SUPERCOMPACT PRIKRY FORCING CAN ADD A COHEN GENERIC

TOM BENHAMOU AND GABRIEL GOLDBERG

1. Intro

2. Preliminaries

3. Constructing generic filters over ultrapowers

Proposition 3.1 (Woodin)**.** *Assume $\diamondsuit(S^{\kappa^+}_\kappa)$, and let U be a nonprincipal κ-complete ultrafilter over κ. Let $\mathbb{P}$ be any κ-closed forcing of cardinality κ^+. (e.g. $Add(\kappa^+,1), Col(\kappa,\kappa^+)$) Then there is an M_U-generic filter for $j_U(\mathbb{P})$.*

Proof. Let $\langle A_\alpha \mid \alpha \in S^{\kappa^+}_\kappa\rangle$ be a diamond sequence. Enumerate $\mathbb{P} = \{p_\beta \mid \beta < \kappa^+\}$ and let $\mathbb{P}_\alpha = \{p_\beta \mid \beta < \alpha\}$. We construct a descending sequence $\langle q_\xi \mid \xi < \kappa^+\rangle \subseteq j_U(\mathbb{P})$ such that for every dense set $D \subseteq j_U(\mathbb{P})$ in M_U, there is $\xi < \kappa^+$ such that $q_\xi \in D$. We will also fix (arbitrarily) at stage ξ of the construction a function $f_\xi : \kappa \to \mathbb{P}$ such that $[f_\xi]_U = q_\xi$.

At successor steps we set $q_{\alpha+1} = q_\alpha$, and at stages of cofinality less than κ, we set $q_\alpha = \bigcup_{\beta<\alpha} q_\beta$ using the $j_U(\kappa)$-closure of $j_U(\mathbb{P})$ and the closure of M_U under κ-sequences.

Suppose that $cf(\alpha) = \kappa$. We say that α *is an active stage* if:

(1) Assumption 1: $\bigcup_{\beta<\alpha}\{f_\beta(\gamma) \mid \beta < \alpha, \gamma < \kappa\} \subseteq \mathbb{P}_\alpha$.
(2) Assumption 2: the diamond sequence A_α codes a sequence of dense subsets $\langle E_\nu \mid \nu < \kappa\rangle \subseteq \mathbb{P}_\alpha$.
(3) Assumption 3: $\mathbb{P}_\alpha$ is $<\kappa$-closed.

Assumption 1 ensures that for each $\beta < \alpha$, $q_\beta = j(f_\beta)([\mathrm{id}]_U) \in j_U(\mathbb{P}_\alpha)$. By Assumption 2, $E^* := [\nu \mapsto E_\nu]_U$ is dense in $j_U(\mathbb{P}_\alpha)$, and by Assumption 3, there is some $q_\alpha \in E^*$ such that $q_\alpha \leq q_\beta$ for every $\beta < \alpha$.

Is $cf(\alpha) = \kappa$ and α is not an active stage, take q_α be a lower bound for q_β which exists by $j_U(\kappa)$-closure of $j_U(\mathbb{P})$.

This concludes the definition of $\langle q_\alpha \mid \alpha < \kappa^+\rangle$. Now let $D \in M_U$, $D \subseteq j_U(\mathbb{P})$ be dense open. Suppose that $D = [\nu \to D_\nu]_U$, where each $D_\alpha \subseteq \mathbb{P}$ is dense open. For each ν there is a club $C_\nu \subseteq \kappa^+$ such that for every $\alpha \in C_\nu$, $D_\nu \cap \mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$ (Take C_ν to be the club of closure points of the function mapping ζ to an index ζ' such that $p_\zeta \geq p_{\zeta'} \in D_\alpha$). Let $C^* = \bigcap_{\nu<\kappa} C_\nu$, then any $\alpha \in C_\alpha$ will have all the $D_\nu \cap \mathbb{P}_\alpha$ dense in $\mathbb{P}_\alpha$. It is easy to see that Assumption 1 also occurs on a club, and that Assumption 3 occurs on a club relative to $S^{\kappa^+}_\kappa$. Since $\langle A_\alpha \mid \alpha \in S^{\kappa^+}_\kappa\rangle$ is a diamond sequence, there is some $\alpha \in S^{\kappa^+}_\kappa$ in this club such that $(D_\nu \cap \mathbb{P}_\alpha)_{\nu<\kappa}$ is coded by A_α.Therefore α is

an active stage, so by our construction, $q_\alpha \in [\nu \mapsto (D_\nu \cap \mathbb{P}_\alpha)]_U = D \cap j_U(\mathbb{P}_\alpha)$, as desired. □

Let $\mathcal{D}_p(\mathbb{P})$ denote the filter of dense open sets of of $\mathbb{P}$ above p.

Theorem 3.2. *Suppose that $\diamondsuit(S^{\kappa^+}_\kappa)$ holds and let $\mathcal{U}$ be a κ-complete normal fine ultrafilter on $P_\kappa(\kappa^+)$. Then for every κ-closed forcing $\mathbb{P}$ of cardinality κ^+, $\bigcap_{D\in\mathcal{D}_p(\mathbb{P})} j_{\mathcal{U}}(D) \neq \emptyset$.*

Proof. Let U be the projection of $\mathcal{U}$ to a normal κ-complete ultrafilter over κ. Let $i : V \to M_U$, $j : V \to M_{\mathcal{U}}$ and $k : M_U \to M_{\mathcal{U}}$ be the usual commutative diagram. Let $G \in V$ be M_U-generic for $i(\mathbb{P})$ and note that $|G| = |i(\mathbb{P}))| = \kappa^+$ by the diamond assumption. By closure, we can find a descending sequence $\langle q_\alpha : \alpha < \kappa^+\rangle \subseteq G$ that generates G. As $M_{\mathcal{U}}$ is closed under κ^+-sequences of its elements, it follows that $\langle k(q_\alpha) : \alpha < \kappa^+\rangle \in M_{\mathcal{U}}$. Since $j(\mathbb{P})$ is $j(\kappa)$-closed, there is a lower bound $q \in j(\mathbb{P})$ for $\langle k(q_\alpha) : \alpha < \kappa^+\rangle$. Let us show that q is as desired. Let $D \in \mathcal{D}_p(\mathbb{P})$. Then $i(D) \in M_U$ is dense in $i(\mathbb{P})$, and by M_U-genericity of G, there is $p \in i(D) \cap G$. Hence $k(p) \in j(D)$, and since D is open and $q \leq k(p)$, we conclude that $q \in j(D)$, as desired. □

Corollary 3.3. *Assume $\diamondsuit(S^{\kappa^+}_\kappa)$, then every κ-closed forcing $\mathbb{P}$ of cardinality κ^+ is a projection of the κ^+-supercompact Prikry forcing.*

Proof. We have several proofs for that. □

Suppose that $2^\kappa > \kappa^+$. Then $Add(\kappa^+, 1)$ collapses 2^κ to κ^+. However, even without GCH, the κ^+-supercompact Prikry forcing is κ^{++}-cc (since $(\kappa^+)^{<\kappa} = \kappa^+$) and therefore does not collapse κ^{++}. Hence if $2^\kappa > \kappa^+$, $Add(\kappa^+, 1)$ cannot be a projection of κ^+-supercompact Prikry forcing.

More generally, if λ is regular and $2^{<\lambda} > \lambda$, then $Add(\lambda, 1)$ collapses $2^{<\lambda}$ to λ, so we should at least assume $2^{<\lambda} = \lambda$.

