_i$ we let $\bar{\alpha} \in \mathscr{T}_\zeta$ iff $\{\alpha_0, \ldots, \alpha_{n-1}, \zeta\}$ is 1-monochromatic under c.

By our assumption each $\mathscr{T}_\zeta$ is well-founded with respect to the reversed order. Therefore, one can define a rank function rk_ζ over Seq_i, for every $\zeta \in \Delta_i$, by the following procedure. If $\bar{\alpha} \in Seq_i - \mathscr{T}_\zeta$ then let $rk_\zeta(\bar{\alpha}) = -1$. If $\bar{\alpha} \in \mathscr{T}_\zeta$ then $rk_\zeta(\bar{\alpha}) = \xi$ iff for every $\varepsilon \in \xi$ one has $rk_\zeta(\bar{\alpha}) \neq \varepsilon$ and there exists an ordinal β for which $rk_\zeta(\bar{\alpha}^\frown\langle\beta\rangle) = \varepsilon$.

The idea of this rank function is to express the degree of maximality of finite sequences below ζ. For a maximal 1-monochromatic sequence below ζ, rk_ζ assumes the value zero. If there is more room for adding ordinals above $\max(\bar{\alpha})$ and keeping the 1-monochromaticity with ζ, then the rank grows. Notice that for every $i < \kappa$ and every $\zeta \in \Delta_i$ the range of rk_ζ is bounded in λ_{i+1} since it is an initial segment of λ_{i+1}.

Let $\Delta_i^{end} \subseteq \Delta_i$ be an end-segment such that for every $\bar{\alpha} \in Seq_i$ and $\gamma < \lambda_i$, if there is $\zeta \in \Delta_i^{end}$ such that $rk_\zeta(\bar{\alpha}) = \gamma$ then there are unboundedly many such ζ's in Δ_i^{end} below λ_{i+1}. The set Δ_i^{end} has a concrete definition: this end-segment is obtained by omitting bounded subsets of Δ_i, which amounts in our case to the intersection of their complements. Since λ_{i+1} is regular, we obtain an end-segment of λ_{i+1}.

We define a set of triples K as follows. A triple $(\bar{\alpha}, Z, f)$ belongs to K iff $Z \in \mathscr{U}, f\colon \kappa \to \mathrm{Ord}$ and for some $0 < i < \kappa, \bar{\alpha} \in Seq_i, \min(Z) > i$ and for every $j \in Z$ there exists $\zeta \in \Delta_j^{end}$ such that $rk_\zeta(\bar{\alpha}) = f(j)$. It is easy to verify that $K \neq \varnothing$.

Let $(\bar{\alpha}^*, Z^*, f^*)$ be a triple in K for which $\neg(f^*)$ is minimal amongst the elements of K. Without loss of generality, all the elements of Z^* are limit ordinals. For every $j \in Z^*$ we isolate a section $\Sigma_j \subseteq \Delta_j^{end}$ by defining $\Sigma_j = \{\zeta \in \Delta_j^{end} \mid rk_\zeta(\bar{\alpha}^*) = f^*(j)\}$. Notice that $|\Sigma_j| = \lambda_{j+1}$. Fix $j \in Z^*$.

We would like to understand what happens between the jth level and upper levels mentioned in Z^*. For every $\ell \in Z^*$ such that $j < \ell$ and every $\zeta \in \Sigma_j$, let $\Sigma^\ell_j(\zeta) = \{\eta \in \Sigma_\ell \mid c(\zeta,\eta) = 1\}$. Let $L_\zeta = \{\ell \in Z^* \mid j < \ell, |\Sigma^\ell_j(\zeta)| = \lambda_{\ell+1}\}$, the set of *large* 1-monochromatic levels above j. Similarly, let $T_\zeta = Z^* - L_\zeta$, the set of *tiny* 1-monochromatic levels above j.

