On the structure of an Ultrafilter Basis

Tom Benhamou

Department of Mathematics,
Rutgers University

This research was supported by the National Science Foundation under Grant No. DMS-2346680





October 26, 2026

Outline

- Motivation
  - Filters & Ultrafilters
  - The Cofinality of Ultrafilters
- Main Results
- Ideas from the Proof
- Further Results

Filters

Filters

Filters unify the meaning of 'largeness' across mathematics:

Filters

Filters unify the meaning of 'largeness' across mathematics:

Probability one events,

Filters

Filters unify the meaning of 'largeness' across mathematics:

Probability one events, Comeager sets,

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$.

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Let S be a set, $\mathcal{F} \subseteq P(S)$ is called a *filter* over S if:

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Let S be a set, $\mathcal{F} \subseteq P(S)$ is called a *filter* over S if:

1. $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Let S be a set, $\mathcal{F} \subseteq P(S)$ is called a *filter* over S if:

1. $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. $\mathcal{F}$ is closed under intersection ($\forall A, B \in \mathcal{F},\ A \cap B \in \mathcal{F}$).

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Let S be a set, $\mathcal{F} \subseteq P(S)$ is called a *filter* over S if:

1. $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. $\mathcal{F}$ is closed under intersection $(\forall A, B \in \mathcal{F},\ A \cap B \in \mathcal{F})$.
3. $\mathcal{F}$ is closed under superset. $(\forall A \in \mathcal{F}\ \forall B,\ A \subseteq B \Rightarrow B \in \mathcal{F})$.

Filters

Filters unify the meaning of 'largeness' across mathematics:
Probability one events, Comeager sets, cofinite sets ...

Idea: (Cartan, Bourbaki)

Let S be a set, $\mathcal{F} \subseteq P(S)$. $X \in \mathcal{F} \equiv$ "X is $\mathcal{F}$-large".

Definition 1

Let S be a set, $\mathcal{F} \subseteq P(S)$ is called a *filter* over S if:

1. $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. $\mathcal{F}$ is closed under intersection $(\forall A, B \in \mathcal{F},\ A \cap B \in \mathcal{F})$.
3. $\mathcal{F}$ is closed under superset. $(\forall A \in \mathcal{F}\ \forall B,\ A \subseteq B \Rightarrow B \in \mathcal{F})$.

$X^c \in \mathcal{F} \equiv$ "X is small".

Example 2

The collection of events of probability 1 is a filter over the sample space.

Example 2

The collection of events of probability 1 is a filter over the sample space.

Example 3

Given a point x in a topological space X, the collection $\mathcal{N}(x)$ consisting of the open neighborhoods of x is a filter over X.

Example 2

The collection of events of probability 1 is a filter over the sample space.

Example 3

Given a point x in a topological space X, the collection $\mathcal{N}(x)$ consisting of the open neighborhoods of x is a filter over X.

Example 4

The Fréchet filter is the collection of all cofinite sets:

$$\mathcal{FR} = \{X \subseteq \mathbb{N} \mid X^c \text{ is finite}\}$$

Example 2

The collection of events of probability 1 is a filter over the sample space.

Example 3

Given a point x in a topological space X, the collection $\mathcal{N}(x)$ consisting of the open neighborhoods of x is a filter over X.

Example 4

The Fréchet filter is the collection of all cofinite sets:

$$\mathcal{FR} = \{X \subseteq \mathbb{N} \mid X^c \text{ is finite}\}$$

Filters do not in general determine whether every subset is large or small.

Ultrafilters

Ultrafilters are notions of largeness which are *complete*.

Ultrafilters

Ultrafilters are notions of largeness which are *complete*.

Definition 5 (Cartan, Bourbaki)

Let S be a set, $\mathcal{U} \subseteq P(S)$ is called a *ultrafilter* over S if:

1. $\mathcal{U}$ is a filter ($S \in \mathcal{U}$, closed under $\cap$, upward closed wrt. to $\subseteq$).
2. For every $X \subseteq S$, either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$.

Ultrafilters

Ultrafilters are notions of largeness which are *complete*.

Definition 5 (Cartan, Bourbaki)

Let S be a set, $\mathcal{U} \subseteq P(S)$ is called a *ultrafilter* over S if:

1. $\mathcal{U}$ is a filter ($S \in \mathcal{U}$, closed under $\cap$, upward closed wrt. to $\subseteq$).
2. For every $X \subseteq S$, either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$.

Ultrafilters generalize probabilistic '0–1 laws' to arbitrary sets

Ultrafilters

Ultrafilters are notions of largeness which are *complete*.

Definition 5 (Cartan, Bourbaki)

Let S be a set, $\mathcal{U} \subseteq P(S)$ is called a *ultrafilter* over S if:

1. $\mathcal{U}$ is a filter ($S \in \mathcal{U}$, closed under $\cap$, upward closed wrt. to $\subseteq$).
2. For every $X \subseteq S$, either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$.

Ultrafilters generalize probabilistic '0–1 laws' to arbitrary sets

Given a filter $\mathcal{F}$, if X is not determined by $\mathcal{F}$, we can add X to $\mathcal{F}$ and close under intersections and inclusion.

Ultrafilters

Ultrafilters are notions of largeness which are *complete*.

Definition 5 (Cartan, Bourbaki)

Let S be a set, $\mathcal{U} \subseteq P(S)$ is called a *ultrafilter* over S if:

1. $\mathcal{U}$ is a filter ($S \in \mathcal{U}$, closed under $\cap$, upward closed wrt. to $\subseteq$).
2. For every $X \subseteq S$, either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$.

Ultrafilters generalize probabilistic '0–1 laws' to arbitrary sets

Given a filter $\mathcal{F}$, if X is not determined by $\mathcal{F}$, we can add X to $\mathcal{F}$ and close under intersections and inclusion.

Lemma 6 (The Ultrafilter Lemma)

Under the axiom of choice, any filter can be completed to an ultrafilter

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

- Topology (Stone-Čech compactification)

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

- Topology (Stone-Čech compactification)
- Measure theory (two-valued measures)

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

- Topology (Stone-Čech compactification)
- Measure theory (two-valued measures)
- Model theory (ultraproduct construction).

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

- Topology (Stone-Čech compactification)
- Measure theory (two-valued measures)
- Model theory (ultraproduct construction).
- More (Geometric group theory, Combinatorics,...)

Example 7

Given $s \in S$, the principal ultrafilter is $\{A \subseteq S \mid s \in A\}$.

Example 8

Any ultrafilter extending the Frećhet filter $\mathcal{FR}$ is a non-principal ultrafilter.

Throughout this talk we will consider only non-principal ultrafilters which appear naturally in:

- Topology (Stone-Čech compactification)
- Measure theory (two-valued measures)
- Model theory (ultraproduct construction).
- More (Geometric group theory, Combinatorics,...)

Ultrafilters lie at the foundations of every area of modern set theory.

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.
- True in 1st-ctbl spaces:

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.
- True in 1st-ctbl spaces: the filter $\mathcal{N}(x)$ admits a ctbl. filter-basis,

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.
- True in 1st-ctbl spaces: the filter $\mathcal{N}(x)$ admits a ctbl. filter-basis, that is,

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.
- True in 1st-ctbl spaces: the filter $\mathcal{N}(x)$ admits a ctbl. filter-basis, that is, a countable collection $\mathcal{A} \subseteq \mathcal{N}(x)$

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 9

A function $f : X \to Y$ is **continuous in the sequential sense** if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ converges to x, also $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

- A function $f : \mathbb{R} \to X$ is continuous iff it's sequentially continuous.
- True in 1st-ctbl spaces: the filter $\mathcal{N}(x)$ admits a ctbl. filter-basis, that is, a countable collection $\mathcal{A} \subseteq \mathcal{N}(x)$ such that every $U \in \mathcal{N}(x)$ contains some $A \in \mathcal{A}$.

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point,

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is,

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$,

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.

Counterexample: sequential continuity $\not\Rightarrow$ continuity

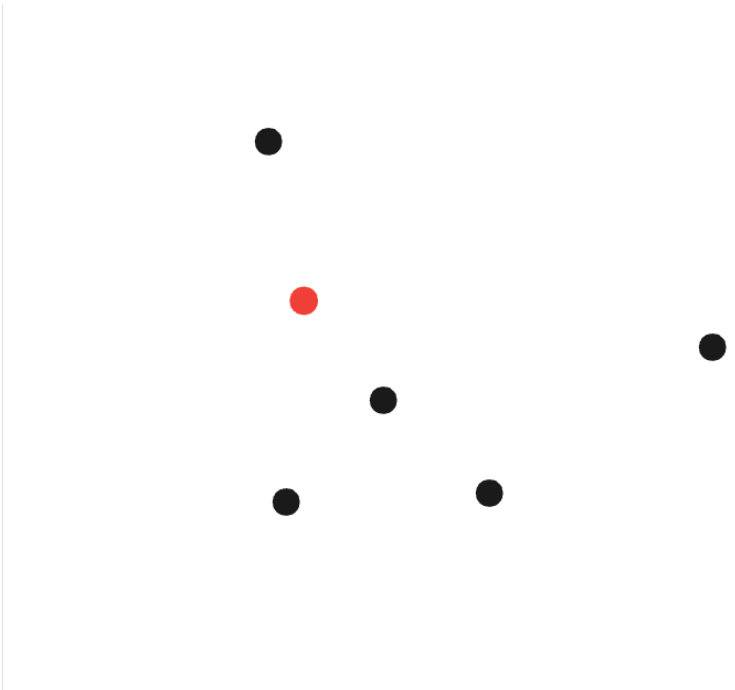
Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