3.1. $<\kappa$-strategically closed forcings. Generalization of Woodin

Proposition 3.4. *Suppose κ is λ-supercompact and $cf(\lambda) \geq \kappa$. Assume $\diamondsuit(S^{\lambda^+}_\kappa)$ holds, and let U be a λ-supercompact measure on $P_\kappa(\lambda)$. Let $\mathbb{P}$ be any κ-closed forcing of cardinality λ^+. Then there is an M_U-generic filter for $j_U(\mathbb{P})$.*

Proof. Let $\langle A_\alpha \mid \alpha \in S^{\lambda^+}_\kappa\rangle$ be a diamond sequence. Enumerate $\mathbb{P} = \{p_\beta \mid \beta < \lambda^+\}$ and let $\mathbb{P}_\alpha = \{p_\beta \mid \beta < \alpha\}$. We construct a descending sequence $\langle q_\xi \mid \xi < \lambda^+\rangle \subseteq j_U(\mathbb{P})$ such that for every dense set $D \subseteq j_U(\mathbb{P})$ in M_U, there is $\xi < \lambda^+$ such that $q_\xi \in D$. We will also fix (arbitrarily) at stage ξ of the construction a function $f_\xi : P_\kappa(\lambda) \to \mathbb{P}$ such that $[f_\xi]_U = q_\xi$.

At successor steps we set $q_{\alpha+1} = q_\alpha$, and at stages of cofinality which is not κ, we set $q_\alpha = \bigcup_{\beta<\alpha} q_\beta$ using the $j_U(\kappa)$-closure of $j_U(\mathbb{P})$ (indeed $j_U(\kappa) > \lambda$) and the closure of M_U under λ-sequences.

Suppose that $cf(\alpha) = \kappa$. We say that α *is an active stage* if:

(1) Assumption 1: $\bigcup_{\beta<\alpha}\{f_\beta(x) \mid \beta < \alpha, x \in P_\kappa(\lambda)\} \subseteq \mathbb{P}_\alpha$.

(2) Assumption 2: the diamond sequence A_α codes a sequence of dense subsets $\langle E_x \mid x \in P_\kappa(\lambda)\rangle \subseteq \mathbb{P}_\alpha$.
(3) Assumption 3: $\mathbb{P}_\alpha$ is $<\kappa$-closed.

Assumption 1 ensures that for each $\beta < \alpha$, $q_\beta = j(f_\beta)([\mathrm{id}]_U) \in j_U(\mathbb{P}_\alpha)$. By Assumption 2, $E^* := [x \mapsto E_x]_U$ is dense in $j_U(\mathbb{P}_\alpha)$, and by Assumption 3, $j_U(\mathbb{P}_\alpha)$ is $j_U(\kappa)$-closed, and so there is some $q_\alpha \in E^*$ such that $q_\alpha \leq q_\beta$ for every $\beta < \alpha$.

If $cf(\alpha) = \kappa$ and α is not an active stage, take q_α be a lower bound for q_β which exists by $j_U(\kappa)$-closure of $j_U(\mathbb{P})$.

This concludes the definition of $\langle q_\alpha \mid \alpha < \lambda^+\rangle$. Now let $D \in M_U$, $D \subseteq j_U(\mathbb{P})$ be dense open. Suppose that $D = [x \to D_x]_U$, where each $D_x \subseteq \mathbb{P}$ is dense open. For each x there is a club $C_x \subseteq \lambda^+$ such that for every $\alpha \in C_x$, $D_x \cap \mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$ (Take C_x to be the club of closure points of the function mapping ζ to an index ζ' such that $p_\zeta \geq p_{\zeta'} \in D_x$). Let $C^* = \bigcap_{x \in P_\kappa(\lambda)} C_\nu$. By Solovay, $\lambda^{<\kappa} = \lambda$, and therefore C^* is a club in λ^+. Then any $\alpha \in C^*$ will have all the $D_x \cap \mathbb{P}_\alpha$ dense in $\mathbb{P}_\alpha$. It is easy to see that Assumption 1 also occurs on a club, and that Assumption 3 occurs on a club relative to $S^{\lambda^+}_\kappa$. Since $\langle A_\alpha \mid \alpha \in S^{\lambda^+}_\kappa\rangle$ is a diamond sequence, there is some $\alpha \in S^{\lambda^+}_\kappa$ in this club such that $(D_x \cap \mathbb{P}_\alpha)_{x \in P_\kappa(\lambda)}$ is coded by A_α. Therefore α is an active stage, so by our construction, $q_\alpha \in [x \mapsto (D_x \cap \mathbb{P}_\alpha)]_U = D \cap j_U(\mathbb{P}_\alpha)$, as desired. □

Let $\mathcal{D}_p(\mathbb{P})$ denote the filter of dense open sets of of $\mathbb{P}$ above p.

Theorem 3.5. *Suppose that $cf(\lambda) \geq \kappa$ and let $\mathcal{U}$ be a κ-complete normal fine ultrafilter on $P_\kappa(\lambda^+)$. Then for every $<\kappa$-strategically closed forcing $\mathbb{P}$ of cardinality λ^+, $\bigcap_{D \in \mathcal{D}_p(\mathbb{P})} j_{\mathcal{U}}(D) \neq \emptyset$.*

By Shelah, $\diamondsuit(S^{\lambda^+}_\kappa)$ holds.

(1) Let $\mathbb{P}$ be a $< \kappa$-strategically closed of size λ^+, let $\vec{\sigma} = \langle \sigma_\alpha \mid \alpha < \kappa\rangle$ be a sequence of winning strategies and $\diamondsuit(S^{\lambda^+}_\kappa)$. Then for any normal ultrafilter U on $P_\kappa(\lambda)$ there is a decreasing sequence $(q_\alpha \mid \alpha < \lambda^+)$ played according to $j_U(\vec{\sigma})_{\lambda^+ + 1}$ (i.e. all the restriction are according to the strategy) which generates an M_U-generic for $j_U(\mathbb{P})$.
(2) If $\mathbb{P}$ is κ-strategically closed of size λ^+, and $\mathcal{U}$ is $P_\kappa(\lambda^+)$-supercompact measure, then $\bigcap j''_U \mathcal{D}(\mathbb{P}) \neq \emptyset$.

Proof. Proof of 1. Let $\langle A_\alpha \mid \alpha \in S^{\lambda^+}_\kappa\rangle$ be a diamond sequence and let . Fix an internally approachable sequence of elementary submodels $(N_i \mid i < \lambda^+)$ accumulating to λ^+, such that for each limit point i, $N_i \cap \lambda^+ = i$ and there is a κ-strategy σ with $\vec{\sigma}, \mathbb{P}, P_\kappa(\lambda), U \in N_0$. For every i, $N_i^{<\lambda} \subseteq N_{i+1}$. Suppose that $\mathbb{P}_\alpha = \mathbb{P} \cap N_\alpha$. We construct a descending sequence $\langle q_\xi \mid \xi < \lambda^+\rangle \subseteq j_U(\mathbb{P})$ using the strategy $\sigma^* := j_U(\vec{\sigma})_{\lambda^+ + 1}$ (we note that λ^+ is computed correctly in M_U and that $j_U(\kappa) > \lambda^+$ so that the strategy σ^* exists), such that for every dense set $D \subseteq j_U(\mathbb{P})$ in M_U, there is $\xi < \lambda^+$ such that $q_\xi \in D$. We will

also fix (arbitrarily) at stage ξ of the construction a function $f_\xi : P_\kappa(\lambda) \to \mathbb{P}$ such that $[f_\xi]_U = q_\xi$.

Let $q_0 = \sigma^*(\langle\rangle)$. At successor steps we set $q_{\alpha+1} = \sigma^*((q_\beta)_{\beta<\alpha})$, and at stages of cofinality which is not κ, we set $q_\alpha \leq q_\beta$ for all $\beta < \alpha < \lambda^+$ which is given by the strategy and the closure of M_U under λ-sequences. Note that is $(q_\beta \mid \beta < \alpha) \subseteq j_{\mathcal{U}}(N_\alpha)$, then $q_\alpha \in j_U(N_{\alpha+1})$ (since $j_U(N_{\alpha+1})$ is closed under λ-sequences from $j_U(N_\alpha)$ and $\sigma \in N_{\alpha+1}$). By Łos, we can choose $f_\alpha : P_\kappa(\lambda) \to \mathbb{P} \cap N_{\alpha+1}$ such that $[f_\alpha]_U \in N_{\alpha+1}$.