Our goal is to garner many ordinals ζ with big T_ζ, since then we will be able to remove the "1" edges (there will be only a few of them) and create a 0-monochromatic union of size λ. We claim, therefore, that $L_\zeta \notin \mathscr{U}$ (and parallely, $T_\zeta \in \mathscr{U}$) for every $j \in Z^*, \zeta \in \Sigma_j$. For suppose not. Fix $i \in Z^*, \beta \in \Sigma_i$ such that $L_\beta \in \mathscr{U}$. Define $\bar{\alpha}' = \bar{\alpha}^* {}^\frown \langle\beta\rangle, Z' = L_\beta$ and for $j \in Z'$ let $f'(j) = \min\{rk_\eta(\bar{\alpha}') \mid \eta \in \Sigma^j_i(\beta)\}$ and $f'(j) = 0$ otherwise. Notice that $(\bar{\alpha}', Z', f') \in K$. We claim that $\daleth(f') < \daleth(f^*)$. Indeed, for every $j \in Z' = L_\beta$ one has $f'(j) = rk_\eta(\bar{\alpha}')$ for some $\eta \in \Sigma^j_i(\beta)$, so $f'(j) = rk_\eta(\bar{\alpha}^* {}^\frown \langle\beta\rangle) < rk_\eta(\bar{\alpha}^*) = f^*(j)$. Thus, $\daleth(f') < \daleth(f^*)$, contradicting the minimality of $\daleth(f^*)$.

We conclude, therefore, that $T_\zeta \in \mathscr{U}$ for every $j \in Z^*, \zeta \in \Sigma_j$. Fix now $j \in Z^*$.

Let $\mathcal{A}^j = \langle T_\zeta \mid \zeta \in \Sigma_j\rangle \subseteq \mathscr{U}$, apply $\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda_{j+1}, \lambda_{j+1})$ and find $M_j \subseteq \Sigma_j$ of cardinality λ_{j+1} such that $\cap_{\zeta\in M_j} T_\zeta =: Y_j \in \mathscr{U}$. We emphasize that we do not need the axiom of choice in order to define these objects, because of definability. We render this process at every $j \in Z^*$. Finally, let $Y = \Delta\{Y_j \mid j \in Z^*\} \in \mathscr{U}$.

For every $j \in Y$ we wish to define $A_j \subseteq M_j$ such that $|A_j| = \lambda_{j+1}$. The sets A_j will be 0-monochromatic, and our goal is to show that their union is 0-monochromatic as well. We define these sets by induction on $j \in Y$. So fix $j \in Y$ and define $A_j = \{\xi \in M_j \mid \forall i \in Y \cap j, \forall \zeta \in A_i, \xi \notin \Sigma^j_i(\zeta)\}$. We claim that $|A_j| = \lambda_{j+1}$. To see this, observe that if $i \in Y \cap j$ then for every $\zeta \in A_i$ the set $\Sigma^j_i(\zeta)$ is bounded in λ_{j+1}, hence $\lambda_{j+1} - \Sigma^j_i(\zeta)$ contains a final segment of λ_{j+1}. Thus, by regularity of λ_{j+1}, $\bigcap\{(\lambda_{j+1} - \Sigma^j_i(\zeta)) \mid \zeta \in A_i\}$ contains a final segment. Notice that A_j is obtained by intersecting this set with M_j, hence $|A_j| = \lambda_{j+1}$.

Define $A = \bigcup_{j\in Y} A_j$, so $|A| = \lambda$. By proving that $c``[A]^2 = \{0\}$ we will be done. Pick $\alpha, \beta \in A$ such that $\alpha < \beta$. If there exists $j \in \omega_1$ such that $\alpha, \beta \in A_j$ then $c(\alpha,\beta) = 0$ since $A_j \subseteq A^j_0 \subseteq \Sigma_j$ and $c``[\Sigma_j]^2 = \{0\}$. If not, then there are $i < j < \omega_1$ such that $\alpha \in A_i, \beta \in A_j$, and $i, j \in Y$. By definition, $\beta \notin \Sigma^j_i(\alpha)$ and hence $c(\alpha,\beta) = 0$ and the proof is accomplished. □

The above proof also gives the following corollary in ZFC, where the club filter over regular cardinals is always normal.