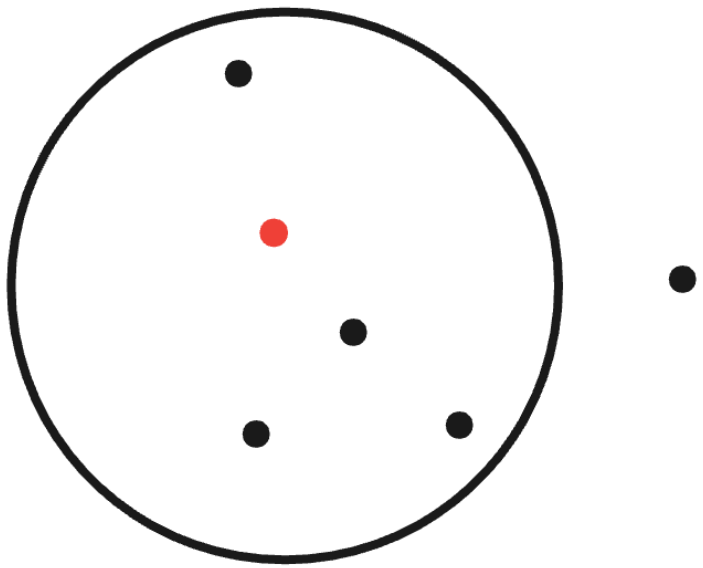
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

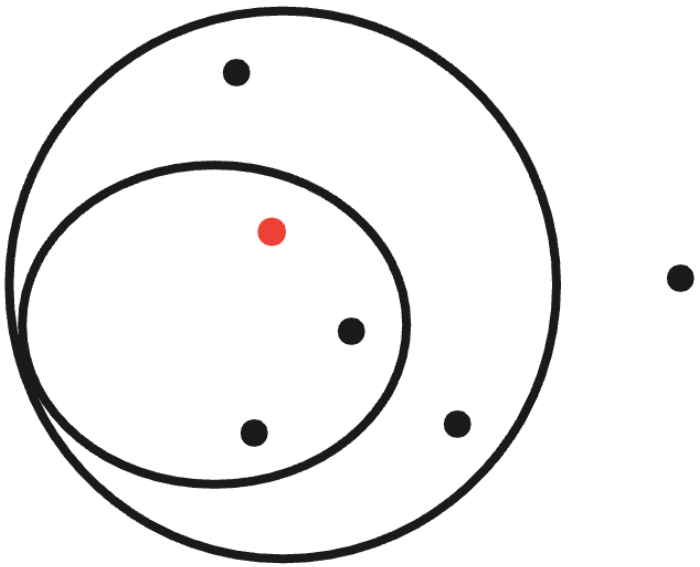
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

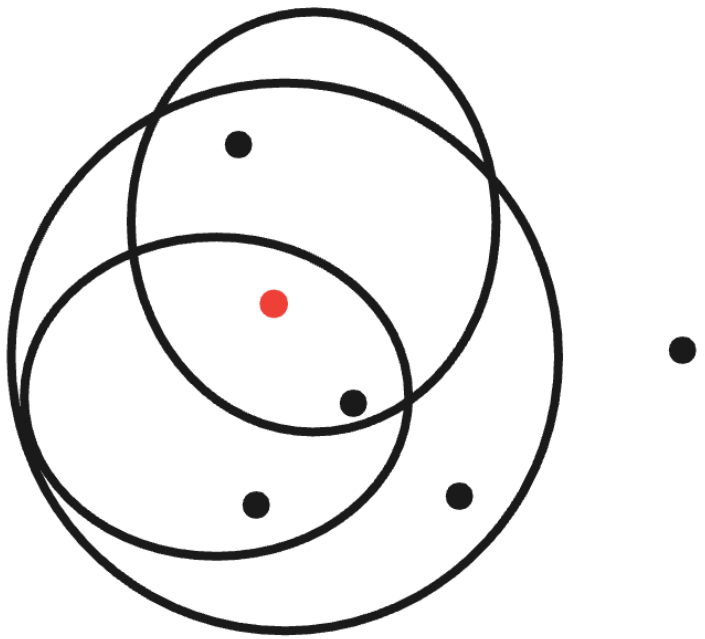
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

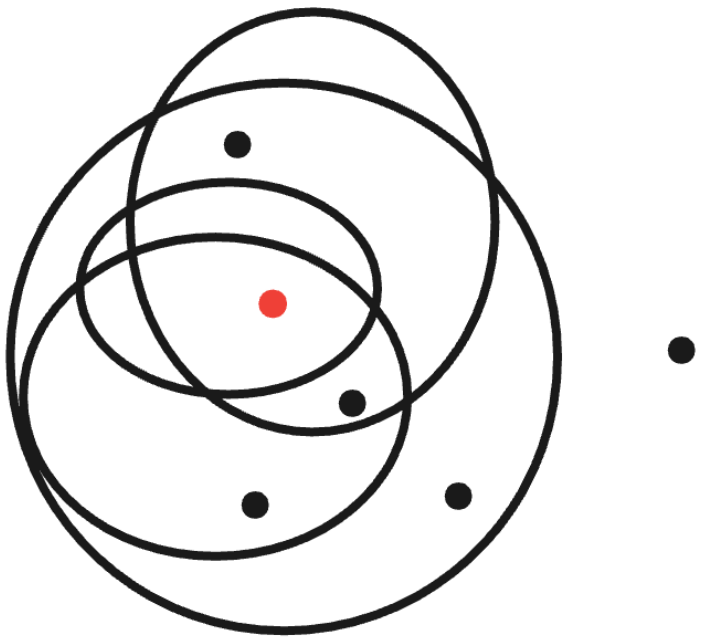
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

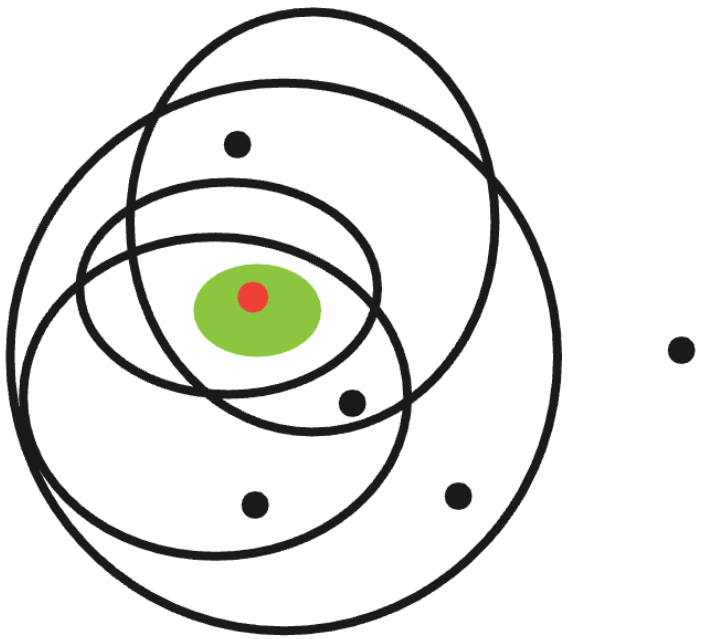
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

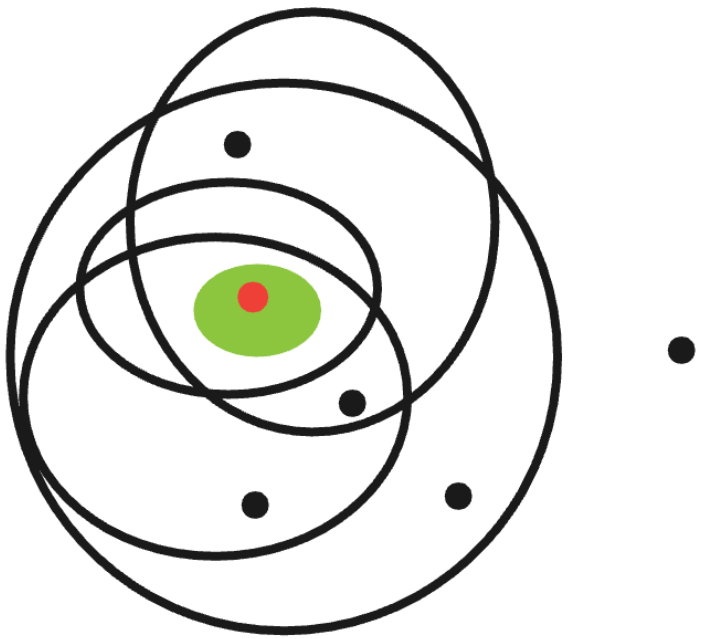
1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.

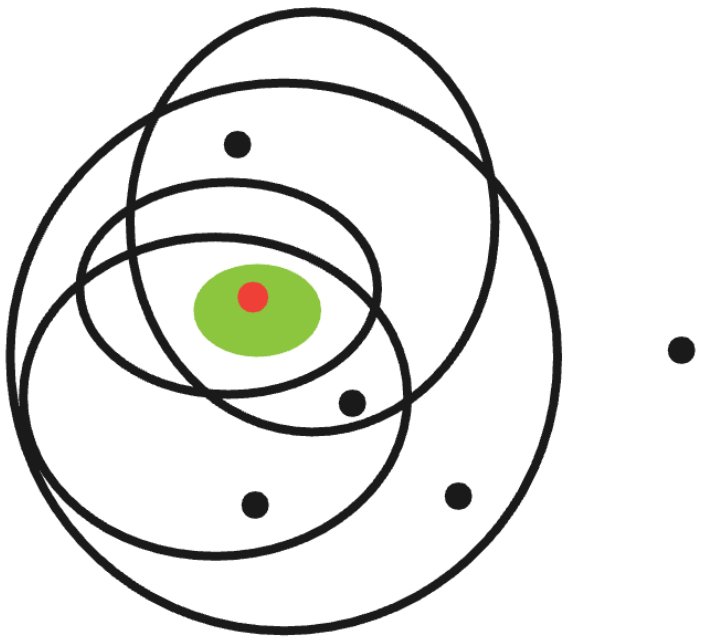


Problem: (ctbl.) sequences capture only countably many sets in $\mathcal{N}(x)$.

Counterexample: sequential continuity $\not\Rightarrow$ continuity

Suppose $x \in X$ is a p-point, that is, for every countable $\mathcal{A} \subseteq \mathcal{N}(x)$, there exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$ for all $A \in \mathcal{A}$
(e.g. $X = \omega_1 + 1,\ x = \omega_1$)

1. Define $f(x) = 1$ and $f(y) = 0$ for $y \neq x$.
2. Non-continuous but sequentially continuous.



Problem: (ctbl.) sequences capture only countably many sets in $\mathcal{N}(x)$.
This invites the study of a the structure of a filter basis.