Suppose that $cf(\alpha) = \kappa$, we say that α is an *active* stage if:

(∗) the diamond sequence A_α codes a sequence of dense subsets $\langle E_x \mid x \in P_\kappa(\lambda)\rangle \subseteq \mathbb{P}_\alpha$.

By construction, for every $\beta < \alpha$, we chose $q_\beta = j_U(f_\beta)([\mathrm{id}]_U) \in j_U(\mathbb{P}_\alpha)$. Hence $(q_\beta \mid \beta < \alpha) \in M_U$ is a $< j_U(\kappa)$-sequence of elements in $j_U(N_\alpha)$. Since $cf(\alpha) = \kappa$, N_α is closed under $< \kappa$-sequences and therefore $j_U(N_\alpha)$ is closed under $< j_U(\kappa)$-sequences, and in particular $(q_\beta \mid \beta < \alpha) \in N_\alpha$. Also note that since $\kappa \subseteq N_\alpha$, $\lambda^+ + 1 \subseteq j_U(N_\alpha)$ and therefore $\sigma^* \in j_U(N_\alpha)$. Hence we can play the strategy σ^* to find a lower bound $q'_\alpha \leq q_\beta$ for all $\beta < \alpha$ inside $j_U(N_\alpha)$. Letting $E^* := [x \mapsto E_x]_U$, we use (∗) to see that E^* is dense in $j_U(\mathbb{P}_\alpha)$, thus there is some $p_\alpha \leq q'_\alpha$, $p_\alpha \in E^*$, and so the strategy gives q_α which is played against the bad player's p_α move.

If $cf(\alpha) = \kappa$ and α is not an active stage, take q_α be a lower bound for q_β given by the strategy.

This concludes the definition of $\langle q_\alpha \mid \alpha < \lambda^+\rangle$. Now let $D \in M_U$, $D \subseteq j_U(\mathbb{P})$ be dense open. Suppose that $D = [x \to D_x]_U$, where each $D_x \subseteq \mathbb{P}$ is dense open. For each x there is a club $C_x \subseteq \lambda^+$ such that for every $\alpha \in C_x$, $D_x \cap \mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$ (Take C_x to be the club of closure points of the function mapping ζ to an index ζ' such that $p_\zeta \geq p_{\zeta'} \in D_x$). Let $C^* = \bigcap_{x \in P_\kappa(\lambda)} C_x$. By Solovay, $\lambda^{<\kappa} = \lambda$, and therefore C^* is a club in λ^+. Then any $\alpha \in C^*$ will have all the $D_x \cap \mathbb{P}_\alpha$ dense in $\mathbb{P}_\alpha$. It is easy to see that Assumption 1 also occurs on a club, and that Assumption 3 occurs on a club relative to $S^{\lambda^+}_\kappa$. Since $\langle A_\alpha \mid \alpha \in S^{\lambda^+}_\kappa\rangle$ is a diamond sequence, there is some $\alpha \in S^{\lambda^+}_\kappa$ in this club such that $(D_x \cap \mathbb{P}_\alpha)_{x \in P_\kappa(\lambda)}$ is coded by A_α. Therefore α is an active stage, so by our construction, $q_\alpha \in [x \mapsto (D_x \cap \mathbb{P}_\alpha)]_U = D \cap j_U(\mathbb{P}_\alpha)$, as desired. □

Proof of 2. Let k be the factor map as before and note that $k[\langle q_\alpha \mid \alpha < \lambda^+\rangle] \in M_{\mathcal{U}}$, is played according to $k(\sigma^*) = k(j_U(\vec{\sigma})_{\lambda^++1})$. Indeed, the critical point of k is $(\lambda^{++})^{M_U}$, we have that $k(j_U(\vec{\sigma})_{\lambda^++1}) = j_{\mathcal{U}}(\vec{\sigma})_{\lambda^++1}$. Also, for every $\xi < \lambda^+$ $k(\langle q_\alpha \mid \alpha < \xi\rangle) = \langle k(q_\alpha) \mid \alpha < \xi\rangle$ and $k(q_\xi) = k(j_U(\sigma)(\langle q_\alpha \mid \alpha < \xi\rangle) = j_{\mathcal{U}}(\sigma)(\langle k(q_\alpha) \mid \alpha < \xi\rangle)$. Since $j_{\mathcal{U}}(\vec{\sigma})_{\lambda^++1}$ is a winning strategy for longer game, we can find $q^* \leq k(q_\alpha)$ for every $\alpha < \lambda^+$. □

3.2. **New condition.** Suppose $\mathbb{P}$ is a poset, p is a condition in $\mathbb{P}$, and κ is a cardinal. Given a $\subseteq$-increasing sequence $\vec{N} = \langle N_i \mid i < \gamma \rangle$ of elementary submodels of H_θ for some large enough θ such that:

(1) $\gamma < \kappa$;
(2) $(\bigcup_{j<i} N_j)^{<\kappa} \subseteq N_i$;
(3) $\lambda, p, \mathbb{P} \in N_0$.

We define $G(\vec{N}, \mathbb{P})$ to be the game between two players playing conditions $p_i, q_i \in \mathbb{P} \cap N_i$ respectively. The game is of length γ such that at stages $i < \gamma$, Good plays first, and chooses $p_i \in N_i$ bounding $\langle p_j, q_j \mid j < i \rangle$ and then Bad plays $q_i \leq p_i$ also in N_i. If no such p_i exists, the run terminates (and Bad wins the run). Good wins the run iff they survive all stages $i < \gamma$ and there is a lower bound to the sequence $\langle p_i \mid i < \gamma \rangle$.

A sequence satisfying $(1) - (3)$ is called a $(p, \mathbb{P})$-*playful* sequence of elementary submodels of H_θ.

Definition 3.6. Let $\mathbb{P}$ be a forcing notion of cardinality λ. We say that $\mathbb{P}$ is $<\kappa$-filtration-distributive if for all sufficiently large θ, and for all $(p, \mathbb{P})$-playful sequences of elementary submodels of H_θ, Bad does not have a winning strategy in $G(\vec{N}, \mathbb{P}/p)$.

Refer to similar definitions of Shelah

Theorem 3.7. *Suppose $\kappa < \lambda$ is either inaccessible or a successor of a strong limit singular of cofinality $\geq \kappa$. Suppose that $\mathbb{P}$ is $<\kappa$-filtration distributive, $|\mathbb{P}| = \lambda$, and that $\diamondsuit(S^\lambda_\kappa)$ holds. Let $\mathcal{U}$ be normal measure on $P_\kappa(\lambda)$. Then there is a condition $q \in j_{\mathcal{U}}(\mathbb{P})$ such that $q \in \bigcap j_{\mathcal{U}}\text{“}\mathcal{D}(\mathbb{P})$.*

Proof. Let $\langle A_\alpha \mid \alpha \in S^{\lambda^+}_\kappa \rangle$ be a diamond sequence, and let us assume that $\mathbb{P} = \lambda$. Then $\langle A_\alpha \mid \alpha \in S^\lambda_\kappa \rangle \in M_{\mathcal{U}}$. In V, fix a sequence of elementary submodels $\vec{N} = \langle N_i \mid i < \lambda \rangle$ be a sequence of lementary submodels of H_λ such that:

(1) $\vec{N}$ is a $\subseteq$-increasing and continuous at points of cofinality$\geq \kappa$.
(2) for each $i < \lambda$, $|N_i| = \lambda$.
(3) $(\bigcup_{j<i} N_j)^{<\kappa} \subseteq N_i$.
(4) $\lambda, \mathbb{P} \in N_0$ and $\kappa + 1 \subseteq N_0$.
(5) $\bigcup_{i<\lambda} N_i = H_\lambda$.