Corollary 3.5 (ZFC)**.** *Suppose that $\mathscr{U}$ is a normal measure over κ, λ is singular with $\mathrm{cf}(\lambda) = \kappa$ and that $\mathrm{Gal}(\mathscr{U}, \lambda_i, \lambda_i)$ holds for cofinaly many regular cardinals below λ. Then, $\lambda \to (\lambda, \omega+1)^2$*

The situation described in the previous corollary is a consequence of a small ultrafilter number: Suppose that κ is measurable, and $\mathscr{U}$ is a normal

ultrafilter over κ with a base of $\mathscr{U}$ of size κ^+. One says, in this case, that $\mathfrak{u}_\kappa^{\rm nor} = \kappa^+$. It is possible to force $\mathfrak{u}_\kappa^{\rm nor} = \kappa^+$ even if 2^κ is arbitrarily large, see e.g. [22] and [21].

It is easy to verify that if $\mathscr{U}$ witnesses $\mathfrak{u}_\kappa^{\rm nor} = \kappa^+$ then $\mathrm{Gal}(\mathscr{U}, \mu, \mu)$ holds whenever $\mu = \mathrm{cf}(\mu) > \kappa^+$, see e.g. [7].

Corollary 3.6. *If κ is measurable and $\mathfrak{u}_\kappa^{\rm nor} = \kappa^+$ then $\lambda \to (\lambda, \omega+1)^2$ whenever $\kappa = \mathrm{cf}(\lambda) < \lambda$.*

The corollary shows that the positive relation $\lambda \to (\lambda, \omega+1)^2$ may hold even if $2^\kappa > \lambda$. This fact was known already to Shelah and Stanley [36]. It is shown there that if λ is a strong limit singular of uncountable cofinality then many versions of Cohen forcing preserve the positive relation $\lambda \to (\lambda, \omega+1)^2$; in particular, adding many Cohen subsets of κ in a way that $2^\kappa > \lambda$. Our corollary generalizes these results, since any κ-complete forcing notion will preserve the statement $\mathfrak{u}_\kappa^{\rm nor} = \kappa^+$.

In the above statements we used Galvin's property in order to prove $\lambda \to (\lambda, \omega+1)^2$. The connection between ordinary partition relations and the structure of normal filters seems to be helpful in the opposite direction as well. That is, from the assumption that $\lambda \to (\lambda, \omega+1)^2$ one can learn something about the Galvin's property.

Proposition 3.7. *Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and let $\mathscr{F}$ be a normal filter over κ. Suppose that $\neg_{\rm st}\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$ is witnessed by $\mathcal{C} = \{C_\alpha \mid \alpha < \kappa^+\}$. Then one can find $a = \{\alpha_n \mid n < \omega\} \subseteq \kappa^+$ such that $\bigcap\{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\} \neq \emptyset$.*

Proof. Assume toward contradiction that $\mathcal{C} = \{C_\alpha \mid \alpha < \kappa^+\}$ witnesses the strong failure of $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$, yet the complements are not overlapping in the sense that every infinite collection of them has empty intersection. Define a coloring $c : [\kappa^+]^2 \to 2$ as follows. For $\alpha < \beta < \kappa^+$ let $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$.