Cofinality of filters

Cofinality of filters

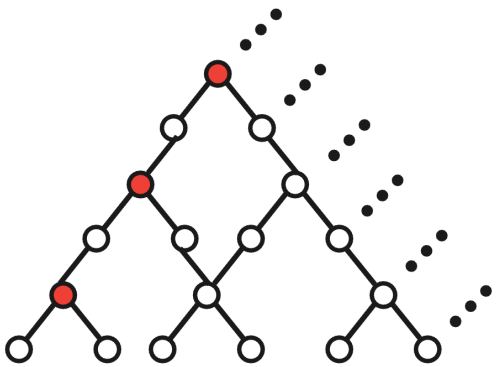
Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.

Cofinality of filters

Definition 10

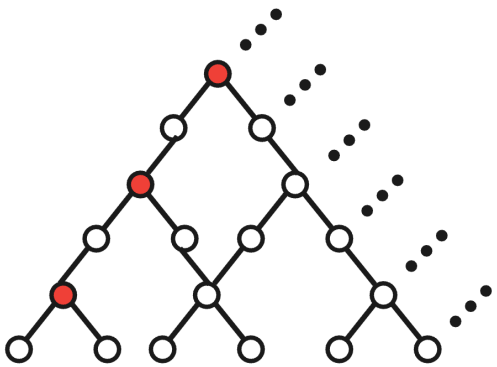
Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.



Cofinality of filters

Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.

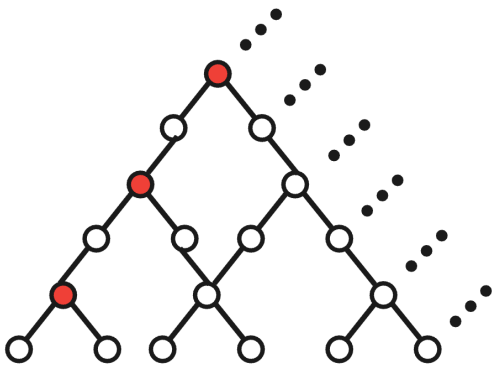


- For $(\mathcal{F}, \supseteq)$, $\mathcal{A}$ is called a **filter-basis** ($\forall B \in \mathcal{F}\ \exists A \in \mathcal{A},\ A \subseteq B$).

Cofinality of filters

Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.

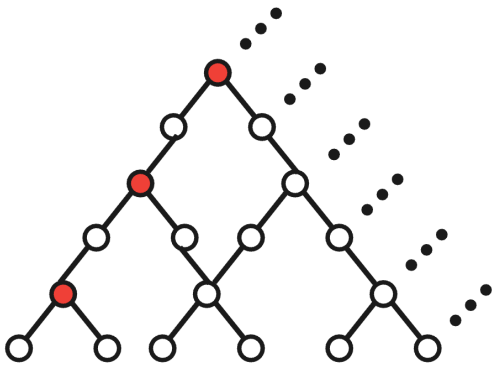


- For $(\mathcal{F}, \supseteq)$, $\mathcal{A}$ is called a **filter-basis** ($\forall B \in \mathcal{F}\ \exists A \in \mathcal{A},\ A \subseteq B$).
- Terminology: $\mathcal{A}$ generates $\mathcal{F}$/is a generatic set for $\mathcal{F}$.

Cofinality of filters

Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.

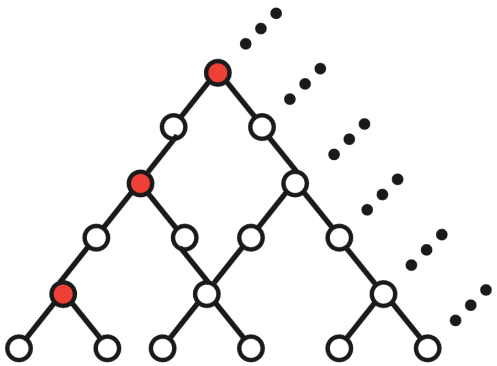


- For $(\mathcal{F}, \supseteq)$, $\mathcal{A}$ is called a **filter-basis** ($\forall B \in \mathcal{F}\ \exists A \in \mathcal{A},\ A \subseteq B$).
- Terminology: $\mathcal{A}$ generates $\mathcal{F}$/is a generatic set for $\mathcal{F}$.
- The **Character of** $\mathbb{P}$/**Cofinality of** $\mathbb{P}$ is defined by:

Cofinality of filters

Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.

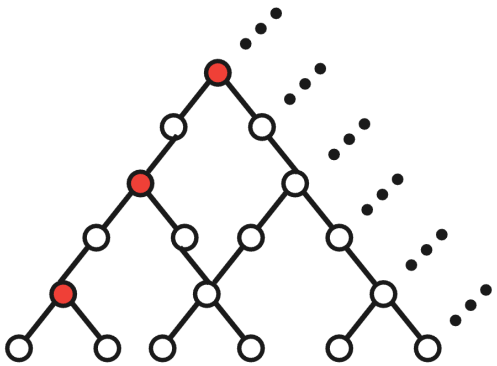


- For $(\mathcal{F}, \supseteq)$, $\mathcal{A}$ is called a **filter-basis** ($\forall B \in \mathcal{F}\ \exists A \in \mathcal{A},\ A \subseteq B$).
- Terminology: $\mathcal{A}$ generates $\mathcal{F}$/is a generatic set for $\mathcal{F}$.
- The **Character of** $\mathbb{P}$/**Cofinality of** $\mathbb{P}$ is defined by:
$$\chi(\mathbb{P}) := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is cofinal in } \mathbb{P}\}$$

Cofinality of filters

Definition 10

Given a directed set $\mathbb{P} = (P, \leq)$, a $\mathbb{P}$-**cofinal** set is a subset $\mathcal{A} \subseteq P$ such that for every $p \in P$ there is $a \in \mathcal{A}$ such that $p \leq a$.



- For $(\mathcal{F}, \supseteq)$, $\mathcal{A}$ is called a **filter-basis** ($\forall B \in \mathcal{F}\ \exists A \in \mathcal{A},\ A \subseteq B$).
- Terminology: $\mathcal{A}$ generates $\mathcal{F}$/is a generatic set for $\mathcal{F}$.
- The **Character of** $\mathbb{P}$/**Cofinality of** $\mathbb{P}$ is defined by:
$$\chi(\mathbb{P}) := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is cofinal in } \mathbb{P}\}$$

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p \in P}$.

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p\in P}$.

Theorem 11

A function $f : X \to Y$ is continuous iff for every direct $\mathbb{P} = (P, \leq)$, whenever x is a limit of a $\mathbb{P}$-net $(x_a)_{p\in P}$, $f(x)$ is a limit of $(f(x_p))_{p\in A}$.

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p\in P}$.

Theorem 11

A function $f : X \to Y$ is continuous iff for every direct $\mathbb{P} = (P, \leq)$, whenever x is a limit of a $\mathbb{P}$-net $(x_a)_{p\in P}$, $f(x)$ is a limit of $(f(x_p))_{p\in A}$.

At the price of complicating sequences, we characterized continuity.

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p\in P}$.

Theorem 11

A function $f : X \to Y$ is continuous iff for every direct $\mathbb{P} = (P, \leq)$, whenever x is a limit of a $\mathbb{P}$-net $(x_a)_{p\in P}$, $f(x)$ is a limit of $(f(x_p))_{p\in A}$.

At the price of complicating sequences, we characterized continuity.

Question

How to get rid of "redundant" directed sets?

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p\in P}$.

Theorem 11

A function $f : X \to Y$ is continuous iff for every direct $\mathbb{P} = (P, \leq)$, whenever x is a limit of a $\mathbb{P}$-net $(x_a)_{p\in P}$, $f(x)$ is a limit of $(f(x_p))_{p\in A}$.

At the price of complicating sequences, we characterized continuity.

Question

How to get rid of "redundant" directed sets? Set-Theoretic question.

Moore-Smith Convergence 1922

Idea: $\mathbb{P}$-nets

Change the index sets of a sequence by a general directed set: $(x_p)_{p\in P}$.

Theorem 11

A function $f : X \to Y$ is continuous iff for every direct $\mathbb{P} = (P, \leq)$, whenever x is a limit of a $\mathbb{P}$-net $(x_a)_{p\in P}$, $f(x)$ is a limit of $(f(x_p))_{p\in A}$.

At the price of complicating sequences, we characterized continuity.

Question

How to get rid of "redundant" directed sets? Set-Theoretic question.

Answer

By comparing the cofinal complexity i.e. The Tukey order.

The Tukey order: a comparison of cofinal complexity

Let $\mathbb{P} = (P, \leq_P), \mathbb{Q} = (Q, \leq_Q)$ be two directed sets.

Definition 12 (Tukey '40)

$\mathbb{P} \leq_T \mathbb{Q}$ iff there is a monotone $f : Q \to P$ with $\mathrm{Im}(f)$ cofinal in $\mathbb{P}$.

The Tukey order: a comparison of cofinal complexity

Let $\mathbb{P} = (P, \leq_P), \mathbb{Q} = (Q, \leq_Q)$ be two directed sets.

Definition 12 (Tukey '40)

$\mathbb{P} \leq_T \mathbb{Q}$ iff there is a monotone $f : Q \to P$ with $\mathrm{Im}(f)$ cofinal in $\mathbb{P}$.

- If $\mathbb{P} \leq_T \mathbb{Q}$, then every $\mathbb{P}$-net can be replaced by a $\mathbb{Q}$-net.

The Tukey order: a comparison of cofinal complexity

Let $\mathbb{P} = (P, \leq_P), \mathbb{Q} = (Q, \leq_Q)$ be two directed sets.

Definition 12 (Tukey '40)

$\mathbb{P} \leq_T \mathbb{Q}$ iff there is a monotone $f : Q \to P$ with $\mathrm{Im}(f)$ cofinal in $\mathbb{P}$.

- If $\mathbb{P} \leq_T \mathbb{Q}$, then every $\mathbb{P}$-net can be replaced by a $\mathbb{Q}$-net.
- The character is an invariant of cofinal complexity: If $\mathbb{P} \leq_T \mathbb{Q}$, then $\chi(\mathbb{P}) \leq \chi(\mathbb{Q})$.

The Tukey order: a comparison of cofinal complexity

Let $\mathbb{P} = (P, \leq_P), \mathbb{Q} = (Q, \leq_Q)$ be two directed sets.

