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ORIGINAL PAPER

ASIP in random environment with disasters

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Abstract

The Asymmetric Inclusion Process (ASIP) is an *n*-site stochastic tandem network where single particles arrive randomly at the first site and move in clusters, simultaneously and unidirectionally, from one site to the next until exiting the system. Each site has a buffer of unlimited size and a gate in front of it that opens from time to time according to some specified stochastic rules. Various ASIP models, as well as generalized ASIP systems, were extensively investigated. Independently, queues in random environment were also studied in the queueing literature. Operating under 'random environment', the system as a whole alternates stochastically between several phases (environments), while residing in each phase a random time with a phase-specific characteristic. In this work, we consider a combined system–ASIP in random environment–where the system alternates between two environments, residing in phase k ($k = 1, 2$) an exponentially distributed time with mean $\frac{1}{\eta_k}$. When in environment k, single particles arrive at the first site following a Poisson process with rate λ_k, and propagate in clusters unidirectionally through the network's sites. The gate of site j ($j = 1, 2, \ldots, n$) opens randomly, independently of the other gates, as follows: when the system is in phase k, gate j opens every exponentially distributed time with mean $\frac{1}{\mu_{jk}}$, causing all particles accumulated in site j to move forward instantaneously and simultaneously, as a batch, to site $j + 1$ ($j = 1, 2, \ldots, n - 1$). When gate n opens, all particles present there exit the system. Moreover, at each change of phase, all particles present in the entire system are cleared (a disaster). We study this combined system – ASIP in random environment with disasters – and derive the following: (i) the multi-dimensional probability generating function (PGF) of site occupancies; (ii) the mean occupancy of each site; (iii) the probability that site j ($j = 1, 2, \ldots, n$) is occupied; (iv) the mean sojourn time of a particle in the system; (v) the PGF and mean of the overall load of the first m sites ($m = 1, 2, \ldots, n$); (vi) the mean duration of a busy period; (vii) the probability that the first occupied site is site m ($m = 1, 2, \ldots, n$); and (viii) the mean length of the draining time (the time elapsing from the moment the arrival process

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stops until the first moment thereafter that the system becomes completely empty). The results are compared with those of the classical single-environment ASIP.

Keywords Asymmetric inclusion process (ASIP) · Random environment · Disasters · Performance measures

1 Introduction

Complex queueing networks are prevalent in real-life situations. Among these, the class of tandem stochastic systems is of great importance – practically and analytically. A Tandem Stochastic System (TSS) is an n-site network in series, where particles (customers, messages, calls, jobs, products, molecules, etc.) arrive randomly in time at the leftmost site of the network, and propagate unidirectionally from one site to the next, (under specific stochastic rules) along the one-dimensional array of sites (queues, servers, stations, etc.), until exiting the system. This underlying model, with different variations, has been widely studied in the literature. In a TSS, each site has a buffer and a gate in front of it that opens stochastically, allowing particles to move forward. Varying assumptions on site buffer sizes, as well as on gate (service) capacities and inter-gate opening instants, lead to different TSS-type models. A well-known TSS model is the Tandem Jackson Network (TJN) where the buffer capacity of each site is unlimited, but only a single particle is allowed to move forward when a site's gate opens. Another studied TSS model is the Asymmetric Exclusion Process (ASEP), in which each buffer's capacity is limited to a single particle only, and each site's gate capacity is also a single particle. A recently introduced TSS model (Reuveni et al. 2011), that bridges the gap between the TJN and the ASEP, is the so-called Asymmetric Inclusion Process (ASIP). In that system, the buffer size of each site, where particles accumulate, is unlimited, and when the gate in front of a site opens (service at the site is completed), all particles accumulated there move instantaneously and simultaneously as a cluster to the next site. References to those models (Reuveni et al. 2011, 2012a, b, 2014; Lauber Bonomo and Reuveni 2019; Boxma et al. 2016, 2021; Yeger and Yechiali 2022a, b, 2024; Yeger et al. 2023; Geelen et al. 2025) can be found in the literature. An illustration of the ASIP tandem network is presented in Fig. 1.

In the ASIP system, particles arrive at the first site following a Poisson process, while gate j $(j = 1, 2, \ldots, n)$ opens, independently of all other gates, every exponentially distributed time with a site-dependent rate. The complexity of the ASIP network was exhibited and analyzed (Reuveni et al. 2012a), while performance measures, along with limit laws, were studied (Reuveni et al. 2012b, 2014). Lauber Bonomo and Reuveni (2019) calculated occupancy correlations in the ASIP system. Boxma et al. (2016) extended the analysis of ASIP systems and investigated a generalized ASIP (G-ASIP) model where: (i) times between gate openings are determined by a renewal process, and (ii) particles arrive to all sites during the time interval between two instants of gate openings. The focus was on derivation of the multi-dimensional joint PGF of site occupancies. Expressions for various performance measures in G-ASIP systems were derived a following paper by Yeger and Yechiali (2022a): tra-

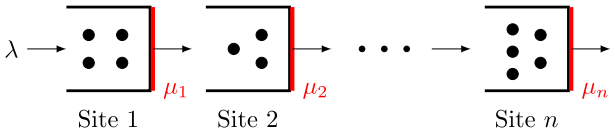


Fig. 1 The ASIP tandem network

versal time, overall load, busy period, first occupied site, and system's draining time. A matrix approach to derive the joint site-occupancy PGF in an n-site generalized ASIP (G-ASIP) systems was developed in a later study (Yeger and Yechiali 2022b), replacing the tedious successive substitution method commonly used to derive the multi-dimensional PGF of the site occupancies. A generalized ASIP with arrivals to all sites and particle movements in all directions was further studied (Yeger and Yechiali 2024), while an ASIP system with consumption is analyzed by Yeger et al. (2023). Recently, Geelen, Vlasiou and Yeger applied analytical and numerical approximation methods to evaluate the distribution of the number of particles in an n-site ASIP system (Geelen et al. 2025).