Such a sequence exists since $\lambda^{<\kappa} = \lambda$ and the cardinal arithmetic assumptions. Note that condition $(1), (3)$ are compatible.

Let $\mathbb{P}_\alpha = \mathbb{P} \cap N_\alpha = \lambda \cap N_\alpha$. Work in $M_{\mathcal{U}}$, then by elementarity $j_{\mathcal{U}}(\vec{N})$ has the same properties, and we may consider the subsequence $j''_{\mathcal{U}}\vec{N} = j_{\mathcal{U}}(\vec{N}) \restriction j''_{\mathcal{U}}\lambda$. Note that this a $(1_{\mathbb{P}}, \mathbb{P})$-playful sequence $\subseteq$-increasing (non-continuous) sequence in $M_{\mathcal{U}}$ of length $\lambda < j_{\mathcal{U}}(\kappa)$ such that $j_{\mathcal{U}}(\mathbb{P}) \in j_{\mathcal{U}}(N_0)$, and $(\bigcup_{j<i} j_{\mathcal{U}}(N_j))^{<j_{\mathcal{U}}(\kappa)} \cap M_{\mathcal{U}} \subseteq j_{\mathcal{U}}(N_i)$. Hence Bad has no winning strategy in $M_{\mathcal{U}}$ for the game $G(j''_{\mathcal{U}}\vec{N}, j_{\mathcal{U}}(\mathbb{P}))^{M_{\mathcal{U}}}$.

Let us define a strategy for Bad in $G(j_{\mathcal{U}}\text{“}\vec{N}, j_{\mathcal{U}}(\mathbb{P}))^{M_{\mathcal{U}}}$ as follows. Suppose that as stage $\alpha < \lambda$, we have recursively obtained a run $\langle p_i, q_i \mid i < \alpha\rangle$ which is played according to the strategy, and Good plays $p_\alpha \in j_{\mathcal{U}}(N_\alpha)$.

We say that α is an *active* stage if α is a limit ordinal of cofinality κ and the following holds:

(∗) the diamond sequence A_α codes (i.e. equals) a dense open subset $E \subseteq \mathbb{P}_\alpha$.

By elementarity, $E^* := j_{\mathcal{U}}(E)$ is dense in $j_{\mathcal{U}}(\mathbb{P}_\alpha)$, and so Bad can play some $q_\alpha \leq p_\alpha$ in E^*. Note that this is a legitimate move inside the model $j_{\mathcal{U}}(N_\alpha)$.

If α is not an active stage, and Good played $p_\alpha \in j_{\mathcal{U}}(\mathbb{P}_\alpha)$, then Bad plays $q_\alpha = p_\alpha$.

This concludes the definition of the strategy for Bad. Fix $\langle q_\alpha, p_\beta \mid \alpha < \lambda, \beta \leq \lambda\rangle$ witnessing that the strategy defined above is not winning for Bad in $G(j_{\mathcal{U}}\text{“}\vec{N}, j_{\mathcal{U}}(\mathbb{P}))^{M_{\mathcal{U}}}$.

Claim 3.8. $p_\lambda \in \bigcap j''_{\mathcal{U}}\mathcal{D}(\mathbb{Q})$.

Proof. Let $D \subseteq \mathbb{P}$ be dense open. Let C_0 be a club relative to cofinality $\geq \kappa$ such that for every $\alpha \in C_0$, $\mathbb{P}_\alpha = N_\alpha \cap \lambda \in \lambda$ (here we use the continuity of the sequence at points of cofinality $\geq \kappa$). There is a club C such that for every $\alpha \in C$, $D \cap \mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$ (Take C to be the club of closure points of the function mapping $\zeta < \lambda$ to an index $\zeta' \in D$ such that $\zeta \geq_{\mathbb{P}} \zeta'$).

Since $\langle A_\alpha \mid \alpha \in S^\lambda_\kappa\rangle$ is a diamond sequence, there is some $\alpha \in S^\lambda_\kappa \cap C \cap C_0$ such that $D \cap \mathbb{P}_\alpha = A_\alpha$. Therefore α is an active stage, so by our construction, $q_\alpha \in j_{\mathcal{U}}(D \cap \mathbb{P}_\alpha)$. Since $p_\lambda \leq q_\alpha$, $p_\lambda \in j_{\mathcal{U}}(D)$. □

□

Proposition 3.9. *Let κ be supercompac. Suppose that $\mathbb{P}$ is a forcing of cardinality λ which is either inaccessible or a successor of a strong limit singular of cofinality $\geq \kappa$. The $\mathbb{P}$ is $< \kappa$-strategically closed implies $\mathbb{P}$ is $< \kappa$-filtration distributive implies $\mathbb{P}$ is κ-distributive.*

Proof. Suppose that $\mathbb{P}$ is $< \kappa$-strategically closed, and let us prove that it is $< \kappa$-filtration distributive. Let $p \in \mathbb{P}$ and let $\vec{N} = \langle N_i \mid i < \gamma\rangle$ a $(p, \mathbb{P})$-playful sequence. By elementarity, and since $H_\theta \models \mathbb{P}$ is $<\kappa$-strategically closed, there is $\vec{\sigma} = (\sigma_i \mid i < \kappa) \in N_0$ such that each σ_i is a winning strategy for Good in games of length i.

Suppose that τ is a strategy for $G(\vec{N}, \mathbb{P}/p)$. Let us show that τ is not winning for Bad. The point is to play τ against $\sigma_{\gamma+1}$, and produce a winning run for Good. Formally, we start with $p_0 = \sigma_{\gamma+1}(\langle\rangle)$. Note that $p_0 \in N_0$ (since $\sigma_{\gamma+1} \in N_0$), and let $q_0 = \tau(\langle p_0\rangle) \in N_0$ (by the rules of the game). Suppose that $\langle p_i, q_i \mid i < \beta\rangle$ was defined, such that $p_i = \sigma_{\gamma+1}(\langle p_j, q_j \mid j < i\rangle)$ and $q_j = \tau(\langle p_j, q_j \mid j < i\rangle^\frown p_i)$. Note that $\langle p_i, q_i \mid i < \beta\rangle \subseteq \bigcup_{i<\beta} N_{\alpha_i}$ and therefore by condition (3), $\langle p_i, q_i \mid i < \beta\rangle \in N_\beta$. Since $\sigma_{\gamma+1} \in N_\beta$,$\sigma_{\gamma+1}\langle p_i, q_i \mid i < \beta\rangle) = p_\beta \in N_\beta$ and also $q_\beta = \tau(\langle p_i, q_i \mid i < \beta\rangle^\frown p_\beta) \in N_\beta$.

Since $\sigma_{\gamma+1}$ is a winning strategy for Good, there is $p_{\gamma+1}$ bounding this sequence at the end, showing that τ is not winning for Bad.

Next, suppose that $\mathbb{P}$ is $< \kappa$-filtration distributive and let us show it is κ-distributive. witnessing the $< \kappa$-filtration distributivity of $\mathbb{P}$. Let $\vec{D} = (D_\alpha \mid \alpha < \gamma)$ be a collection of $\gamma < \kappa$ many dense open subsets of $\mathbb{P}$. We would like to prove that $\bigcap_{i<\gamma} D_i$ is dense. Towards this, let $p \in \mathbb{P}$. Let $\vec{N}$ be a sequence as in Definition 3.6 such that $\vec{D}, p \in N_0$. Note that by elementarity, for each $\alpha < \gamma$, $D_\alpha \cap N_\alpha$ is dense in $\mathbb{P} \cap N_\alpha$. Let us define a strategy for Bad in $G(\vec{N}, \mathbb{P}/p)$ (which must then be non-winning).