Definition 12 (Tukey '40)

$\mathbb{P} \leq_T \mathbb{Q}$ iff there is a monotone $f : Q \to P$ with $\mathrm{Im}(f)$ cofinal in $\mathbb{P}$.

- If $\mathbb{P} \leq_T \mathbb{Q}$, then every $\mathbb{P}$-net can be replaced by a $\mathbb{Q}$-net.
- The character is an invariant of cofinal complexity: If $\mathbb{P} \leq_T \mathbb{Q}$, then $\chi(\mathbb{P}) \leq \chi(\mathbb{Q})$.

Theorem 13 (Ultrafilters of maximal cofinal complexity-Isbell '65)

There is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that for any directed $\mathbb{P} = (P, \leq)$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$, provided that P has size at most continuum ($\mathfrak{c} = |\mathbb{R}|$).

The Tukey order: a comparison of cofinal complexity

Let $\mathbb{P} = (P, \leq_P), \mathbb{Q} = (Q, \leq_Q)$ be two directed sets.

Definition 12 (Tukey '40)

$\mathbb{P} \leq_T \mathbb{Q}$ iff there is a monotone $f : Q \to P$ with $\mathrm{Im}(f)$ cofinal in $\mathbb{P}$.

- If $\mathbb{P} \leq_T \mathbb{Q}$, then every $\mathbb{P}$-net can be replaced by a $\mathbb{Q}$-net.
- The character is an invariant of cofinal complexity: If $\mathbb{P} \leq_T \mathbb{Q}$, then $\chi(\mathbb{P}) \leq \chi(\mathbb{Q})$.

Theorem 13 (Ultrafilters of maximal cofinal complexity-Isbell '65)

There is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that for any directed $\mathbb{P} = (P, \leq)$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$, provided that P has size at most continuum ($\mathfrak{c} = |\mathbb{R}|$).

- These are called *Tukey-top* ultrafilters.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small filter-base $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small filter-base $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is possible that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX can be realized along with φ, then φ is consistent relative to AX.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

- $\mathbb{Q}$ is a model of LO satisfying φ.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

- $\mathbb{Q}$ is a model of LO satisfying φ. Hence LO does not refute φ.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

- $\mathbb{Q}$ is a model of LO satisfying φ. Hence LO does not refute φ.
- $\mathbb{Z}$ is a model of LO not satisfying φ.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

- $\mathbb{Q}$ is a model of LO satisfying φ. Hence LO does not refute φ.
- $\mathbb{Z}$ is a model of LO not satisfying φ. Hence LO does not prove φ.

A word on consistency results

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$. i.e. the existence of $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{c}$ cannot be refuted.

Generally, a statement φ is said to be **consistent** relative to the axioms AX if AX does not refute φ.

Lemma 16 (Soundness Lemma)

If AX has a model satisfying φ, then φ is consistent relative to AX.

Example 17

Let LO denote the axioms of linear orders. Consider the statement $\varphi =$"the order is dense"

- $\mathbb{Q}$ is a model of LO satisfying φ. Hence LO does not refute φ.
- $\mathbb{Z}$ is a model of LO not satisfying φ. Hence LO does not prove φ.

We say that φ is independent of LO.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

- Kunen produced a *ZFC*-model satisfying $\exists \mathcal{U}$ on $\mathbb{N}$,$\chi(\mathcal{U}) < \mathfrak{c}$.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

- Kunen produced a *ZFC*-model satisfying $\exists \mathcal{U}$ on $\mathbb{N}$,$\chi(\mathcal{U}) < \mathfrak{c}$.
- Consistently, $\exists \mathcal{U}$ on $\mathbb{N}$, $\mathcal{U}$ is non-Tukey-top.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

- Kunen produced a *ZFC*-model satisfying $\exists \mathcal{U}$ on $\mathbb{N}$,$\chi(\mathcal{U}) < \mathfrak{c}$.
- Consistently, $\exists \mathcal{U}$ on $\mathbb{N}$, $\mathcal{U}$ is non-Tukey-top.
- Dobrinen-Todorcevic('14): p-point ult. on $\mathbb{N}$ are non-Tukey-top.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

- Kunen produced a *ZFC*-model satisfying $\exists \mathcal{U}$ on $\mathbb{N}$,$\chi(\mathcal{U}) < \mathfrak{c}$.
- Consistently, $\exists \mathcal{U}$ on $\mathbb{N}$, $\mathcal{U}$ is non-Tukey-top.
- Dobrinen-Todorcevic('14): p-point ult. on $\mathbb{N}$ are non-Tukey-top.
- Cancino-Zapletal('25): Consistently, $\forall \mathcal{U}$ on $\mathbb{N}$ $\mathcal{U}$ is Tukey-top.

Question 1 (Isbell's question '65)

Is every ultrafilter on $\mathbb{N}$ Tukey-top?

Since the character is an invariant of cofinal complexity, we have:

Fact 14 (Small character $\Rightarrow$ non-Tukey-top)

If $\mathcal{U}$ is Tukey-top on $\mathbb{N}$ then $\chi(\mathcal{U}) = \mathfrak{c}$.

Theorem 15 (Kunen '70's)

It is consistent that there is an ultrafilter on $\mathbb{N}$ with $\chi(\mathcal{U}) < \mathfrak{c}$.

- Kunen produced a *ZFC*-model satisfying $\exists \mathcal{U}$ on $\mathbb{N}$,$\chi(\mathcal{U}) < \mathfrak{c}$.
- Consistently, $\exists \mathcal{U}$ on $\mathbb{N}$, $\mathcal{U}$ is non-Tukey-top.
- Dobrinen-Todorcevic('14): p-point ult. on $\mathbb{N}$ are non-Tukey-top.
- Cancino-Zapletal('25): Consistently, $\forall \mathcal{U}$ on $\mathbb{N}$ $\mathcal{U}$ is Tukey-top.

Conclusion: Isbell's problem is independent of ZFC

Question 2

Is it consistent for an ultrafilter over an uncountable set to be generated by fewer than the maximal number of sets?

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over an uncountable set to be generated by fewer than the maximal number of sets?

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than the maximal number of sets?

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Theorem 18 (B.-Moore-Serafin '25)

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Theorem 18 (B.-Moore-Serafin '25)

No.

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Theorem 18 (B.-Moore-Serafin '25)

No. Consistently.

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Theorem 18 (B.-Moore-Serafin '25)

No. Consistently. (Relative to large cardinals).

Recall that $\aleph_1 =$ the first uncountable cardinal.

Question 2 (Kunen's question- 70's)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by fewer than $2^{\aleph_1}$-many sets?

By "Small character $\Rightarrow$ non-Tukey-top", this is closely related to:

Question 3 (Isbell's question at $\aleph_1$)

Is every ultrafilter over a set of size $\aleph_1$ Tukey-top?

Theorem 18 (B.-Moore-Serafin '25)

No. Consistently. (Relative to large cardinals).

Theorem 19 (B.-Moore-Serafin '25)

Isbell's problem at $\aleph_1$ is independent of ZFC.

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

A major attempt on Kunen's problem was made by Raghavan-Shelah:

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

A major attempt on Kunen's problem was made by Raghavan-Shelah:

Definition 20

Let $\mathcal{U}$ be an ultrafilter on $\aleph_1$. A family of sets $\mathcal{A}$ **generates** $\mathcal{U}$ if:

1. $\mathcal{A} \subseteq \mathcal{U}$.
2. for every $U \in \mathcal{U}$ there is $A \in \mathcal{A}$ such that $A \subseteq U$.

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

A major attempt on Kunen's problem was made by Raghavan-Shelah:

Definition 20

Let $\mathcal{U}$ be an ultrafilter on $\aleph_1$. A family of sets $\mathcal{A}$ **ext.-generates** $\mathcal{U}$ if:

1. $\mathcal{A} \subseteq \mathcal{U}$.
2. for every $U \in \mathcal{U}$ there is $A \in \mathcal{A}$ such that $A \subseteq U$.

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

A major attempt on Kunen's problem was made by Raghavan-Shelah:

Definition 20

Let $\mathcal{U}$ be an ultrafilter on $\aleph_1$. A family of sets $\mathcal{A}$ **ext.-generates** $\mathcal{U}$ if:

1. $\mathcal{A} \subseteq [\aleph_1]^{\aleph_1}$.
2. for every $U \in \mathcal{U}$ there is $A \in \mathcal{A}$ such that $A \subseteq U$.

Question 2 (Kunen's question)

Is it consistent for an ultrafilter over $\aleph_1$ to be generated by $< 2^{\aleph_1}$ sets?

A major attempt on Kunen's problem was made by Raghavan-Shelah:

Definition 20

Let $\mathcal{U}$ be an ultrafilter on $\aleph_1$. A family of sets $\mathcal{A}$ **ext.-generates** $\mathcal{U}$ if:

1. $\mathcal{A} \subseteq [\aleph_1]^{\aleph_1} := \{X \subseteq \aleph_1 \mid |X| = \aleph_1\}$.
2. for every $U \in \mathcal{U}$ there is $A \in \mathcal{A}$ such that $A \subseteq U$.

Question 2 (Kunen's question)

Is it consistent for an ultrafilter on $\aleph_1$ to be generated by $<2^{\aleph_1}$ sets?

Question 3 (Raghavan-Shelah)

Is it consistent for an ultrafilter on $\aleph_1$ to be generated by $<2^{\aleph_1}$ sets?

Question 3 (Raghavan-Shelah)

Is it consistent for an ultrafilter on $\aleph_1$ to be ext.-generated by $<2^{\aleph_1}$ sets?

Question 3 (Raghavan-Shelah)

Is it consistent for an ultrafilter on $\aleph_1$ to be ext.-generated by $<2^{\aleph_1}$ sets?

Theorem 21 (B.-Eskew 25+)

Yes.

Some details

"A couple of months in the laboratory can frequently save a couple of hours in the library"- Frank Henry Westheimer.

Some details

"A couple of months in the laboratory can frequently save a couple of hours in the library"- Frank Henry Westheimer.

A variation for the working mathematician:

Some details

"A couple of months in the laboratory can frequently save a couple of hours in the library"- Frank Henry Westheimer.

A variation for the working mathematician:

"A couple of months in the library can frequently save a couple of hours in front of a paper"- Menachem Magidor.