Queueing systems in random environment were also widely studied in the literature. An M/M/1-type queue in a 2-phase, and in n-phase, random environment was introduced by Yechiali and Naor (1971); Yechiali (1973), respectively. They were further investigated in a later work (Gupta et al. 2006). Following, an M/M/1-type queue with an n-phase random environment, with disaster, was studied by Paz and Yechiali (2014), while an M/M/∞ queue with n-phase random environment was analyzed (Paz and Yechiali 2007). Neuts devoted a full chapter to the analysis of queues in an n-phase random environment, employing matrix-geometric methods to derive the multi-dimensional joint phase-occupancy PGF (Neuts 1981) . In particular, and interestingly, it was shown for the M/M/1 queue (Yechiali and Naor 1971; Yechiali 1973) in random environment, as well as for the M/M/∞ queue (Paz and Yechiali 2007) in random environment, that they each fully resemble the corresponding classical M/M/1 queue and the M/M/∞ queue, respectively, if and only if the ratios between arrival rate and service rate are the same in all environments.

The ASIP system has been extensively studied due to its applicability in modeling diverse processes such as production lines, traffic systems, and nutrient transport between cells in microbiology. However, prior research has focused exclusively on single-phase systems, overlooking scenarios in which arrival and service rates vary significantly due to external processes that influence the arrival and service rates. Examples are production lines with fluctuating workloads, alternating periods of heavy and light traffic, or biological systems that operate differently at different time periods(Reuveni et al. 2011; Yeger et al. 2023). Analysis of costumers' equilibrium balking strategies for a non-ASIP, single site, clearing queueing system operating in alternating environment was carried out by Economou and Manou (2013).

The current paper addresses the aforementioned issues while widening the scope of analysis of ASIP-type networks. Specifically, it introduces random environment into the ASIP model by considering an n-site ASIP system operating randomly in a two-phase environment. An augment feature is that the system is subject to random disasters that cause the loss (clearing) of all particles present in the network at

instants of a disaster. The focus of this paper is on deriving analytical expressions for various key performance measures as detailed in the abstract.

The model is defined in sect. 2, while sect. 3 deals with the stochastic dynamics of the combined system. In sect. 4, the multi-dimensional joint PGF of site occupancies is constructed, along with calculations of mean site occupancies. The following performance measures are investigated in sect. 5: A particle's mean sojourn time in the system; overall load; the probability that the system is empty; busy period; the probability that site *m* $(m = 1, 2, ...n)$ is the first occupied site; and the draining time: the length of time it takes from the moment the arrival process stops until the first instant thereafter that the system becomes completely empty. Section 6 concludes the paper with a concise discussion.

2 The model

Consider an *n*-site ASIP system operating in a two-phase random environment, subject to system failure (disaster) at each phase. The system alternates between the two environments (phases), where the duration of time the system resides in phase *k* $(k = 1, 2)$, denoted by O_k, is exponentially distributed with mean $\frac{1}{\eta_k}$. When in phase *k*, particles arrive at the system following a Poisson process with rate λ_k, while times between consecutive openings of gate *j* $(j = 1, 2, ..., n)$ are exponentially distributed with mean $\frac{1}{\mu_{jk}}$, independently of all other gates. All processes are independent of each other. It follows that in steady state the probability that the system is in phase *k* is

$$P(\text{phase } k) := \frac{\frac{1}{\eta_k}}{\frac{1}{\eta_1} + \frac{1}{\eta_2}}, \quad k = 1, 2. \tag{1}$$

A unique feature of the current model is that each time a change of phase occurs, all present particles are cleared from the system. In the subsequent analysis, for the sake of convenience, we adopt a generalized notation referring to an arbitrary phase *k* and its counterpart phase $\bar{k}$, $k = 1, 2$.

The illustration in Fig. 2 shows the dynamic behavior of ASIP within a random environment. During phase 1, the system mimics the characteristics of a single-phase ASIP. Upon environment change from phase 1 to phase 2, the system undergoes a complete clearance of all particles, initiating the process anew from phase 2. A similar reset occurs at a transition instant from phase 2 back to phase 1.

3 Stochastic dynamics

At an arbitrary time *t* $(t \geq 0)$, let $X_{jk}(t)$ denote the number of particles (occupancy) present at site *j* when the system is in phase *k*. Denote the vector

$$(X_{1k}(t), X_{2k}(t), ..., X_{nk}(t)) := (X_1(t), X_2(t), ..., X_n(t))_k.$$

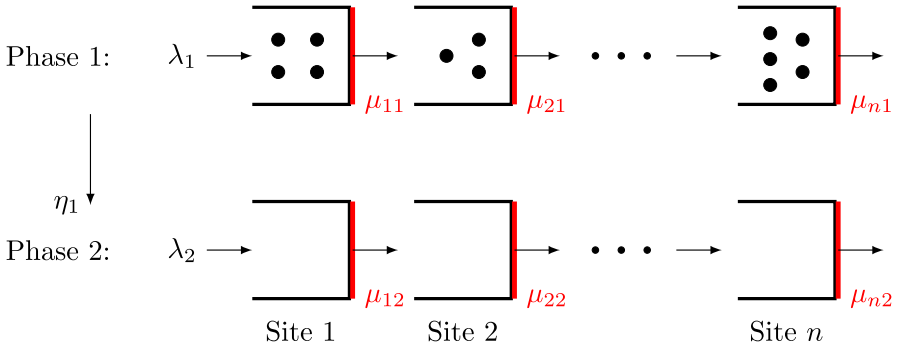


Fig. 2 An ASIP tandem queue in random environment with phase-transition rate η_1 from phase 1 to phase 2. Upon phase transition, all particles present in the system are cleared and lost. While in phase k, the Poisson arrival rate is λ_k and the service rate of site j is μ_{jk}, $k = 1, 2.; j = 1, 2, \ldots, n$. A similar process rules when the system is in phase 2