Notice that there is no 1-monochromatic sequence of length $\omega + 1$ under c. For if $\langle \alpha_n \mid n \leq \omega \rangle$ is such a sequence then $\alpha_\omega \in \bigcap\{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\}$, in contrary to our assumption at the beginning of the proof. Likewise, there is no 0-monochromatic set $A \in [\kappa^+]^{\kappa+\kappa}$. For if $A = A_0 \cup A_1$ with $\sup(A_0) < \min(A_1)$ and $\mathrm{otp}(A_0) = \mathrm{otp}(A_1) = \kappa$, then $A_1 \subseteq \bigcap\{C_\alpha \mid \alpha \in A_0\}$. This is impossible since $\{C_\alpha \mid \alpha \in A_0\} \subseteq \mathcal{C}$. Thus, the coloring c witnesses $\kappa^+ \nrightarrow (\kappa^+, \omega+1)^2$, which is an absurd since κ^+ is regular and uncountable. □

A similar argument becomes more powerful at successors of large cardinals. By [8] one can force a κ-complete ultrafilter $\mathscr{U}$ over a measurable cardinal κ such that $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ fails.

Proposition 3.8. *Suppose that κ is measurable and $\mathscr{U}$ is a κ-complete ultrafilter over κ for which $\neg_{\rm st}\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ is witnessed by the sequence $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$. Then there is $S \in [\kappa]^\kappa$ such that $\bigcap\{(\kappa \setminus C_\alpha) \mid \alpha \in S\}$ is non-empty.*

Proof. We define $c : [\kappa^+]^2 \to 2$ as before, by letting $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$. Of course, the definition is needed at $\alpha < \beta < \kappa^+$ only, since the coloring is symmetric. Assume toward contradiction that there is no $S \in [\kappa]^\kappa$ such that $\bigcap\{(\kappa - C_\alpha) \mid \alpha \in S\} \neq \emptyset$. By the same argument as in the previous claim, c witnesses the negative relation $\kappa^+ \not\to (\kappa + \kappa, \kappa + 1)^2$. However, this relation holds if κ is measurable, as proved in [17], a contradiction. □

The above statements show that there is a limitation on forcing empty intersection over the complements of sets which witness the strong failure of the Galvin's property. One can try, however, to force empty intersection only over certain families. The following is tailored to the goal of obtaining a negative relation at singular cardinals.

Given $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$ and a cofinal sequence $\langle \lambda_i \mid i < \kappa \rangle$ in λ, let $\Delta_0 := [0, \lambda_0)$ and $\Delta_{1+i} := [\lambda_i, \lambda_{i+1})$.

Proposition 3.9. *Suppose that:*

- ($\aleph$) $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$ *and* $\mathscr{F}$ *is a normal filter over* κ.
- ($\beth$) $\{C_\alpha \mid \alpha < \lambda\} \subseteq \mathscr{F}$ *witnesses* $\neg_{\mathrm{st}}\mathrm{Gal}(\mathscr{F}, \kappa, \lambda)$.
- ($\gimel$) $\bigcap\{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\} = \emptyset$ *whenever* $\alpha_n \in \Delta_{i_n}$ *for every* $n < \omega$, *and* $m \neq n \Rightarrow \alpha_m \neq \alpha_n$.

Then $\lambda \not\to (\lambda, \omega + 1)^2$.

Proof. For $\alpha < \beta < \lambda$, if there exists $i \in \kappa$ such that $\alpha, \beta \in \Delta_i$ then let $c(\alpha, \beta) = 0$. If $\alpha \in \Delta_i, \beta \in \Delta_j$ and $i < j$, let $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$. One can verify that $c : [\lambda]^2 \to 2$ witnesses $\lambda \not\to (\lambda, \omega + 1)^2$, so we are done. □

On a different path, Theorem 3.10 can be applied also to models of AD, where the distance between regular and measurable is quite short. This distance is even shorter under the extra assumption that $V = L(\mathbb{R})$, where every regular cardinal below Θ is measurable, as proved in [37].

Corollary 3.10 (AD + $V = L(\mathbb{R})$)**.** *Suppose that* $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$. *If* λ *is a limit of regular cardinals then* $\lambda \to (\lambda, \omega + 1)^2$.