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}$

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}$ $\forall \mathcal{B} \in [\mathcal{A}]^\mu$

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}\ \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}\ \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Theorem 13 (Isbell '65)

There is an ultrafilter on $\mathbb{N}$ such that for every directed set A with $|A| \leq 2^{\aleph_0}$, $A \leq_T \mathcal{U}$.

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}\ \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Theorem 13 (Isbell '65)

There is an ultrafilter on $\mathbb{N}$ such that $\exists \mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}} \forall \mathcal{B} \in [\mathcal{A}]^{\aleph_0},\ \bigcap \mathcal{B} \notin \mathcal{U}$.

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}\ \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Theorem 13 (Isbell '65)

There is an ultrafilter on $\mathbb{N}$ such that $\exists \mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}} \forall \mathcal{B} \in [\mathcal{A}]^{\aleph_0},\ \bigcap \mathcal{B} \notin \mathcal{U}$.

Definition 23 (Tukey, Kanamori, Galvin (all independently))

We say that $\mathcal{U}$ is μ-**Tukey-top** if $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa} \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

Let $\mathcal{U}$ be an ultrafilter over κ and let μ be a cardinal.

Theorem 22 (Tukey for $\mu = \omega$, B.-Dobrinen for $\mu > \omega$)

TFAE:

1. *for every μ-directed $\mathbb{P}$ of cardinality $\leq 2^\kappa$, $\mathbb{P} \leq_T (\mathcal{U}, \supseteq)$.*
2. $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa}\ \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

μ-directed: every $\mathcal{A} \in [\mathbb{P}]^{<\mu}$ is bounded in $\mathbb{P}$.
Recall $[X]^\lambda = \{Y \subseteq X \mid |Y| = \lambda\}$

Theorem 13 (Isbell '65)

There is an ultrafilter on $\mathbb{N}$ such that $\exists \mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}} \forall \mathcal{B} \in [\mathcal{A}]^{\aleph_0},\ \bigcap \mathcal{B} \notin \mathcal{U}$.

Definition 23 (Tukey, Kanamori, Galvin (all independently))

We say that $\mathcal{U}$ is μ-**Tukey-top** if $\exists \mathcal{A} \in [\mathcal{U}]^{2^\kappa} \forall \mathcal{B} \in [\mathcal{A}]^\mu,\ \bigcap \mathcal{B} \notin \mathcal{U}$.

- Tukey-top$=$ $\aleph_0$-Tukey-top.

$$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ *implies that every ultrafilter* $\mathcal{U}$ *on* κ *is* μ*-Tukey-top.*

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

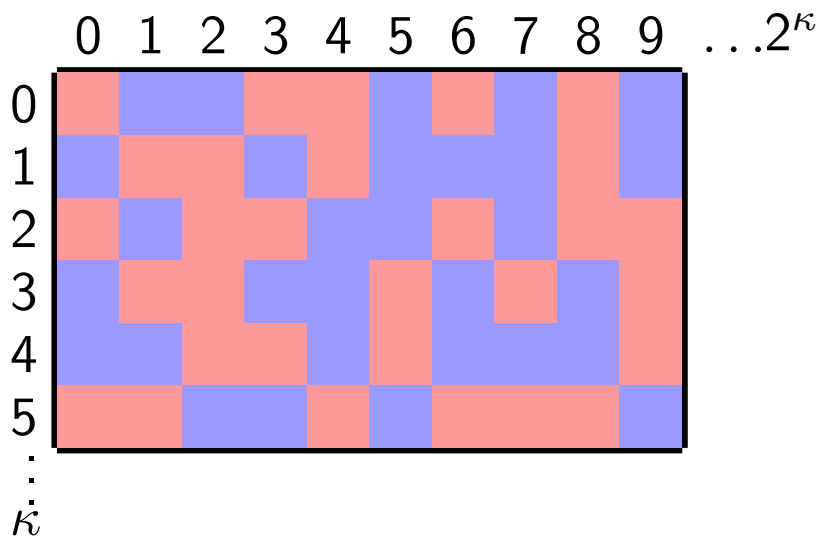
Proof.

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring

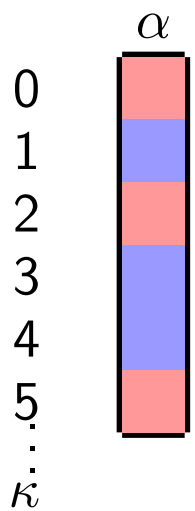


$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

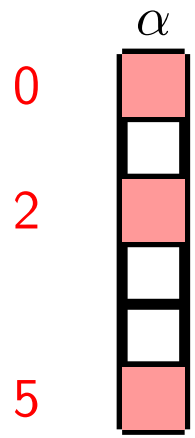


$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

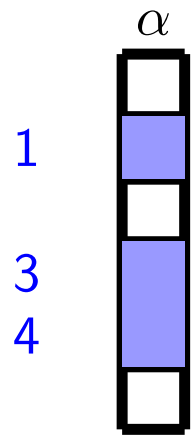


$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.



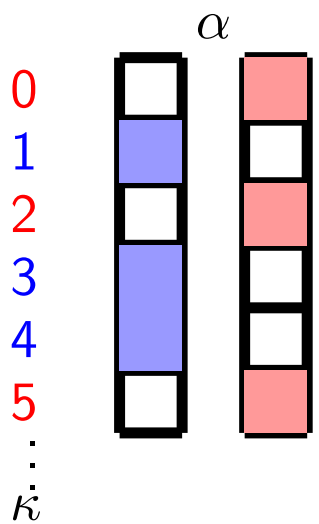
$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult.,



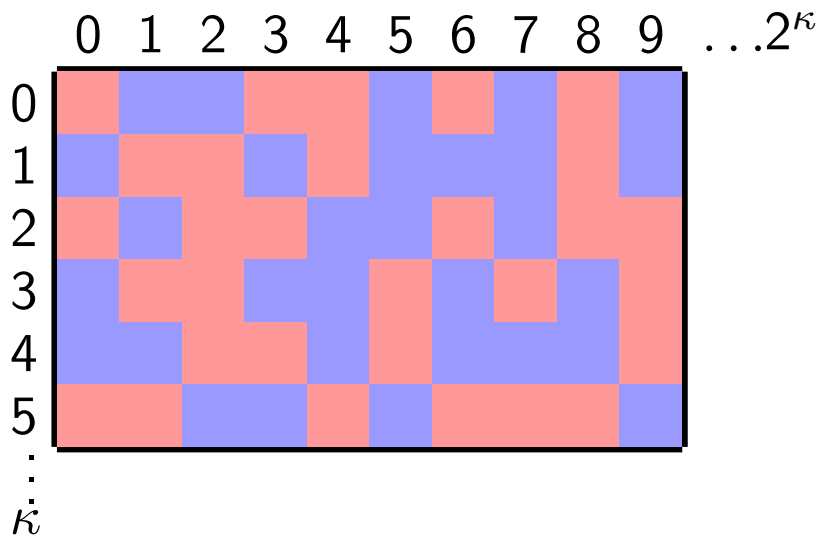
$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

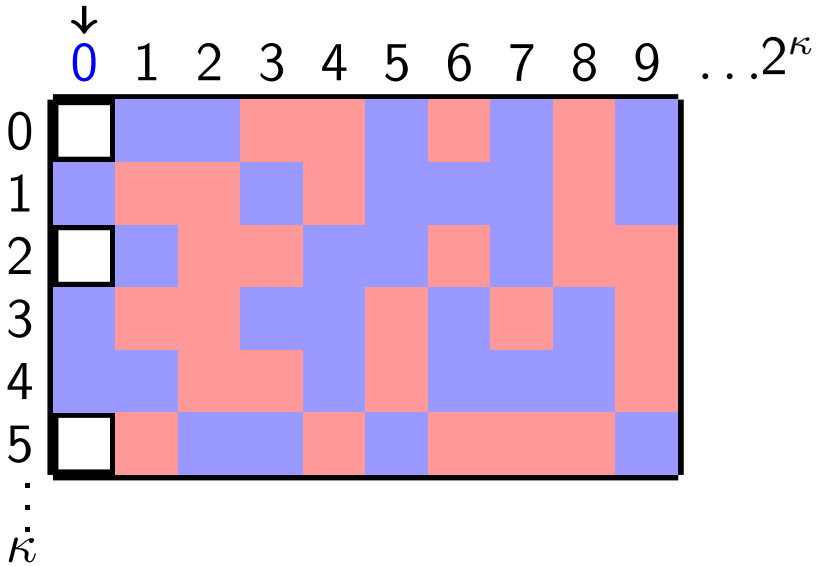
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

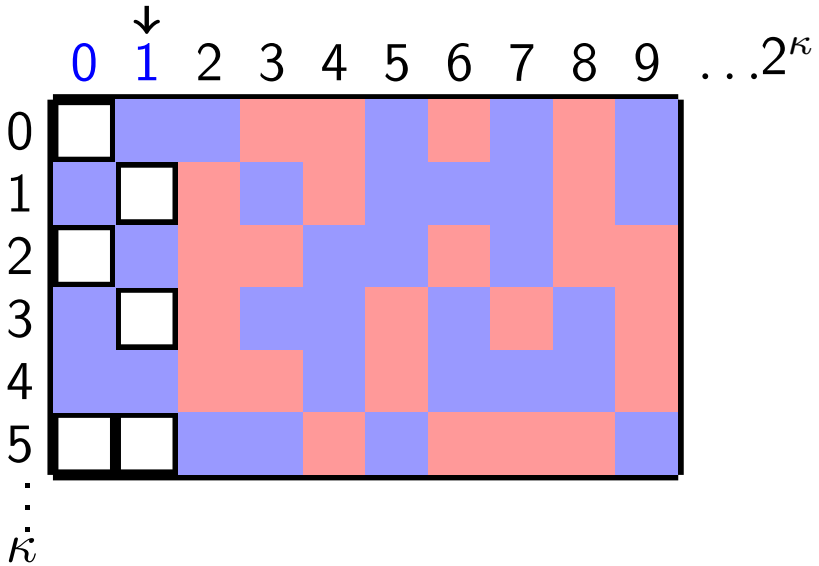
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