Consider the system at times t and $t + \Delta t$ and denote the shorthand notation $X_j = X_j(t)$ and $X'_j = X_j(t + \Delta t)$. Let Δt be small enough, such that, in a Δt time length, at most a single event occurs: particle arrival, gate opening or phase transition.

The Markovian dynamics of the stochastic process is given by

$$(X'_1, X'_2, ..., X'_n)_k = \begin{cases} (X_1, X_2, ..., X_n)_k & \text{w.p.} \quad 1 - (\lambda_k + \sum_{j=1}^{n} \mu_{jk} + \eta_k)\Delta t + o(\Delta t) \\ (X_1 + 1, X_2, ..., X_n)_k & \text{w.p.} \quad \lambda_k \Delta t + o(\Delta t) \\ (0, X_1 + X_2, ..., X_n)_k & \text{w.p.} \quad \mu_{1k}\Delta t + o(\Delta t) \\ (X_1, 0, X_2 + X_3, ..., X_n)_k & \text{w.p.} \quad \mu_{2k}\Delta t + o(\Delta t) \\ \quad \vdots & \quad \vdots \\ (X_1, X_2, ..., X_{n-1}, 0)_k & \text{w.p.} \quad \mu_{nk}\Delta t + o(\Delta t) \\ (0, 0, ..., 0)_{\bar{k}} & \text{w.p.} \quad \eta_k \Delta t + o(\Delta t) \end{cases} \tag{2}$$

The occupancy vector $(X'_1, X'_2, ..., X'_n)_{\bar{k}}$ is similar to $(X'_1, X'_2, ..., X'_n)_k$ while interchanging k and $\bar{k}$. Note that, when change of phase occurs all particles are cleared from the system and the occupancy vector is $(0, 0, \ldots, 0)$. With the stochastic dynamics defined in Eq. (2), we next derive the multi-dimensional PGF of the sites occupancies and their means.

4 Site occupancies: PGF and means

4.1 PGF

The Markovian dynamics described in Eq. (2) leads to the following multidimensional probability generating function (PGF) of the site occupancies at steady state:

$$\begin{aligned} G_k(\vec{z}) \equiv & G_k(z_1, z_2, ..., z_n) = E[\prod_{j=1}^{n} z_j^{X'_j}] = E[E[\prod_{j=1}^{n} z_j^{X'_j}|X]] \\ =&(1 - \Delta t(\lambda_k + \sum_{j=1}^{n} \mu_{jk} + \eta_k))E[\prod_{j=1}^{n} z_j^{X_j}] \\ &+ \Delta t\lambda_k E[z_1 \prod_{j=1}^{n} z_j^{X_j}] + \Delta t\mu_{1k} E[z_2^{X_1} \prod_{j\neq 1}^{n} z_j^{X_j}] \\ &+ \Delta t\mu_{2k} E[z_3^{X_2} \prod_{j\neq 2}^{n} z_j^{X_j}] + ... + \Delta t\mu_{nk} E[\prod_{j\neq n}^{n} z_j^{X_j}] + \Delta t\eta_k E[\prod_{j=1}^{n} z_j^{0}] + o(\Delta t). \end{aligned} \tag{3}$$

$G_k(\vec{z})$ above follows the events represented in Eq. (2): The first term refers to occurrence of no event at time Δt. The second refers to an arrival, the third to the opening of gate 1, the fourth to the opening of gate 2, etc., while the last refers to the event of phase alternation.

Dividing by Δt, rearranging terms and taking $\Delta t \to 0$, Eq. (3) becomes

$$\begin{aligned} \big(\lambda_k(1 - z_1) + \sum_{j=1}^{n} \mu_{jk} + \eta_k\big) & G_k(z_1, z_2, ..., z_n) \\ =&\mu_{1k} G_k(z_2, z_2, ..., z_n) + \mu_{2k} G_k(z_1, z_3, ..., z_n) \\ &+ ... + \mu_{nk} G_k(z_1, z_2, ..., z_{n-1}, 1) + \eta_k G_{\bar{k}}(1, 1, ..., 1). \end{aligned} \tag{4}$$

Eq. (4) implicitly defines the site occupancies PGF in steady state. An explicit expression for this PGF can be obtained either through successive substitutions, similar to the approach taken in Reuveni et al. (2011) for a non-random environment ASIP, or by utilizing the more efficient matrix methodology outlined in Yeger and Yechiali (2022a). Nonetheless, these and prior studies have demonstrated that the computational complexity involved in obtaining the PGF, along with the resulting PGF expression, renders the methods too cumbersome for large systems where the number of sites is $n \geq 3$. Consequently, in the next subsection, we narrow the derivation of an explicit result of the occupancies PGF to the case with $n = 2$ sites (all other results in the sequel are for a general network with $n \geq 2$).