Proof. For transparency, assume that $\kappa = \omega_1$. Let $\lambda = \bigcup_{i<\omega_1} \lambda_i$, where $\langle \lambda_i \mid i < \omega_1 \rangle$ is increasing and continuous, $\omega_1 < \lambda_0$ is measurable and λ_{i+1} is measurable for every $i < \omega_1$. Let $\mathscr{U}$ denote the club filter over ω_1, which is a normal ultrafilter under AD. For every $i < \omega_1$ let $\mathscr{U}_{i+1}$ be the filter generated by the unbounded ω-closed subsets of λ_{i+1}. This is a normal ultrafilter over λ_{i+1} under AD+$V = L(\mathbb{R})$, as shown in [37, Theorem 8.27]. Also, $\mathrm{Gal}_{\mathrm{Def}}(\mathscr{U}, \lambda_{i+1}, \lambda_{i+1})$ holds by virtue of Proposition 2.34. Thereby, the assumptions of Theorem 3.10 are met. □

It turns out that if $\lambda = \aleph_{\omega_1}$ we can do it better:

Corollary 3.11 (AD)**.** *The relation* $\aleph_{\omega_1} \to (\aleph_{\omega_1}, \omega + 1)^2$ *holds.*

Proof. For each limit $\alpha < \omega_1$ denote by δ^1_α the supremum of the lengths of all $\mathbf{\Delta}^1_\alpha$ prewellorderings. It follows that, under AD, the sequence $\langle \delta^1_{\alpha+1} \mid$

$\alpha \in \mathrm{acc}(\omega_1)\rangle$ satisfies the following property: for each $\alpha \in \mathrm{acc}(\omega_1)$ the filter generated by the ω-closed subsets of $\delta^1_{\alpha+1}$ is a normal measure [26, §5.4]. Also $\sup_{\alpha<\omega_1} \delta^1_{\alpha+1} = \aleph_{\omega_1}$. Now we can argue as in the previous corollary that $\aleph_{\omega_1} \to (\aleph_{\omega_1}, \omega+1)^2$. □

We finalize the current section with a ZFC question:

Question 3.12. Does $\lambda \not\to (\lambda, \omega+1)^2$ have any consistency strength?

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References

1. U. Abraham and S. Shelah, *On the intersection of closed unbounded sets*, J. Symbolic Logic **51** (1986), no. 1, 180–189. MR 830084
2. Joan Bagaria, *$C^{(n)}$-cardinals*, Archive for Mathematical Logic **51** (2012), no. 3, 213–240.
3. Joan Bagaria and Alejandro Poveda, *More on the preservation of large cardinals under class forcing*, The Journal of Symbolic Logic (2021), 1–38.
4. J. E. Baumgartner, A. Hajnal, and A. Mate, *Weak saturation properties of ideals*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, 1975, pp. 137–158. Colloq. Math. Soc. János Bolyai, Vol. 10. MR 0369081
5. Omer Ben-Neria, Shimon Garti, and Yair Hayut, *Weak prediction principles*, Fund. Math. **245** (2019), no. 2, 109–125.
6. Tom Benhamou, *Prikry Forcing and Tree Prikry Forcing of Various Filters*, Arch. Math. Logic **58** (2019), 787—-817.
7. Tom Benhamou, Shimon Garti, and Alejandro Poveda, *Negating the Galvin property*, J. London Math. Soc. (2021).
8. Tom Benhamou, Shimon Garti, and Saharon Sehlah, *Kurepa trees and the failure of the Galvin property*, Proc. Amer. Math. Soc. **151** (2023), 1301–1309.
9. Tom Benhamou and Moti Gitik, *Intermediate models of Magidor-Radin forcing-Part II*, Ann. Pure Appl. Logic **173** (2022), no. 6, Paper No. 103107. MR 4393795
10. ———, *On Cohen and Prikry Forcing Notions*, preprint (2022), arXiv:2204.02860.
11. E. Bilinsky and M. Gitik, *A model with a measurable which does not carry a normal measure*, Arch. Math. Logic **51** (2012), 863–876.
12. Andrew D. Brooke-Taylor, Vera Fischer, Sy-David Friedman, and Diana Carolina Montoya, *Cardinal characteristics at κ in a small u(κ) model*, Ann. Pure Appl. Log. **168** (2017), 37–49.
13. W. Chan, Private communication (2022).
14. James Cummings, Mirna Džamonja, Menachem Magidor, Charles Morgan, and Saharon Shelah, *A framework for forcing constructions at successors of singular cardinals*, Transactions of the American Mathematical Society **369** (2017), no. 10, 7405–7441.
15. Ben Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610. MR 4862
16. Paul Erdős, András Hajnal, Attila Máté, and Richard Rado, *Combinatorial set theory: partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, vol. 106, North-Holland Publishing Co., Amsterdam, 1984. MR 795592
17. Matthew Foreman and Andras Hajnal, *A partition relation for successors of large cardinals*, Math. Ann. **325** (2003), no. 3, 583–623. MR MR1968607 (2004b:03063)