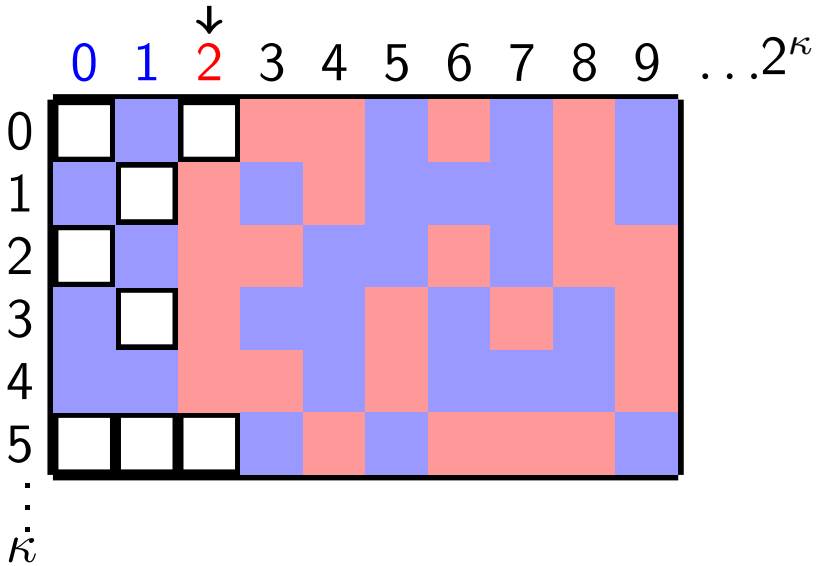
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

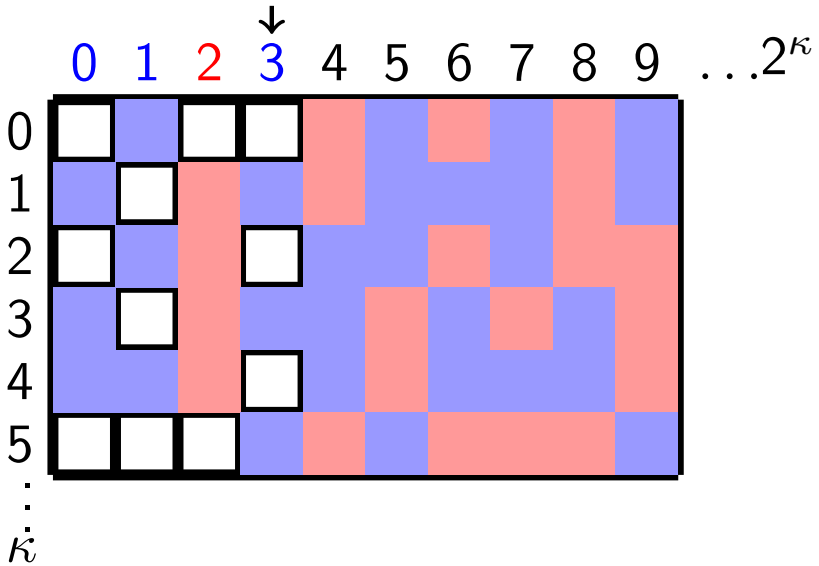
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

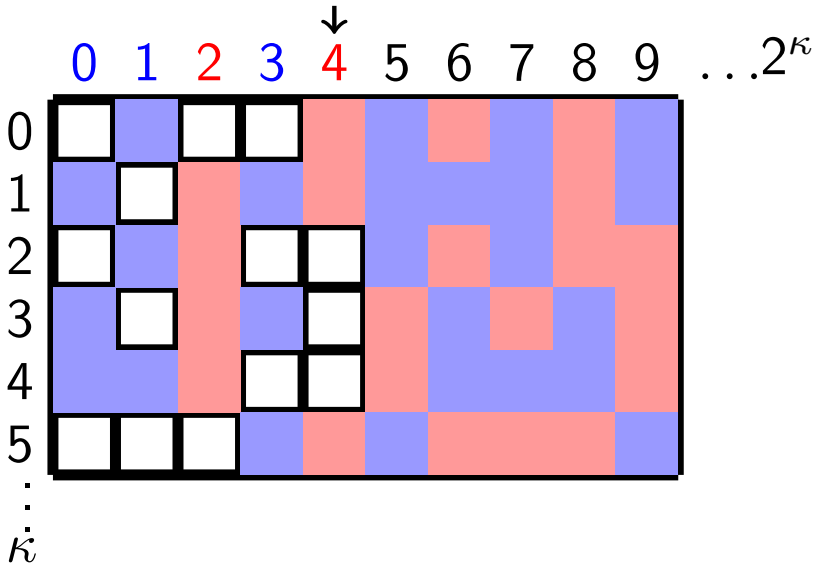
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

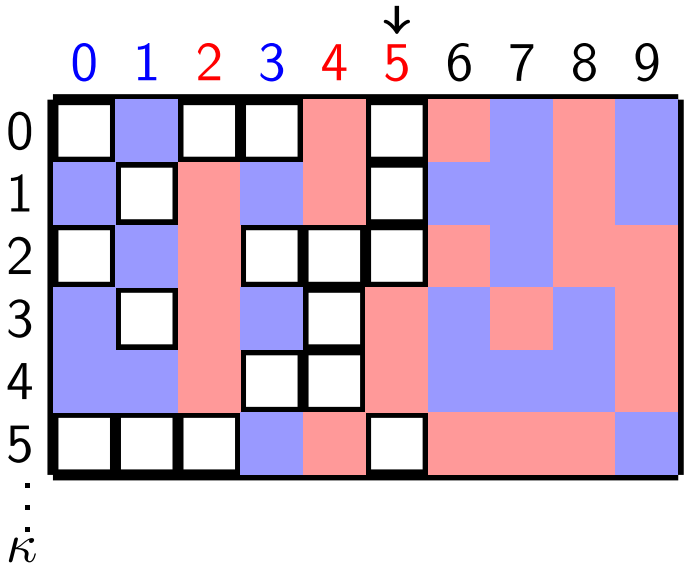
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

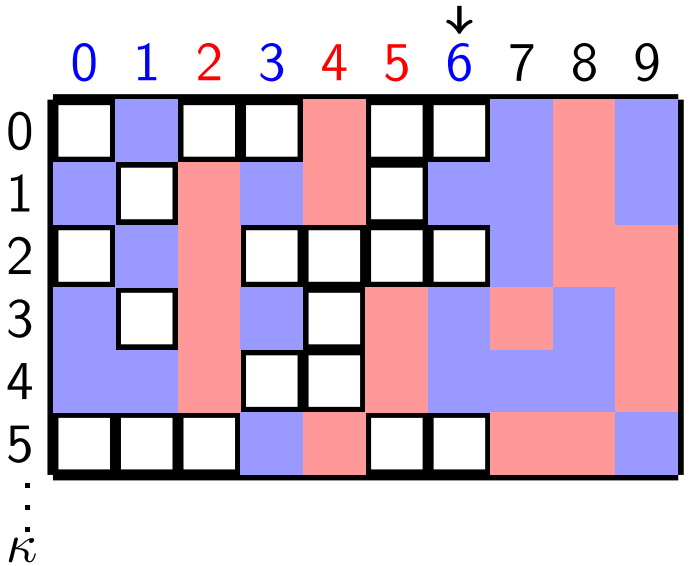
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

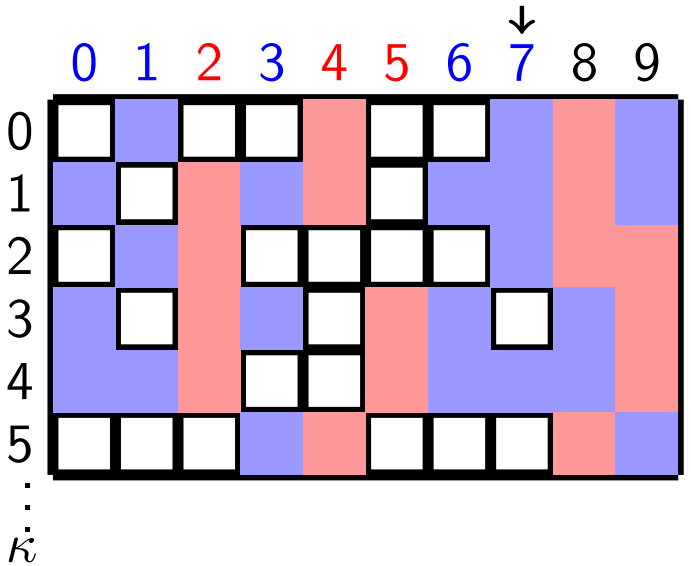
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

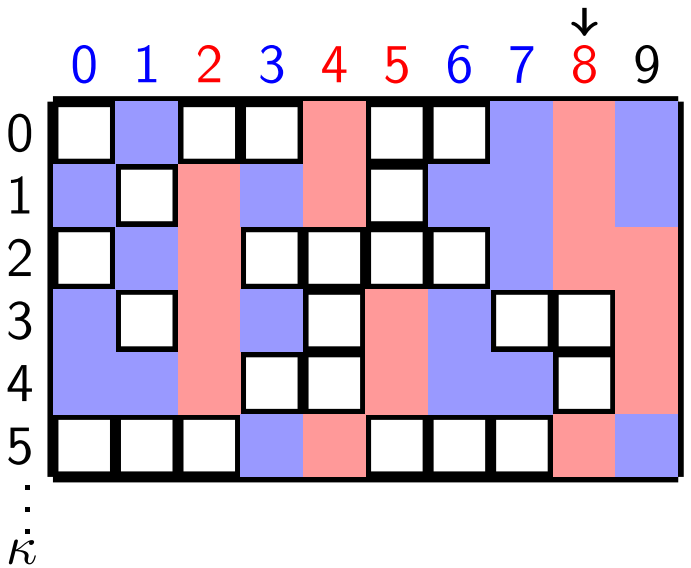
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_8 = \{0, 1, 2, 5, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