4.1.1 $n = 2$ sites

Consider the case of $n = 2$ sites. Then, Eq. (4) becomes

$$\big[\lambda_k(1 - z_1) + \mu_{1k} + \mu_{2k} + \eta_k\big] G_k(z_1, z_2) = \mu_{1k} G_k(z_2, z_2) + \mu_{2k} G_k(z_1, 1) + \eta_k G_{\bar{k}}(1, 1). \tag{5}$$

Substituting $z_1 = 0, z_2 = 1$ in Eq. (5) and using $G_{\bar{k}}(1, 1) = 1$ yields

$$G_k(0, 1) = \frac{\mu_{1k} + \eta_k}{\lambda_k + \mu_{1k} + \eta_k}. \tag{6}$$

Eq. (6) asserts that the likelihood of the initial site being vacant arises if either gate 1 opens or a phase change (clearance) occurs, before a new particle arrives.

Substituting $z_2 = 1$ in Eq. (5) yields

$$G_k(z_1, 1) = \frac{\mu_{1k} + \eta_k}{\lambda_k(1 - z_1) + \mu_{1k} + \eta_k}. \tag{7}$$

Then, substituting $z_2 = z_1$ in Eq. (5) results in

$$\big[\lambda_k(1 - z_1) + \mu_{2k} + \eta_k\big] G_k(z_1, z_1) = \mu_{2k} G_k(z_1, 1) + \eta_k. \tag{8}$$

Substituting $G(z_1, 1)$ from Eq. (7) into Eq. (8) yields

$$G_k(z_1, z_1) = \mu_{2k}(\mu_{1k} + \eta_k) \prod_{j=1}^{2} \frac{1}{\lambda_k(1 - z_1) + \mu_{jk} + \eta_k} + \frac{\eta_k}{\lambda_k(1 - z_1) + \mu_{2k} + \eta_k}. \tag{9}$$

We first set $z_1 = z_2$ in Eq. (9). Then, we substitute Eqs. (9) and (7) in Eq. (5). This yields an explicit expression for the sought for PGF, $G(z_1, z_2)$:

$$\begin{aligned} &\big[\lambda_k(1 - z_1) + \mu_{1k} + \mu_{2k} + \eta_k\big] G_k(z_1, z_2) \\ &\quad = (\mu_{1k} + \eta_k) \prod_{j=1}^{2} \frac{\mu_{jk}}{\lambda_k(1 - z_2) + \mu_{jk} + \eta_k} \\ &\qquad + \frac{\mu_{1k}\eta_k}{\lambda_k(1 - z_2) + \mu_{2k} + \eta_k} + \frac{\mu_{2k}(\mu_{1k} + \eta_k)}{\lambda_k(1 - z_1) + \mu_{1k} + \eta_k} + \eta_k. \end{aligned} \tag{10}$$

When $\eta_k \to 0$, Eq. (10) reduces to Eq. (39) in Reuveni et al. (2011) (for the single-phase ASIP), namely,

$$\big[\lambda_k(1 - z_1) + \mu_{1k} + \mu_{2k}\big] G_k(z_1, z_2) = \mu_{1k} \prod_{j=1}^{2} \frac{\mu_{jk}}{\lambda_k(1 - z_2) + \mu_{jk}} + \frac{\mu_{1k}\mu_{2k}}{\lambda_k(1 - z_1) + \mu_{1k}}. \tag{11}$$

Ultimately, applying the law of total probability, and considering the two possible phases yields the desired PGF $G(z_1, z_2)$

$$G(z_1, z_2) = G_1(z_1, z_2)P(O_1) + G_2(z_1, z_2)P(O_2). \tag{12}$$

Note that $G_k(z_1, z_2)$ is given in Eq. (11) for $k = 1, 2$.

$$G(z_1,z_2) = \eta_2 \frac{\mu_{11}\prod_{j=1}^{2}\frac{\mu_{j1}}{\lambda_1(1-z_2)+\mu_{j1}} + \frac{\mu_{11}\mu_{21}}{\lambda_1(1-z_1)+\mu_{11}}}{(\lambda_1(1-z_1)+\mu_{11}+\mu_{21})(\eta_1+\eta_2)} + \eta_1 \frac{\mu_{12}\prod_{j=1}^{2}\frac{\mu_{j2}}{\lambda_2(1-z_2)+\mu_{j2}} + \frac{\mu_{12}\mu_{22}}{\lambda_2(1-z_1)+\mu_{12}}}{(\lambda_2(1-z_1)+\mu_{12}+\mu_{22})(\eta_1+\eta_2)}. \tag{13}$$

Substituting $z_1 = z_2 = 0$ in Eq. (13) gives the probability of an empty system

$$G(0,0) = \eta_2 \frac{\mu_{11}\prod_{j=1}^{2}\frac{\mu_{j1}}{\lambda_1+\mu_{j1}} + \frac{\mu_{11}\mu_{21}}{\lambda_1+\mu_{11}}}{(\lambda_1+\mu_{11}+\mu_{21})(\eta_1+\eta_2)} + \eta_1 \frac{\mu_{12}\prod_{j=1}^{2}\frac{\mu_{j2}}{\lambda_2+\mu_{j2}} + \frac{\mu_{12}\mu_{22}}{\lambda_2+\mu_{12}}}{(\lambda_2+\mu_{12}+\mu_{22})(\eta_1+\eta_2)}. \tag{14}$$