18. Shimon Garti, *Weak diamond and Galvin's property*, Period. Math. Hungar. **74** (2017), no. 1, 128–136. MR 3604115
19. _____, *Tiltan*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 4, 351–359. MR 3787522
20. Shimon Garti and Saharon Shelah, *Strong polarized relations for the continuum*, Annals of Combinatorics **16** (2012), no. 2, 271–276.
21. _____, *Partition calculus and cardinal invariants*, J. Math. Soc. Japan **66** (2014), no. 2, 425–434. MR 3201820
22. Moti Gitik and Saharon Shelah, *On densities of box products*, Topology Appl. **88** (1998), no. 3, 219–237. MR 1632081
23. András Hajnal, *On some combinatorial problems involving large cardinals*, Fundamenta Mathematicae **69** (1970), no. 1, 39–53.
24. Joel David Hamkins, *Gap forcing: Generalizing the lévy-solovay theorem*, The Bulletin of Symbolic Logic **5** (1999), no. 2, 264–272.
25. _____, *The Lottery Preparation*, Annals of Pure and Applied Logic **101** (2000), no. 2-3, 103–146.
26. Steve Jackson, *Structural consequences of AD*, Handbook of set theory, Springer, 2010, pp. 1753–1876.
27. Thomas Jech, *Set Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513
28. Akihiro Kanamori, *Ultrafilters over a measurable cardinal*, Annals of Mathematical Logic **10** (1976), 315–356.
29. _____, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Large cardinals in set theory from their beginnings. MR 1994835 (2004f:03092)
30. A. Karagila, *Fodor's lemma can fail everywhere*, Acta Math. Hungar. **154** (2018), 231–242.
31. Péter Komjáth and Vilmos Totik, *Ultrafilters*, The American Mathematical Monthly **115** (2008), no. 1, 33–44.
32. Kenneth Kunen, *Set theory: An introduction to independence proofs*, North-Holland, 1980.
33. Adrian Mathias, *Happy Families*, Annals of Mathematical Logic **12** (1977), 59–111.
34. F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. **S2-30**, no. 1, 264. MR 1576401
35. Saharon Shelah, *The Erdos-Rado arrow for singular cardinals*, Canad. Math. Bull. **52** (2009), no. 1, 127–131. MR 2494318
36. Saharon Shelah and Lee J. Stanley, *Filters, Cohen sets and consistent extensions of the Erdös-Dushnik-Miller theorem*, J. Symbolic Logic **65** (2000), no. 1, 259–271. MR 1782118
37. John R. Steel, *An outline of inner model theory*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1595–1684. MR 2768698
38. Neil H. Williams, *Combinatorial set theory*, Studies in Logic and the Foundations of Mathematics, vol. 91, North-Holland Publishing Co., Amsterdam, 1977. MR 3075383

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