The strategy is defined such that at stage $\alpha < \gamma$, if $(p_i, q_i \mid i < \alpha)$ is a partial run and $p_\alpha \in N_\alpha$ was the response of Good, then Bad chooses $q_\alpha \leq p_\alpha$ inside $D_\alpha \cap N_\alpha$. This is possible, since $p_\alpha \in N_\alpha \cap \mathbb{P}$ and $D_\alpha \cap N_\alpha$ is dense in $\mathbb{P} \cap N_\alpha$. By the assumption there is a winning run according to this strategy for Good which is below p. This produces a lower bound $p^* \in \bigcap_{\alpha<\gamma} D_\alpha$ such that $p^* \leq p$.

□

4. Successors of singulars

Lemma 4.1. *Suppose λ is a limit cardinal, $\mathbb{B}$ is a λ^+-distributive complete Boolean algebra, and U is an ultrafilter on $\mathbb{B}$ that is γ-indecomposable for all $\gamma \geq \lambda$. Then there is a maximal antichain $A \subseteq \mathbb{B}$ such that for any refinement B of A, there is some $u \in U$ such that $B \mid u = A \mid u$.*

Proof. Assume not. We construct a sequence $\langle A_\alpha : \alpha \leq \lambda \rangle$ of maximal antichains of $\mathbb{B}$ by choosing $A_{\alpha+1}$ so that there is no $u \in U$ with $A_{\alpha+1} \mid u = A_\alpha \mid u$, and taking least common refinements at limit stages.

By indecomposability, take $u \in U$ such that $|A_\lambda \mid u| < \lambda$. Let $\gamma = |A_\lambda \mid u|$, and note that $|A_\alpha \mid u| \leq \gamma$ for all α.

If $a \in A_{\gamma^+} \mid u$ and $\alpha < \gamma^+$, denote by a_α the unique $\bar{a} \in A_\alpha \mid u$ such that $a \leq \bar{a}$. For distinct $a, b \in A_{\gamma^+} \mid u$, let $\alpha_{a,b} < \gamma$ be the least α such that $a_\alpha \neq b_\alpha$. Note that $\alpha_{a,b}$ exists since $A_{\gamma^+} \mid u$ is the least common refinement of $\langle A_\alpha \mid u : \alpha < \gamma^+ \rangle$. Let $\beta < \gamma^+$ bound the γ-sized set of ordinals $\alpha_{a,b}$. Then $A_\beta \mid u = A_{\gamma^+} \mid u$, which is a contradiction. □

Lemma 4.2. *Suppose λ is a singular limit of λ-supercompact cardinals. If $\mathbb{B}$ is a λ^+-distributive λ^{++}-cc complete Boolean algebra, there are no $\mathrm{cf}(\lambda)^+$-complete λ^+-indecomposable ultrafilters on $\mathbb{B}$.*

Proof. Assume U is a counterexample. Fix an antichain $A \subseteq \mathbb{B}$ as in Lemma 4.1. By restricting to a measure 1 set, we can assume $|A| < \lambda$. Let $\eta < \lambda$ be a λ-supercompact cardinal with $|A| < \eta$. Let W be the projection of U onto $P(A)$, so W is an ultrafilter over A. Note that $\mathrm{Ult}(V, U) = \mathrm{Ult}(V, W)$. By Woodin's approximation lemma, $\mathrm{Ult}(V, W)$ has the η-approximation property in V, so there can be no $j_W(\mathbb{B})$-generic set in V, contradiction. □

Theorem 4.3. *Suppose λ is a singular limit of λ-supercompact cardinals and $\mathcal{U}$ is a normal fine κ-complete ultrafilter on $P_\kappa(\lambda^+)$ where $\kappa > \mathrm{cf}(\lambda)$. Then Prikry forcing with $\mathcal{U}$ does not add a Cohen subset of λ^+.*

Proof. Assume it does. Let $j_{0\omega} : V \to M$ be the ω-th iterate of $\mathcal{U}$. Let $\mathbb{B}$ be the Boolean completion of $\mathrm{Add}(\lambda^+, 1)$, and let G be an M-generic ultrafilter on $j_{0\omega}(\mathbb{B})$. Let N be the transitive collapse of $\mathrm{Hull}^{M[G]}(j_{0\omega}[V] \cup \{G\})$, and let $k : N \to M[G]$ be the inverse of the transitive collapse. Let $\bar{M} = k^{-1}[M]$, $i : V \to \bar{M}$ be $k^{-1} \circ j_{0\omega}$, and $\bar{G} = k^{-1}(G)$, so $N = \bar{M}[\bar{G}]$.

Since $i(\lambda^+)$ is regular in N, and N is closed under κ-sequences, $i(\lambda^+)$ has uncountable cofinality. In particular, $\nu = \sup k[i(\lambda^+)]$ is strictly below $j_{0\omega}(\lambda^+)$. Again since $i(\lambda^+)$ is regular in N, we have

$$\nu = j_{0\omega}(\lambda^+) \cap \mathrm{Hull}^{M[G]}(k[N] \cup \nu)$$

Let $\langle \xi_n : n < \omega \rangle$ be the Prikry sequence over M, viewed as a sequence of ordinals below $j_{0\omega}(\lambda^+)$. Thus $\xi_n = \sup j_{n\omega}[j_{0n}(\lambda^+)]$. Let n be least such that $\xi_n \geq \nu$. We have

$$\xi_n \subseteq j_{0\omega}(\lambda^+) \cap \mathrm{Hull}^{M[G]}(j_{n\omega}[M_n] \cup \{G\}) \subseteq j_{0\omega}(\lambda^+) \cap \mathrm{Hull}^{M[G]}(k[N] \cup \nu) = \nu$$

It follows that $\xi_n \leq \nu$, so $\xi_n = \nu$.

Let $U = j_{n\omega}^{-1}[G]$. Then U is a κ_n-complete ultrafilter on $\mathbb{B}^* = j_{0n}(\mathbb{B})$. Applying Lemma 4.2 in M_n, we will obtain a contradiction by verifying that U is $j_{0n}(\lambda^+)$-indecomposable in M_n. For this, suppose $\langle a_\xi : \xi < j_{0n}(\lambda^+) \rangle \in M_n$ enumerates a maximal antichain of $\mathbb{B}^*$. We will show there is some $\beta < j_{0n}(\lambda^+)$ such that $\bigvee_{\xi<\beta} a_\xi \in U$. Let $\langle a^*_\xi : \xi < j_{0\omega}(\lambda^+) \rangle = j_{n\omega}(\langle a_\xi : \xi < j_{0n}(\lambda^+) \rangle)$. Then the unique $\zeta < j_{0\omega}(\lambda^+)$ such that $a^*_\zeta \in G$ belongs to $\mathrm{Hull}^{M[G]}(j_{n\omega}[M_n] \cup \{G\})$. It follows that $\zeta < \xi_n = \sup j_{n\omega}[j_0 n(\lambda^+)]$. So fix $\beta < j_{0n}(\lambda^+)$ such that $\zeta < j_{n\omega}(\beta)$. We then have $\bigvee_{\xi<\beta} a_\xi \in U$ since $j_{n\omega}(\bigvee_{\xi<\beta} a_\xi) \in G$. □

Next let us address the same problem but at a successor of a singular of small cofinality which is not a limit of supercompacts. For concretness, let us consider the first such cardinal which is $\kappa^{+\omega+1}$.

Theorem 4.4. *From infinitely many supercompacts it is consistent to have a supercompact cardinal κ such that for every normal measure $\mathcal{U}$ over $P_\kappa(\kappa^{+\omega+1})$, the supercompact Prikry forcing $Pr(\mathcal{U})$ does not add a $\kappa^{+\omega+1}$-Cohen generic function.*

On the other hand, we have:

Theorem 4.5. *It is consistent from a supercompact cardinal that there is a normal measure $\mathcal{U}$ over $P_\kappa(\kappa^{+\omega+1})$ such that $Pr(\mathcal{U})$ adds a $\kappa^{+\omega+1}$-Cohen generic.*

5. Separating examples

5.1. **A consistency result.**

5.2. **A ZFC result.** Goal: A κ-distributive forcing of size κ^+ which is not a projection of the supercompact Prikry forcing.