4.2 Site occupancy means

Denote by $E_k[X_j]$ the mean occupancy of site *j* when the system is in phase *k*. Then, differentiating Eq. (4) with respect to z_1 and substituting $z_1 = z_2 = \ldots = z_n = 1$ yields

$$-\lambda_k + \Big[\sum_{j=1}^{n}\mu_{jk} + \eta_k\Big]E_k[X_1] = \mu_{2k}E_k[X_1] + \ldots + \mu_{nk}E_k[X_1], \tag{15}$$

which implies the mean value occupancy balance of the first site, namely,

$$[\mu_{1k} + \eta_k]E_k[X_1] = \lambda_k. \tag{16}$$

Similarly, differentiating Eq. (4) with respect to z_j and substituting $z_1 = z_2 = \ldots = z_n = 1$ yields

$$\begin{aligned}\Big[\sum_{j=1}^{n}\mu_{jk} + \eta_k\Big]E_k[X_j] =& \mu_{1k}E_k[X_j] + \mu_{2k}E_k[X_j] \\ &+ \ldots + \mu_{j-1,k}E_k[X_{j-1}+X_j] + \mu_{j+1,k}E_k[X_j] + \ldots. + \mu_{nk}E_k[X_j].\end{aligned} \tag{17}$$

Eq. (17) reduces to

$$E_k[X_j] = \frac{\mu_{j-1,k}}{\mu_{jk}+\eta_k}E_k[X_{j-1}], \tag{18}$$

which, again, expresses the mean value occupancy balance regarding site *j*.

Finally,

$$E_k[X_j] = \frac{\lambda_k}{\mu_{jk}}\prod_{i=1}^{j}\frac{\mu_{ik}}{\mu_{ik}+\eta_k}, \quad j = 1,2,\cdots,n. \tag{19}$$

Using the law of total probability, we get for $j = 1,2,\ldots,n$:

$$E[X_j] = E_1[X_j]P(O_1) + E_2[X_j]P(O_2)$$
$$= \frac{\lambda_1\eta_2}{\mu_{j1}(\eta_1+\eta_2)}\prod_{i=1}^{j}\frac{\mu_{i1}}{\mu_{i1}+\eta_1} + \frac{\lambda_2\eta_1}{\mu_{j2}(\eta_1+\eta_2)}\prod_{i=1}^{j}\frac{\mu_{i2}}{\mu_{i2}+\eta_2}. \quad (20)$$

When $\eta_1 \to 0$, Eq. (20) coincides with Eq. (16) in Reuveni et al. (2011), namely, $E[X_j] = \frac{\lambda}{\mu_j}$, for the single-phase ASIP (where μ_j of the single-phase ASIP is equivalent to μ_{j1} in our case).

Summing over the n sites gives the system's mean load

$$E[L] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n}\left[\frac{\lambda_1\eta_2}{\mu_{j1}(\eta_1+\eta_2)}\prod_{i=1}^{j}\frac{\mu_{i1}}{\mu_{i1}+\eta_1} + \frac{\lambda_2\eta_1}{\mu_{j2}(\eta_1+\eta_2)}\prod_{i=1}^{j}\frac{\mu_{i2}}{\mu_{i2}+\eta_2}\right]. \quad (21)$$

5 Performance measures

We analyze five performance measures for a ASIP in random environment with disasters: (i) Traversal time–the random duration of time it takes a particle to traverse through the system, namely, the time elapsing from the first instant a particle is admitted to the first site, until the instant thereafter it exits the system; (ii) Overall load–the total number of particles present in the system in steady state; (iii) Busy period–the random time-duration in which the system is continuously non-empty; (iv) First occupied site–first non-empty site in the system; and (v) Draining time–the duration of time it takes the network to clear, namely, the random time elapsing from the moment the arrival process stops, until the first instant thereafter that the system becomes empty. These performance measures are compared with the results obtained previously for single-phase ASIP and G-ASIP in Reuveni et al. (2012b) and in Yeger and Yechiali (2022a), respectively.

5.1 Particle's traversal time

Consider a particle that arrives when the system is in phase k ($k = 1, 2$). Let T_k denote the traversal time of that particle through the system, starting from its entry until its exit, either upon traversing successfully the entire system, or due to clearance resulting from a phase change, which can occur at any site. We can write,

$$E[T_k] = \frac{1}{\mu_{1k}+\eta_k} + \frac{\mu_{1k}}{\mu_{1k}+\eta_k}\Big(\frac{1}{\mu_{2k}+\eta_k} + \frac{\mu_{2k}}{\mu_{2k}+\eta_k}\Big[\frac{1}{\mu_{3k}+\eta_k} + \ldots$$
$$+ \frac{\mu_{n-1,k}}{\mu_{n-1,k}+\eta_k}\Big(\frac{1}{\mu_{nk}+\eta_k}\Big)\Big]\Big) \quad (22)$$
$$= \sum_{j=1}^{n}\frac{1}{\mu_{jk}}\prod_{i=1}^{j}\Big(\frac{\mu_{ik}}{\mu_{ik}+\eta_k}\Big).$$

The first term in Eq. (22) is the mean time until the first of the following two events occurs at the first site: either a gate opening, or a phase change. This term is the mean of the minimum of two independent exponential variables with parameters μ_{1k} and η_k, respectively. Now, the probability that the first event to occur is a phase transition is $\frac{\eta_k}{\mu_{1k}+\eta_k}$, upon which the particle leaves the system (disaster). On the other hand, if the first event of the two is a gate opening, which occurs with probability $\frac{\mu_{1k}}{\mu_{1k}+\eta_k}$, the batch of particles moves to the next site and the time until the next event is $\frac{1}{\mu_{2k}+\eta_k}$. This process continues sequentially until the last site where the particle leaves the system either after a successful service or following a disaster.