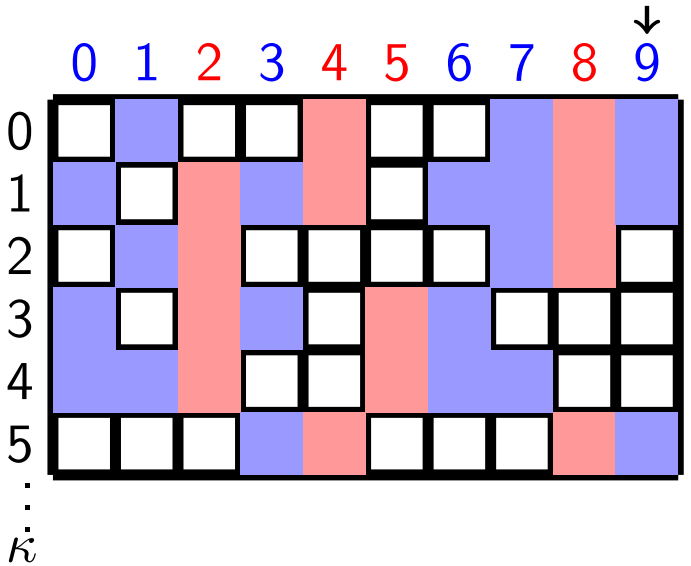
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_8 = \{0, 1, 2, 5, \cdots\}$
$A_9 = \{0, 1, 5, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

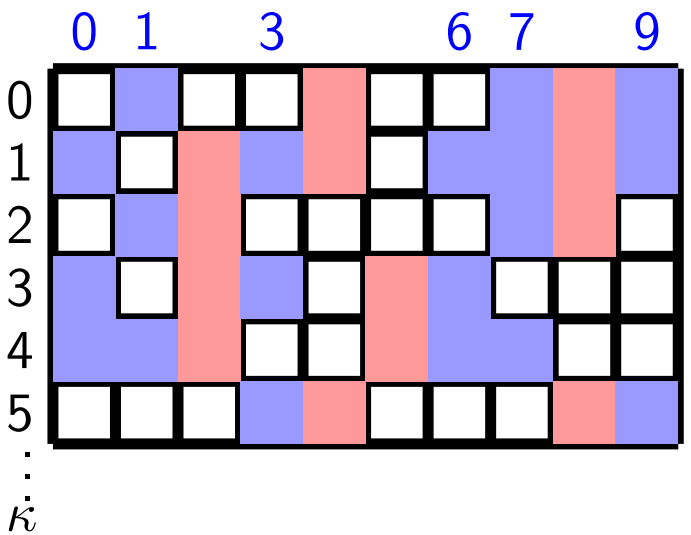
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_8 = \{0, 1, 2, 5, \cdots\}$
$A_9 = \{0, 1, 5, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

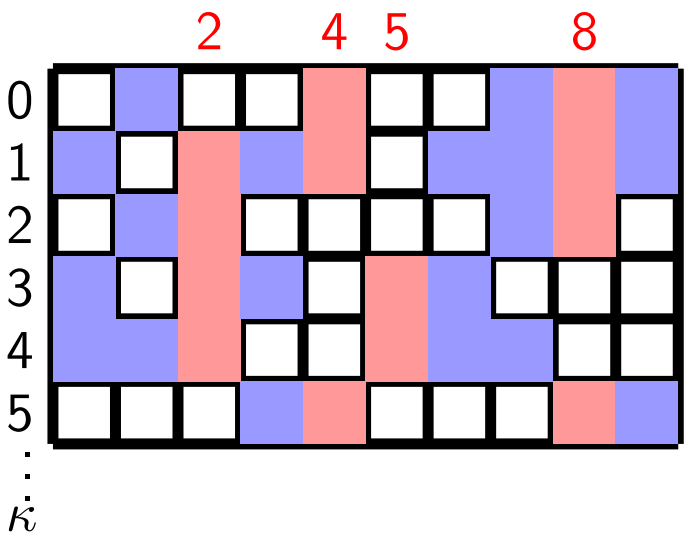
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_8 = \{0, 1, 2, 5, \cdots\}$
$A_9 = \{0, 1, 5, \cdots\}$

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

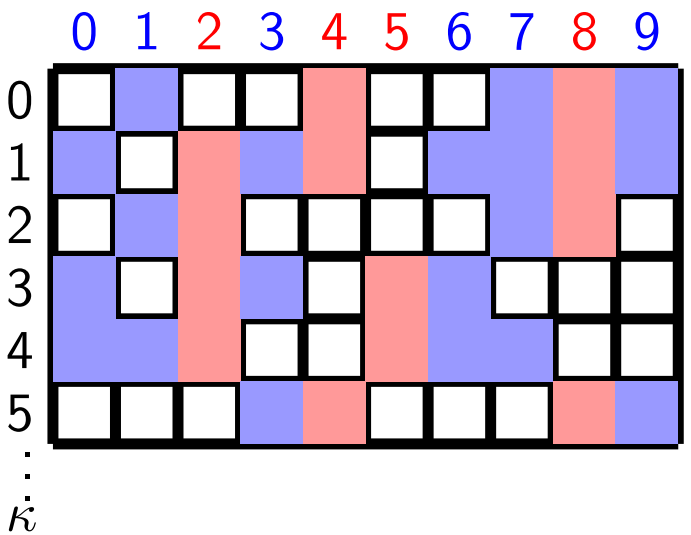
Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_2 = \{1, 2, 3, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_4 = \{0, 1, 5, \cdots\}$



$A_5 = \{0, 1, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_8 = \{0, 1, 2, 5, \cdots\}$
$A_9 = \{0, 1, 5, \cdots\}$

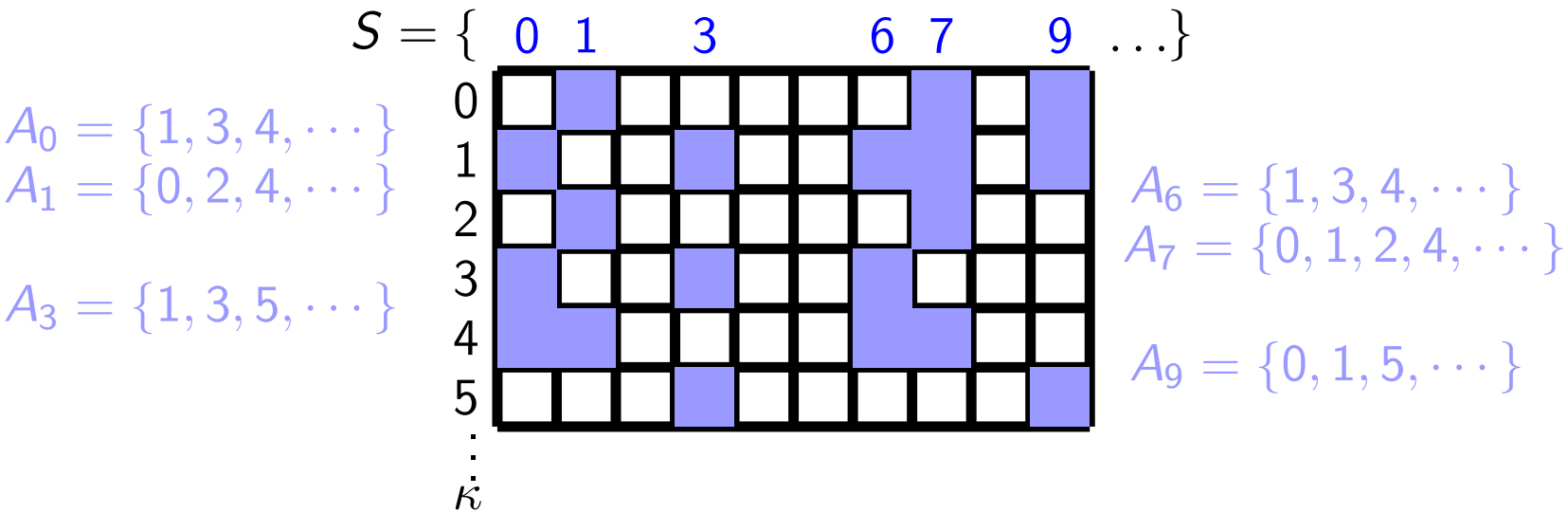
$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.
- There is $S \in [2^\kappa]^{2^\kappa}$ such that for every $\alpha \in S$, $\epsilon_\alpha = 1$ (WLOG).



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

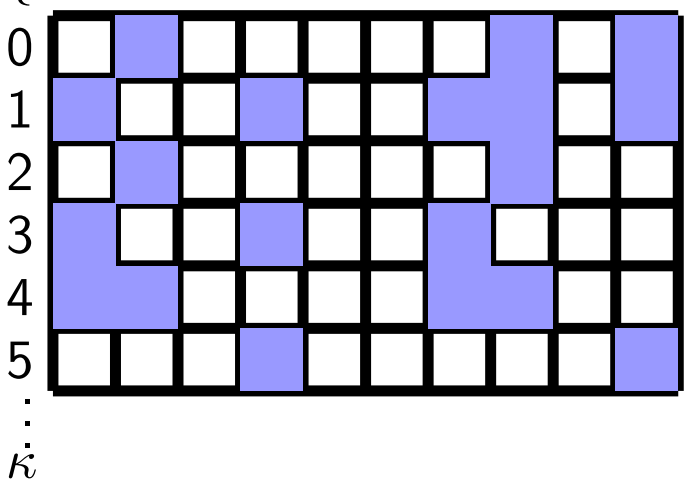
$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.
- There is $S \in [2^\kappa]^{2^\kappa}$ such that for every $\alpha \in S$, $\epsilon_\alpha = 1$ (WLOG).
- Then $\langle A_\alpha \mid \alpha \in S \rangle$ witnesses μ-Tukey-top:

$S = \{ 0 \ 1 \ 3 \ 6 \ 7 \ 9 \ \ldots \}$

$A_0 = \{1, 3, 4, \cdots\}$
$A_1 = \{0, 2, 4, \cdots\}$
$A_3 = \{1, 3, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$
$A_7 = \{0, 1, 2, 4, \cdots\}$
$A_9 = \{0, 1, 5, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