We need a forcing which is not $<\kappa$-strategically closed, κ-distributive. The typical example is shooting a club through a fat stationary set of κ. But this is of small cardinality and therefore can be absorbed into the supercompact Prikry forcing at κ^+ (assuming $2^\kappa = \kappa^+$). So we need something which is not κ^+-cc.

So it is natural to think about club shooting at κ^+. Let us start with the following observesion:

Proposition 5.1. *Suppose that $\kappa^{<\kappa} = \kappa$ and $S \subseteq \kappa^+ \cap cof(<\kappa)$ is stationary and $\mathbb{P}$ is a forcing notion that destroys the stationarity of S, then $\mathbb{P}$ is not $<\kappa$-strategically closed.*

Proof. Suppose otherwise that $\mathbb{P}$ is $<\kappa$-strategically closed and let $\dot{C}$ is a $\mathbb{P}$-name for a club at κ^+ such that $p \Vdash_{\mathbb{P}} \dot{C} \cap S = \emptyset$.

By the stationarity of S, we can find an elementary submodel $M \prec H(\theta)$ of size $< \kappa^+$ such that $\rho_M := M \cap \kappa^+ \in S$ and $p, \dot{C}, \mathbb{P} \in M$. Since $\kappa^{<\kappa} = \kappa$ we can moreover assume that $M^{<\mathrm{cf}(\rho_M)} \subseteq M$. By our assumptions, ρ_M has cofinality $\theta < \kappa$ and there is a winning strategy for the good player in the game of length $\rho_M + 1$. Let $\langle \rho_i \mid i < \theta \rangle$ be cofinal in ρ_M. Note that for every $j < \theta$, $\langle \rho_i \mid i < j \rangle \in M$, by the closure assumption on M. For each $\alpha < \rho_M$, the set D_α of all $q \leq p$ forcing some ordinal $\delta > \alpha$ to be in $\dot{C}$ is dense below p and definable in M. so we can now use the strategy to construct a decreasing sequence below p of conditions $p_i \in D_{\rho_i} \cap M$ with a lower bound q. Then $q \Vdash \rho_M \in \dot{C} \cap S$, contradiction. □

Now the question is when does a club shooting forcing at κ^+ is κ-distributive?

The Abraham Shelah characterization suggest we need some fatness. And here is a natural requirement: A stationary set $S \subseteq \kappa^+$ is κ-fat if for any club $C \subseteq \kappa^+$, and every $\tau < \kappa$, $S \cap C$ contains a closed set of ordertype $\tau + 1$. The following is a small variant of Abraham and Shelah:

Proposition 5.2. *Suppose that $\kappa^{<\kappa} = \kappa$ and $S \subseteq \kappa^+$ is κ-fat. Then the forcing shooting a club through S is κ-distributive.*

Proof. Let $\langle D_i \mid i < \rho \rangle$ for $\rho < \kappa$. As in Abraham and Shelah we let $(M_i \mid i < \kappa^+)$ be a sequence of elementary models which is continuous and sees the relevant objects, such that M_{i+1} contains $< \kappa$-sequences from M_i, and $M_i \cap \kappa^+$ is an ordinal. Let C be the club $\{M_i \cap \kappa^+ \mid i < \kappa^+\}$ and find a closed set $A \subseteq S \cap C$ of ordertype $\rho + 1$. Then we can meet the dense sets D_i with a decreasing p_i so that $(\sup(p_i), \sup(p_{i+1})) \cap A \neq \emptyset$. So that at the end we can take the union of the p_i and add the supremum which is guaranteed to be in A (and so in S). □

Next we would like to construct a κ-fat stationary set $S \subseteq cof(<\kappa) \cap \kappa^+$ such that $cof(<\kappa) \setminus S$ is stationary in κ^+. Then, by the above propositions, the forcing which shoots a club through S is κ-distributive but not

$<\kappa$-strategically closed. It is easily seen to be not κ^+-cc since it kills the stationary of $cof(\kappa) \cap \kappa^+$ and therefore collapses κ^+.

The "easy" way to obtain this is to add a generic Cohen to κ^+ to a model where κ is indestructibly supercompact. Let $A_0 \subseteq \kappa^+$ be the Cohen set. Consider $S = \{\alpha \in A_0 \mid cf(\alpha) < \kappa\}$, we claim it is as wanted: first, if $C \subseteq \kappa^+$ is a club, then there is a Cohen name $\dot{C}$ for it. Fixt any $\tau < \kappa$, then there is a dense set of conditions in the Cohen forcing which forces that there is a closed set of ordertype $\tau + 1$ which is included in $\dot{C} \cap A_0$. Given any condition, we can construct an increasing sequence of conditions of length $\omega \cdot \tau$, such that at successor steps $p_{i+1} \Vdash (\sup(\mathrm{dom}(p_i)), \sup(\mathrm{dom}(p_{i+1}))) \cap \dot{C} \neq \emptyset$. Then at every limit step of the form $\omega \cdot \delta$ for $\delta \leq \tau$, we get to decide to define $p_{\omega\cdot\delta}(\sup_i(\mathrm{dom}(p_i)) = 1$ and also $p_{\omega\cdot\delta} \supseteq \bigcup_{i<\omega\cdot\delta} p_i$. Let $p^* = p_{\omega\cdot\tau}$. Then p^* forces that $\{\max(\mathrm{dom}(p_{\omega\cdot\delta})) \mid \delta \leq \tau\} \subseteq \dot{C} \cap \dot{S}$. By the same argument we see that we get to put stationarily many points of cofinality $< \kappa$ outside A_0. so that $cof(< \kappa) \setminus S$ is stationary in κ^+.

Remark 5.3. Note that since the forcing is κ-distributive, a generic club added by the above forcing is closed under $< \kappa$-sequences from the ground model.

Question 5.4. Maybe there is a ZFC-construction of such a κ-fat stationary set for supercompact κ?

The next proposition shows that the diamond construction from the previous section cannot be carries out for this forcing:

Proposition 5.5. *Let $S \subseteq \kappa^+$ be such that $cf(<\kappa) \setminus S$ is stationary in κ^+. Let $\mathcal{U}$ be a normal measure on $P_\kappa(\kappa^+)$. Let U be the normal ultrafilter derived from $j_{\mathcal{U}}$ and κ, $i = j_U$ be the ultrapower embedding, and $k : M_U \to M_{\mathcal{U}}$ be the factor map $k([f]_U) = j_{\mathcal{U}}(f)(\kappa)$. Then there is no $q^* \in j(Q(S))$ such that for every $D \subseteq Q(S)$ dense open there is $q_D \in i(D)$ such that $q^* \leq k(q_D)$.*

Proof. Otherwise, for each $\alpha < \kappa^+$, let D_α the dense set of all conditions with an element above α. Then for each α, there is $q_\alpha \in i(D_\alpha)$ such that $q^* \geq k(q_\alpha)$. It follows that $q^* \geq \bigcup_{\alpha<\kappa^+} k(q_\alpha) = C$. Then $C \subseteq \sup(j''\kappa^+) \cap j(S)$ is a $< j(\kappa)$-closed. It follows that $j''\kappa^+ \cap C$ is a $< \kappa$-closed and unbounded subset of $j''\kappa^+$. Letting $C^* \subseteq j^{-1}[C]$, we have that $C^* \subseteq S$ is a club relative to $cof(< \kappa) \cap \kappa^+$, contradicting the fact that $cof(< \kappa) \setminus S$ is stationary. □

The argument above shows more generally, that if $q \in \bigcap j''D(Q(S))$, then $\max(q) > \sup(j''\kappa^+)$ (and that $q \cap \sup(j''\kappa^+)$ has to be bounded)

Suppose that $q \in \bigcap j''D(Q(S))$ has minimal $\max(q)$ (which again has to be above $\sup(j''\kappa^+)$). Then for each $\eta < \max(q)$, there is a dense open $D \subseteq Q(S)$ such that $q \cap \eta \notin j(D)$.