Using the law of total probability we have

$$\begin{aligned} E[T] &= E[T_1]P(O_1) + E[T_2]P(O_2) \\ &= \sum_{j=1}^{n} \frac{1}{\mu_{j1}} \prod_{i=1}^{j} \left(\frac{\mu_{i1}}{\mu_{i1}+\eta_1}\right) \frac{\eta_2}{\eta_1+\eta_2} + \sum_{j=1}^{n} \frac{1}{\mu_{j2}} \prod_{i=1}^{j} \left(\frac{\mu_{i2}}{\mu_{i2}+\eta_2}\right) \frac{\eta_1}{\eta_1+\eta_2}. \end{aligned} \tag{23}$$

Now, as expected, Little's law is verified when writing

$$E[L] = \lambda_1 E[T_1]P(O_1) + \lambda_2 E[T_2]P(O_2). \tag{24}$$

5.2 Overall load

At an arbitrary time t, $(t \geq 0)$, let $X_{(m)}(t)$ denote the total number of particles present in the first m sites. That is, $X_{(m)}(t) = \sum_{j=1}^{m} X_j(t)$. At steady state the mean overall load of the first m sites was derived in sect. 4 and is given as $E[L_m] \equiv \sum_{j=1}^{m} E[X_j]$. Here, in what follows we derive the PGF of the overall load of the first m sites. The Markovian dynamics of the stochastic process $[X_{(m)k}(t), t \geq 0]$ when the system is in phase k is given by

$$X_{(m)k}(t+\Delta t) = \begin{cases} X_{(m)k}(t) & \text{w.p.} \quad 1-\Delta t(\lambda_k + \mu_{mk} + \eta_k) \\ X_{(m)k}(t)+1 & \text{w.p.} \quad \Delta t \lambda_k \\ X_{(m-1)k}(t) & \text{w.p.} \quad \Delta t \mu_{mk} \\ 0 & \text{w.p.} \quad \Delta t \eta_k \end{cases} \tag{25}$$

where $X_{(0)}(t) := 0$.

In steady state, let

$$G_{X_{(m)k}}(z) = E[z^{X_{(m)k}}],$$

then,

$$\begin{aligned} G_{X_{(m)k}}(z) &= (1-\Delta t(\lambda_k + \mu_{mk} + \eta_k))E[z^{X_{(m)k}}] \\ &\quad + \Delta t \lambda_k E[z^{X_{(m)k}+1}] + \Delta t \mu_{mk} E[z^{X_{(m-1)k}}] + \Delta t \eta_k. \end{aligned} \tag{26}$$

Dividing by Δt and taking $\Delta t \to 0$, Eq. (26) becomes

$$(\lambda_k(1-z)+\mu_{mk}+\eta_k)G_{X_{(m)k}}(z) = \mu_{mk}G_{X_{(m-1)k}}(z)+\eta_k, \tag{27}$$

which yields,

$$G_{X_{(m)k}}(z) = \frac{\mu_{mk}}{\lambda_k(1-z)+\mu_{mk}+\eta_k}G_{X_{(m-1)k}}(z) + \frac{\eta_k}{\lambda_k(1-z)+\mu_{mk}+\eta_k}, \tag{28}$$

with initial condition

$$G_{X_{(1)k}}(z) = \frac{\mu_{1k}+\eta_k}{\lambda_k(1-z)+\mu_{1k}+\eta_k}, \tag{29}$$

which gives

$$G_{X_{(m)k}}(z) = \frac{\mu_{1k}+\eta_k}{\mu_{1k}}\prod_{j=1}^{m}\frac{\mu_{jk}}{\lambda_k(1-z)+\mu_{jk}+\eta_k} + \sum_{i=2}^{m}\frac{\eta_k}{\mu_{ik}}\prod_{j=i}^{m}\frac{\mu_{jk}}{\lambda_k(1-z)+\mu_{jk}+\eta_k}. \tag{30}$$

Using the law of total probability we get

$$\begin{aligned} G_{X_{(m)}}(z) =& P(O_1)G_{X_{(m)1}}(z) + P(O_2)G_{X_{(m)2}}(z) \\ =& \frac{\eta_2}{\eta_1+\eta_2}\Big[\frac{\mu_{11}+\eta_1}{\mu_{11}}\prod_{j=1}^{m}\frac{\mu_{j1}}{\lambda_1(1-z)+\mu_{j1}+\eta_1} \\ &+ \sum_{i=2}^{m}\frac{\eta_1}{\mu_{i1}}\prod_{j=i}^{m}\frac{\mu_{j1}}{\lambda_1(1-z)+\mu_{j1}+\eta_1}\Big] \\ &+ \frac{\eta_1}{\eta_1+\eta_2}\Big[\frac{\mu_{12}+\eta_2}{\mu_{12}}\prod_{j=1}^{m}\frac{\mu_{j2}}{\lambda_k(1-z)+\mu_{j2}+\eta_2} \\ &+ \sum_{i=2}^{m}\frac{\eta_2}{\mu_{i2}}\prod_{j=i}^{m}\frac{\mu_{j2}}{\lambda_k(1-z)+\mu_{j2}+\eta_2}\Big]. \end{aligned} \tag{31}$$

Note, however, that in contrast to the single-phase ASIP with no disasters, we do not get a product form result for the $G_{X(m)}(z)$. This follows since a disaster clears all particles from the system.