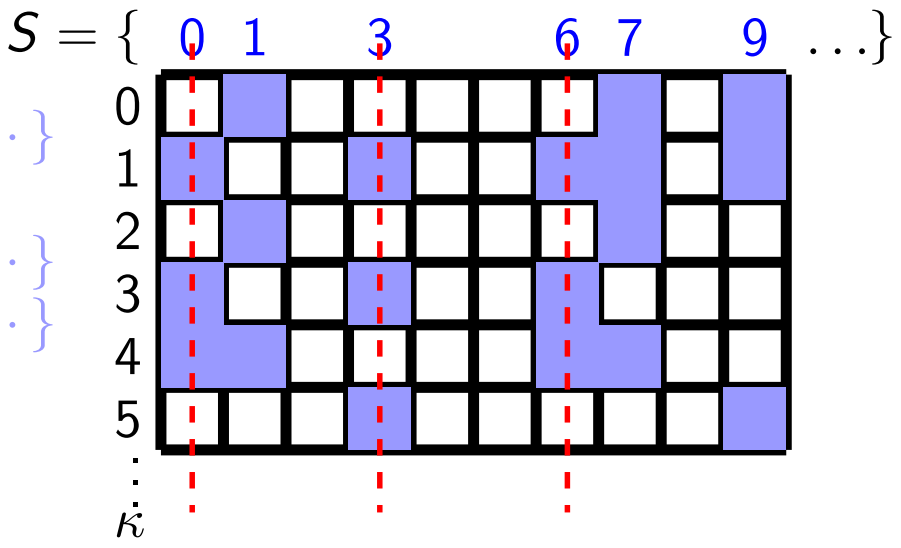
$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa\ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.
- There is $S \in [2^\kappa]^{2^\kappa}$ such that for every $\alpha \in S$, $\epsilon_\alpha = 1$ (WLOG).
- Then $\langle A_\alpha \mid \alpha \in S \rangle$ witnesses μ-Tukey-top:
- $A = \bigcap_{\alpha \in B} A_\alpha \in U$ means that $A \times B$ is hom. with color 1. □

$A_0 = \{1, 3, 4, \cdots\}$

$A_3 = \{1, 3, 5, \cdots\}$
$A_6 = \{1, 3, 4, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Proof. Let c be a witnessing coloring and $c_\alpha : \kappa \to 2$, $c_\alpha(i) = c(\alpha, i)$.

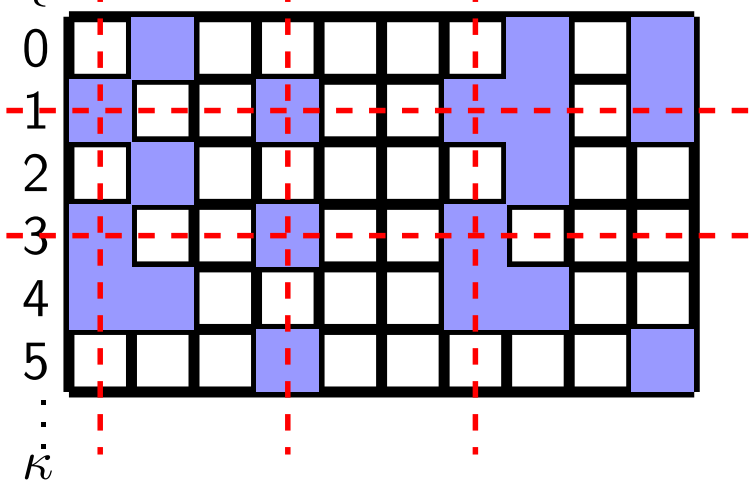
- Since $\mathcal{U}$ is an ult., $\forall \alpha < 2^\kappa \ \exists \epsilon_\alpha < 2$ s.t. $A_\alpha := c_\alpha^{-1}[\{\epsilon_\alpha\}] \in \mathcal{U}$.
- There is $S \in [2^\kappa]^{2^\kappa}$ such that for every $\alpha \in S$, $\epsilon_\alpha = 1$ (WLOG).
- Then $\langle A_\alpha \mid \alpha \in S \rangle$ witnesses μ-Tukey-top:
- $A = \bigcap_{\alpha \in B} A_\alpha \in U$ means that $A \times B$ is hom. with color 1. □

$S = \{ 0 \ 1 \ 3 \ 6 \ 7 \ 9 \ \ldots \}$

$A_0 = \{1, 3, 4, \cdots\}$

$A_3 = \{1, 3, 5, \cdots\}$

$A_6 = \{1, 3, 4, \cdots\}$



$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Corollary 25

$\binom{2^{\aleph_1}}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}$ is consistent (e.g. in the Cohen-model where CH fails). In particular, it is consistent that every ultrafilter on $\aleph_1$ is Tukey-top

$\binom{2^\kappa}{\kappa} \to \binom{\mu}{\kappa} \equiv$ any 2-coloring of $\kappa \times 2^\kappa$ has a monochromatic $\kappa \times \mu$-minor.

Theorem 24

$\binom{2^\kappa}{\kappa} \not\to \binom{\mu}{\kappa}$ implies that every ultrafilter $\mathcal{U}$ on κ is μ-Tukey-top.

Corollary 25

$\binom{2^{\aleph_1}}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}$ is consistent (e.g. in the Cohen-model where CH fails). In particular, it is consistent that every ultrafilter on $\aleph_1$ is Tukey-top

non-Tukey-top ultrafilters

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

Large cardinals=$\diamondsuit$+"there is an ω_1-dense ideal on ω_1".

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

Large cardinals$=\diamondsuit+$"there is an ω_1-dense ideal on ω_1".
Main tool: under these assumptions Laver constructed a special ultrafilter U which is non-regular, answering a central open problem in the 80's.

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

Large cardinals=$\diamondsuit$+"there is an ω_1-dense ideal on ω_1".
Main tool: under these assumptions Laver constructed a special ultrafilter U which is non-regular, answering a central open problem in the 80's. We use that ultrafilter, some mild combinatorics, and a theorem due to Galvin about normal ideal to show that U is not Tukey-top.

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

Large cardinals=$\diamondsuit$+"there is an ω_1-dense ideal on ω_1".
Main tool: under these assumptions Laver constructed a special ultrafilter U which is non-regular, answering a central open problem in the 80's. We use that ultrafilter, some mild combinatorics, and a theorem due to Galvin about normal ideal to show that U is not Tukey-top.

Conjecture 1

If there is a non-Tukey-top ultrafilter over ω_1, then large cardinals consistently exist.

non-Tukey-top ultrafilters

Theorem 26

Assume large cardinals, then there is a non-Tukey-top ultrafilter over ω_1.

Large cardinals=$\diamondsuit$+"there is an ω_1-dense ideal on ω_1".
Main tool: under these assumptions Laver constructed a special ultrafilter U which is non-regular, answering a central open problem in the 80's. We use that ultrafilter, some mild combinatorics, and a theorem due to Galvin about normal ideal to show that U is not Tukey-top.

Conjecture 1

If there is a non-Tukey-top ultrafilter over ω_1, then large cardinals consistently exist.

In particular, I conjecture that a positive answer to Kunen's problem requires large cardinals.

The shape of a basis

The shape of a basis

Question

How arbitrary is a filter-base for an ultrafilter? are there any (non-trivial) restrictions?

The shape of a basis

Question

How arbitrary is a filter-base for an ultrafilter? are there any (non-trivial) restrictions?

On functions $\mathbb{N}^{\mathbb{N}}$ consider the almost everywhere domination order, that is, $f \leq^* g$ iff $\exists N\ \forall n \geq N,\ f(n) \leq g(n)$.

The shape of a basis

Question

How arbitrary is a filter-base for an ultrafilter? are there any (non-trivial) restrictions?

On functions $\mathbb{N}^{\mathbb{N}}$ consider the almost everywhere domination order, that is, $f \leq^* g$ iff $\exists N\ \forall n \geq N,\ f(n) \leq g(n)$.

Theorem 27 (Hechler '74)

Suppose that $\mathbb{D}$ is a well-founded σ-directed post. Then there it is consistent that (i.e. there is a ccc forcing extension in which) there is a $\leq^*$-cofinal family $\mathcal{A} \subseteq \omega^\omega$ s.t. $(\mathcal{A}, \leq^*) \simeq \mathbb{D}$.

The shape of a basis

Question

How arbitrary is a filter-base for an ultrafilter? are there any (non-trivial) restrictions?

On functions $\mathbb{N}^{\mathbb{N}}$ consider the almost everywhere domination order, that is, $f \leq^* g$ iff $\exists N\ \forall n \geq N,\ f(n) \leq g(n)$.

Theorem 27 (Hechler '74)

Suppose that $\mathbb{D}$ is a well-founded σ-directed post. Then there it is consistent that (i.e. there is a ccc forcing extension in which) there is a $\leq^*$-cofinal family $\mathcal{A} \subseteq \omega^\omega$ s.t. $(\mathcal{A}, \leq^*) \simeq \mathbb{D}$.

So $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ can be arbitrary cofinaly complicated.

An Hechler-like theorem for ultrafilters

$A \subseteq^* B$ if and only if $A \setminus B$ is finite (or bounded)

An Hechler-like theorem for ultrafilters

$A \subseteq^* B$ if and only if $A \setminus B$ is finite (or bounded)

Theorem 28 (B.-Cummings-Goldberg-Hayut-Poveda)

Let $\mathbb{D}$ be a well-founded σ-directed poset. Then it is consistent that there is a $\subseteq^*$-filter basis $\mathcal{B}$ for an ultrafilter $\mathcal{U}$ such that $(\mathcal{B}, \supseteq^*) \simeq \mathbb{D}$.

(i.e. in a $|\mathbb{D}|^+$-closed$*$ccc forcing extension)

An Hechler-like theorem for ultrafilters

$A \subseteq^* B$ if and only if $A \setminus B$ is finite (or bounded)

Theorem 28 (B.-Cummings-Goldberg-Hayut-Poveda)

Let $\mathbb{D}$ be a well-founded σ-directed poset. Then it is consistent that there is a $\subseteq^*$-filter basis $\mathcal{B}$ for an ultrafilter $\mathcal{U}$ such that $(\mathcal{B}, \supseteq^*) \simeq \mathbb{D}$.

(i.e. in a $|\mathbb{D}|^+$-closed$*$ccc forcing extension)

Theorem 29 (B.-Cummings-Goldberg-Hayut-Poveda)

Suppose that κ is supercompact cardinal. Let $\mathbb{D}$ be a well-founded κ^+-directed poset. Then there is a forcing extension where there is $\mathcal{B} \subseteq [\kappa]^\kappa$ s.t. $\mathcal{B}$ $\subseteq^*$-filter basis for a normal ultrafilter on κ and $(\mathcal{B}, \supseteq^*) \simeq \mathbb{D}$.

The End!