Proposition 5.6. *Suppose that S is $<\kappa$-fat. For every $\eta < \max(q)$, $q \setminus \eta \in \bigcap j''D(Q(S))$.*

Proof. Otherwise, there are D_1, η_1 such that $q \setminus \eta_1 \notin j(D_1)$. By minimality, there is D_2 such that $q \cap \eta_1 \notin j(D_2)$. Define D^* as the set of all p such that for some $\eta < \max(p)$:

(I) $p \cap \eta \in D_2$.
(II) for every $\alpha \leq \eta$, $p \setminus \alpha \in D_1$.

This is easily seen to be open. To see that it is dense, let p be any condition, extend p to some $p_1 \in D_2$. We fix a sequence $(M_\alpha \mid \alpha < \kappa^+)$ and θ large enough so that:

(1) for each $\alpha < \kappa^+$, $M_\alpha \prec H(\theta)$.
(2) $|M_\alpha| = \kappa$.
(3) The sequence M_α is increasing and continuous.
(4) $\{p_1, S, D_1\} \cup \kappa + 1 \cup \sup(p_1) \subseteq M_0$.
(5) $\gamma_\alpha := M_\alpha \cap \kappa^+ \in \kappa^+$.
(6) $M_\alpha \in M_{\alpha+1}$ and $M_\alpha^{<\kappa} \subseteq M_{\alpha+1}$

Let $C = \{\gamma_\alpha \mid \alpha < \kappa^+\}$. Let $\mathrm{otp}(p_1) = \rho < \kappa$. By fatness of S, $S \cap C$ contains a closed set A of order-type $\rho+1$. Enumerate increasingly $A = \{\gamma_{\alpha_i} \mid i \leq \rho\}$, and $p_1 = \{\delta_i \mid i \leq \rho\}$. Now we define recursively a decreasing sequence r_i for $i \leq \rho$ such that:

(a) $r_i \in M_{\alpha_{i+1}}$.
(b) $\max(r_i) \in [\gamma_{\alpha_i}, \gamma_{\alpha_{i+1}})$.
(c) $r_{i+1} \setminus \delta_i + 1 \in D_1$.

Let $p_1 \geq r_0 \in D_1 \cap M_{\alpha_1}$, with $\max(r_0) \geq \gamma_{\alpha_0}$. Such and r_0 exists, since $M_{\alpha_1} \models$ "D_1 is dense" so p_1 has an extension $r_0' \in D_1 \cap M_{\alpha_1}$ and also $M_{\alpha_1} \models S \setminus \gamma_{\alpha_0} \neq \emptyset$, hence if necessary, we can ass a point from $S \cap M_{\alpha_1}$ to r_0' to form $r_0 \in M_{\alpha_1} \cap D_1$ such that $\max(r_0) \in [\gamma_{\alpha_0}, \gamma_{\alpha_1})$.

At successor steps, suppose that $r_i \in M_{\alpha_i+1}$ was defined, we let $r'_{i+1} \leq r_i$ be an element of $M_{\alpha_{i+2}}$ such that $\max(r'_{i+1}) \geq \gamma_{\alpha_{i+1}}$. Then, let $r''_{i+1} \leq r'_{i+1} \setminus \delta_i + 1$ in $M_{\alpha_{i+2}} \cap D_1$ to a condition in D_1 and let

$$r_{i+1} = r''_{i+1} \cup (p_1 \cap \delta_i + 1) \in M_{\alpha_{i+2}}.$$

Note that since $r_i \leq p_1$, and since $r''_{i+1} \setminus \delta_i + 1 \leq r_i \setminus \delta_i + 1$, we have $r_{i+1} \leq r_i$. Clearly, r_{i+1} satisfies $(a), (b), (c)$.

At limit steps μ, we have constructed a decreasing sequence $r_i \in M_{\alpha_{i+1}} \subseteq M_{\alpha_\mu}$. By condition (6), the sequence $\{r_i \mid i < \mu\} \in M_{\alpha_{\mu+1}}$ and therefore, $\bigcup_{i<\mu} r_i \in M_{\alpha_{\mu+1}}$. By condition (b), we have $\sup(\bigcup_{i<\mu} r_i) = \gamma_{\alpha_\mu} \in S \cap M_{\alpha_{\mu+1}}$, so we can form the condition $r'_\mu = \bigcup_{i<\mu} r_i \cup \{\gamma_{\alpha_\mu}\}$. As in the successor step, we extend $r'_\mu \setminus \delta_\mu + 1$ to a condition $r''_\mu \in D_1 \cap M_{\alpha_{\mu+1}}$, and set $r_\mu = r''_\mu \cup (p_1 \cap (\delta_\mu + 1))$. The concludes the definition of the sequence r_i.

Let $p_2 := r_\rho$. Then $p_2 \leq p_1 \leq p$ and we claim that $p_2 \in D^*$ as witnessed by $\eta = \max(p_1)$. For each $\alpha \leq \max(p_1)$, there is $i < \rho$ such that $p_1 \setminus \alpha = p_1 \setminus (\delta_i + 1)$. The condition $p_2 \setminus \alpha$ is then stronger than $r_{i+1} \setminus \alpha = r_{i+1} \setminus (\delta_i + 1) \in D_1$ (by condition (c)). Hence also $p \setminus \alpha \in D_1$.

So far we have shown that D^* is dense open, and therefore $q \in j(D^*)$. Let $\eta < \max(q)$ witness this. If $\eta \leq \eta_1$, then since $q \cap \eta_1 \notin j(D_2)$, and D_2

is open, $q \cap \eta \notin j(D_2)$ contradicting (I). If $\eta > \eta_1$, then by condition (II), $q \setminus \eta_1 \in j(D_1)$, contradicting the choice of D_1. □

Since[1] $\sup(j''\kappa^+) < \max(q)$, we can use the previous proposition to see that $q^* := a \setminus \sup(j''\kappa^+) \in \bigcap j''D(Q(S))$. Let us show that we can make some more modifications below q^*:

Proposition 5.7. *Let S be $<\kappa$-fat. For every $p \in Q(S)$, $j(p) \cup q^* \in \bigcap j''D(Q(S))$.*

Proof. The proof is essentially homogeneity, but let us add some details. Fix D_1, p_1, and let us show that $j(p_1) \cup q^* \in j(D_1)$. Define D^* as the set of all p such that, $p_1 \cup (p \setminus \max(p_1) + 1) \in D_1$. Suppose we extend $p \in D^*$ to some p', then $p_1 \cup (p \setminus \max(p_1) + 1) \sqsubseteq p_1 \cup (p' \setminus \max(p_1) + 1)$, so $p' \in D^*$, hence D^* is open. To see D^* is dense, take any p, extend it to p' with $\max(p') > \max(p_1)+1$. Find $q \in D$ such that $q \sqsupseteq (p' \setminus (\max(p_1)+1)) \cup p_1$. Set $p'' = q \cup (p' \cap \max(p_1)+1)$. Then $p'' \geq p' \geq p$, and $(p'' \setminus (\max(p_1)+1)) \cup p_1 = q \in D$.

Hence D^* is dense open, and by 5.6 $q^* \in j(D^*)$. This means that $j(p_1) \cup q^* \setminus \max(j(p_1)) + 1 \in j(D_1)$. But $\max(j(p_1)+1) < \sup(j''\kappa^+)$ hence, $j(p_1) \cup q^* \in j(D_1)$. □

We let $f : P_\kappa(\kappa^+) \to Q(S)$ represent q^*, i.e. $j_U(f)(j''\kappa^+) = q^*$.