Again, when $\eta_1 \to 0$, Eq. (31) coincides with Eq. (49) in Reuveni et al. (2011), namely,

$$G_{X_{(m)}}(z) = \prod_{j=1}^{m}\frac{\mu_{j1}}{\lambda_1(1-z)+\mu_{j1}}. \tag{32}$$

5.2.1 Probability of an empty system

Substituting $z = 0$ in Eq. (31) yields the probability that the first *m* sites are empty,

$$\begin{aligned} P(L_m = 0) = & G_{X_{(m)}}(0) = \frac{\eta_2}{\eta_1+\eta_2}\Big[\frac{\mu_{11}+\eta_1}{\mu_{11}}\prod_{j=1}^{m}\frac{\mu_{j1}}{\lambda_1+\mu_{j1}+\eta_1} \\ & + \sum_{i=2}^{m}\frac{\eta_1}{\mu_{i1}}\prod_{j=i}^{m}\frac{\mu_{j1}}{\lambda_1+\mu_{j1}+\eta_1}\Big] \\ & + \frac{\eta_1}{\eta_1+\eta_2}\Big[\frac{\mu_{12}+\eta_2}{\mu_{12}}\prod_{j=1}^{m}\frac{\mu_{j2}}{\lambda_k+\mu_{j2}+\eta_2} \\ & + \sum_{i=2}^{m}\frac{\eta_2}{\mu_{i2}}\prod_{j=i}^{m}\frac{\mu_{j2}}{\lambda_k+\mu_{j2}+\eta_2}\Big]. \end{aligned} \tag{33}$$

Clearly, the probability that the entire system is empty, $P(L_n = 0)$, is obtained by substituting *n* instead of *m* in Eq. (33).

Once again, when substituting $\eta_1 \to 0$ into Eq. (33), and substituting $m = n$, the result coincides with Eq. (8) in Reuveni et al. (2012b) for the classical, single-phase, ASIP:

$$P(L_n = 0) = \prod_{j=i}^{n}\frac{\mu_{j1}}{\lambda_1+\mu_{j1}}. \tag{34}$$

5.3 Busy period

The system as a whole alternates between idle periods and busy periods. Let *B* denote the network's busy period, which is the duration of time elapsing from the instant a particle is admitted to an empty system until the first moment thereafter that the system becomes empty again. Let *IP* denote the length of an idle period. Since all idle periods are i.i.d and all busy periods are i.i.d as well, and idle and busy periods are also independent, then by renewal theory, we can write

$$P(L_n = 0) = \frac{E[IP]}{E[IP]+E[B]}. \tag{35}$$

Clearly, the proportion of time in which the system is idle is equal to the probability that the system is empty.

Eq. (35) is translated into

$$E[B] = \frac{E[IP](1-P(L_n=0))}{P(L_n=0)}. \tag{36}$$

To calculate $E[B]$, the mean duration of a busy period, we argue as follows: Let IP_k denote the length of an idle period starting when the system is in phase k, $(k = 1, 2)$. Then,

$$E[IP_1] = \frac{1}{\lambda_1 + \eta_1} + \frac{\eta_1}{\lambda_1 + \eta_1} E[IP_2], \tag{37}$$

and

$$E[IP_2] = \frac{1}{\lambda_2 + \eta_2} + \frac{\eta_2}{\lambda_2 + \eta_2} E[IP_1]. \tag{38}$$

Hence,

$$E[IP_1] = \frac{1}{\lambda_1 + \eta_1} + \frac{\eta_1}{\lambda_1 + \eta_1} \left(\frac{1}{\lambda_2 + \eta_2} + \frac{\eta_2}{\lambda_2 + \eta_2} E[IP_1] \right), \tag{39}$$

which yields

$$E[IP_1] = \frac{\lambda_2 + \eta_2 + \eta_1}{\lambda_1(\lambda_2 + \eta_2) + \lambda_2 \eta_1}, \quad E[IP_2] = \frac{\lambda_1 + \eta_1 + \eta_2}{\lambda_2(\lambda_1 + \eta_1) + \lambda_1 \eta_2}. \tag{40}$$

A busy period starts either when the system is in phase 1 or when it is in phase 2. The fraction of time the system is empty when in phase k, $P_k(L_n = 0)$, is obtained from Eq. (30), when setting $z = 0$ and $m = n$:

$$P_k(L_n = 0) = \frac{\mu_{1k} + \eta_k}{\mu_{1k}} \prod_{j=1}^{n} \frac{\mu_{jk}}{\lambda_k + \mu_{jk} + \eta_k} + \sum_{i=2}^{n} \frac{\eta_k}{\mu_{ik}} \prod_{j=i}^{n} \frac{\mu_{jk}}{\lambda_k + \mu_{jk} + \eta_k}. \tag{41}$$

Then, applying the law of total probability and utilizing Eq. (36), we get

$$E[B] = \sum_{k=1}^{2} \frac{E[IP_k](1 - P_k(L_n = 0))}{P_k(L_n = 0)} P(O_k). \tag{42}$$

Substituting Eqs. (40) and (41) in Eq. (42) yields

$$E[B] = \sum_{k=1}^{2} \frac{\lambda_k + \eta_1 + \eta_2}{\lambda_1\lambda_2 + \lambda_2\eta_1 + \lambda_1\eta_2} \frac{(1 - (\frac{\mu_{1k}+\eta_k}{\mu_{1k}} \prod_{j=1}^{n} \frac{\mu_{jk}}{\lambda_k+\mu_{jk}+\eta_k} + \sum_{i=2}^{n} \frac{\eta_k}{\mu_{ik}} \prod_{j=i}^{n} \frac{\mu_{jk}}{\lambda_k+\mu_{jk}+\eta_k}))}{\frac{\mu_{1k}+\eta_k}{\mu_{1k}} \prod_{j=1}^{n} \frac{\mu_{jk}}{\lambda_k+\mu_{jk}+\eta_k} + \sum_{i=2}^{n} \frac{\eta_k}{\mu_{ik}} \prod_{j=i}^{n} \frac{\mu_{jk}}{\lambda_k+\mu_{jk}+\eta_k}} \frac{\eta_{\bar{k}}}{\eta_1 + \eta_2}. \tag{43}$$

When $\eta_1 \to 0$,

$$E[B] = \frac{\frac{1}{\lambda_1}(1 - \prod_{j=1}^{n} \frac{\mu_{j1}}{\mu_{j1}+\lambda_1})}{\prod_{j=1}^{n} \frac{\mu_{j1}}{\mu_{j1}+\lambda_1}}, \tag{44}$$

which coincides with Eq. (10) in Reuveni et al. (2012b) (when $\lambda_1 = \lambda$, and $\mu_{j1} = \mu_j$).