Corollary 5.8. *For every dense set $D \subseteq D(Q)$,*

$$E_D := \{x \in P_\kappa(\kappa^+) \mid \forall \vec{y} \prec x,\ (\bigcup_{i \leq |\vec{y}|} f(\vec{y}(i))) \cup f(x) \in D\} \in U.$$

Proof. Let us show that $j''\kappa^+ \in j(E_D)$. Fix any $\vec{y} = (y_1, ..., y_n) \prec j''\kappa^+$. Then for each $1 \leq i \leq n$, there is z_i such that $j(z_i) = j''z_i = y_i$. This means that $j(f)(\vec{y}(i)) = j(f)(j(z_i)) = j(f(z_i))$, and $\bigcup_{i \leq n} j(f)(\vec{y}(i)) = j(\bigcup_{i \leq n} f(z_i))$. Note that $p := \bigcup_{i \leq n} f(z_i)$ is a closed subset of S and therefore $p \in Q(S)$. By the above computation and the previous lemma,

$$\bigcup_{i \leq n} j(f)(\vec{y}(i)) \cup j(f)(j''\kappa^+) = j(p) \cup q^* \in j(D),$$

which by definition means $j''\kappa^+ \in j(E_D)$. □

There is an increasing function[2] $g : \kappa^+ \to \kappa^+$ such that $j(g)(\sup(j''\kappa^+)) \geq \sup(q^*)$.

Claim 5.9. $E = \{x \in P_\kappa(\kappa^+) \mid \forall y \prec x, g(y) < \sup(x)\} \in U$.

Proof. Indeed if $y \prec j''\kappa^+$, $\sup(j(f)(y)) = \sup(j(f(z))) < \sup(j''\kappa^+)$. □

[1]This is where the fact that $cf(< \kappa) \setminus S$ being stationary is crucial.

[2]This is one crucial place where we need the normality.

By forcing below a measure one set, any Prikry generic for $\mathcal{U}$, $(x_n)_{n<\omega}$ will satisfy the following inequalities for each $n < \omega$:

$$\sup(x_n) < \min(f(x_n)) \leq \max(f(x_n)) \leq g(\sup(x_n)) < \sup(x_{n+1}).$$

Let $q_n = f(x_n)$, the above shows in particular that the q_n's form blocks. Let $C := \bigcup_{n<\omega} q_n$.

Proposition 5.10. *C is V-generic for $Q(S)$.*

Proof. Let D be dense open. Then by the corollary 5.8, there is n such that $x_n \in E_D$. This means that for every $(y_1, ..., y_m) \prec x_n$, $\bigcup_{i\leq m} f(y_i) \cup f(x_n) \in D$. In particular, since $(x_0, ..., x_{n-1}) \prec x_n$, we get that $C \cap \max(q_n) + 1 = \bigcup_{i\leq n} q_i = \bigcup_{i\leq n} f(x_i) \in D$. It follows that $C \cap \max(q_n) + 1 \in D \cap G(C)$. □

Finally, to show that no such q^* exists, we will show that the cofinality of κ^+ in $V[C]$ is ω. Let $\gamma_0 = \sup(x_0)$. Then $\delta_0 = g(\gamma_0)$ and let $\gamma_1 = \min(C \setminus \delta_0 + 1)$, let $\delta_1 = g(\gamma_1)$. In general, if δ_n is defined, we let $\gamma_{n+1} = \min(C \setminus \delta_n + 1)$ and $\delta_{n+1} = g(\gamma_{n+1})$. We claim that γ_n (which is clearly defined in $V[C]$) is unbounded in κ^+:

It suffices to show that for every n, $\gamma_n \geq \sup(x_n)$. By induction. For $n = 0$, by definition $\sup(x_0) = \gamma_0$. Suppose this was true for n, then $\gamma_n \geq \sup(x_n)$. Then $g(\gamma_n) = \delta_{n+1} \geq g(\sup(x_n)) \geq \sup(q_n)$ (by monotonicity and the choice of g). Then

$$\gamma_{n+1} = \min(C \setminus \delta_{n+1} + 1) \geq \min(C \setminus \sup(q_n) + 1) = \min(q_{n+1}) \geq \sup(x_{n+1}),$$

as wanted.

Corollary 5.11. *$\mathbb{P}_{\mathcal{U}}$ does not project onto every κ-distributive forcing of cardinality κ^+.*

Proof. I believe that the above argument should also be true if there is a finite iterate of U such that $q^* \in \bigcap j''_{U^n} D(\mathbb{Q}(S))$. Then by the result we have with Seba and Ben, for a κ-distributive forcing, the existence of a (weak) projection is equivalent to such a q^*. □

Remark 5.12. The above corollary is in contrast to the strongly compact Prikry forcing which under slightly weaker assumption than 2^{κ^+}-supercompactness, for every κ-distributive forcing $\mathbb{Q}$ of cardinality κ^+, there is a non-normal fine κ-comnplete measure on $P_\kappa(\kappa^+)$ which does project onto $\mathbb{Q}$.

Can this be pushed further to every non-strategically closed forcing which has similar master condition?

It seems that if we take a forcing $\mathbb{Q}$ of size κ^+, maybe we can look at $\langle \mathbb{Q} \cap M_\alpha \mid \alpha < \kappa^+ \rangle$, then we check whether there is a generic condition q^* for $j_U(\vec{M})_{[id]_U} \cap j(Q)$ in the properness sense, namely, that for every dense set D there is somehow $p' \leq q$ inside the model which is also in $j(D)$. Then we pull back all q such that $j(q) \leq q^*$, this should give a filter which is somehow generic for every $\mathbb{Q} \cap M_\alpha$, and this shouldn't be possible in some situations?

Suppose that $q \in \bigcap j''\mathbb{Q}$. Then q is not going to be above $k''G$ for some filter $G \subseteq i(Q)$ which is M_U-generic.

Suppose Q was $Col(\kappa, \kappa^+)$, and $q^* \in \bigcap j''D(Q)$ and above $k''G$. Then we can look at all q such that $j(q) \geq q^*$. This is a filter. The domain of q^* $k''i(\kappa) = i(\kappa)$, and $q^*(\rho) = k(g(\rho)) \in k''i(\kappa^+)$ is onto. There is no reason why we should have $g(i(\alpha)) \in i''\kappa^+$, so no reason to believe that $k(g(i(\alpha))) = q^*(j(\alpha)) \in j''\kappa^+$. Hence q^* will have only boundedly many elements below it.

We the club this, we must have things blow it in $j''Q$.

So to rule out q^* we see that the condition is that $\{q \mid j(q) \geq q^*\}$ form a filter on Q (always true) which is intersects κ

I want that there is $e : Q \to j(Q)$ is an embedding of forcing (no necessarily the identity) such that e is dense below the set of weaker condition than q^*. The point is that if we take $e = j \restriction Q$, this might avoid being below q^*.

For example, if it was club shooting, we take some club and erase the information outside $j''\kappa^+$, and add the information to the sup of the condition from q^*. Namely, $e(c) = j''c \cup (q^* \cap (j(\max(c)) \setminus j'' \max(c))$.

Given such an e, look at all p such that $e(p) \leq q^*$. This forms a filter. For every $i < \kappa^+$,

$k''G \cap j(D_i) \neq \emptyset$. $e(p) \in j(D_i)$ below q^*.

such that Let $\alpha < \kappa^+$ and consider $j(D_\alpha)$. Then there is $g \in G \cap i(D_\alpha)$ and $k(g) \in q$ Suppose that there $MA_{\kappa^+}(Q)$ fails. Namely, that there are κ^+-many dense sets that no filter meets, (If Q is $< \kappa$-strategically closed of size κ^+,

Otherwise, we define a strategy as follows. Pick Play $(p_n | n < \omega)$ decreasing in M_ω, then $(j(p_n) \mid n < \omega) \in M^*_{j(\omega)}$