5.4 First occupied site

Let I denote the index of the first occupied site such that $P(I = m)$ denotes the probability that the first occupied site in the system is site m. Clearly,

$$P(I = m) = P(X_1 = ... = X_{m-1} = 0) - P(X_1 = ... = X_m = 0). \tag{45}$$

Substituting Eq. (33) in Eq. (45) yields

$$\begin{aligned} P(I = m) =& P(L_{m-1} = 0) - P(L_m = 0) \\ =& \frac{\eta_2}{\eta_1 + \eta_2}[\frac{\mu_{11} + \eta_1}{\mu_{11}} \frac{\lambda_1 + \eta_1}{\lambda_1 + \mu_{m1} + \eta_1} \prod_{j=1}^{m-1} \frac{\mu_{j1}}{\lambda_1 + \mu_{j1} + \eta_1} - \frac{\eta_1}{\lambda_1 + \mu_{m1} + \eta_1}] \\ &+ \frac{\eta_1}{\eta_1 + \eta_2}[\frac{\mu_{12} + \eta_2}{\mu_{12}} \frac{\lambda_2 + \eta_2}{\lambda_2 + \mu_{m2} + \eta_2} \prod_{j=1}^{m-1} \frac{\mu_{j2}}{\lambda_2 + \mu_{j2} + \eta_2} - \frac{\eta_2}{\lambda_2 + \mu_{m2} + \eta_2}]. \end{aligned} \tag{46}$$

Taking the limit $\eta_1 \to 0$ in Eq. (46) yields the result obtained in Eq. (14) in Reuveni et al. (2012b).

$$P(I = m) = \frac{\lambda_1}{\mu_{m1}} \prod_{j=1}^{m} \frac{\mu_{j1}}{\mu_{j1} + \lambda_1}. \tag{47}$$

5.5 Draining time

Let D denote the draining time of the system. That is, the length of time from the moment the arrival process stops until the system is totally empty. If the first occu-

pied site is $I = m$, then the draining time equals the sojourn time from that site until exiting the system. Thus, the mean draining time is given by

$$E[D] = \sum_{m=1}^{n} E[D|I = m]P(I = m). \tag{48}$$

Using Eq. (46) for $P(I = m)$ and Eq. (23) applied to sites m through n yields

$$\begin{aligned} E[D] = & \sum_{m=1}^{n}\Big(\sum_{j=m}^{n}\frac{1}{\mu_{j1}}\prod_{i=1}^{j}\Big(\frac{\mu_{i1}}{\mu_{i1}+\eta_1}\Big)\frac{\eta_2}{\eta_1+\eta_2} + \sum_{j=m}^{n}\frac{1}{\mu_{j2}}\prod_{i=1}^{j}\Big(\frac{\mu_{i2}}{\mu_{i2}+\eta_2}\Big)\frac{\eta_1}{\eta_1+\eta_2}\Big) \\ & \times\Big[\frac{\eta_2}{\eta_1+\eta_2}\Big(\frac{\mu_{11}+\eta_1}{\mu_{11}}\frac{\lambda_1+\eta_1}{\lambda_1+\mu_{m1}+\eta_1}\prod_{j=1}^{m-1}\frac{\mu_{j1}}{\lambda_1+\mu_{j1}+\eta_1} - \frac{\eta_1}{\lambda_1+\mu_{m1}+\eta_1}\Big) \\ & + \frac{\eta_1}{\eta_1+\eta_2}\Big(\frac{\mu_{12}+\eta_2}{\mu_{12}}\frac{\lambda_2+\eta_2}{\lambda_2+\mu_{m2}+\eta_2}\prod_{j=1}^{m-1}\frac{\mu_{j2}}{\lambda_2+\mu_{j2}+\eta_2} - \frac{\eta_2}{\lambda_2+\mu_{m2}+\eta_2}\Big)\Big]. \end{aligned} \tag{49}$$

When $\eta_1 \to 0$, Eq. (49) reduces to

$$E[D] = \sum_{m=1}^{n}\Big(\sum_{j=m}^{n}\frac{1}{\mu_{j1}} \times \frac{\lambda_1}{\mu_{m1}}\prod_{j=1}^{m}\frac{\mu_{j1}}{\mu_{j1}+\lambda_1}\Big). \tag{50}$$

which coincides with Eq. (16) in Reuveni et al. (2012b).

6 Conclusion

In this work, we expand the investigation of the Asymmetric Inclusion Process (ASIP) system, analyzing it in an alternating random environment, with the addition that it is subject to disasters that cause the loss of all particles present in the system.

Our analysis yields several key findings for this combined ASIP system. Specifically, we study the multi-dimensional probability generating function (PGF) of site occupancies, and calculate the mean occupancy of each site in the network.

Additionally, we obtain the mean sojourn time of a particle within the system, the PGF and mean load of the first m ($m = 1, 2, \ldots, n$) sites, and the mean duration of a busy period. We also calculate the probability that the first occupied site is site m, and the mean length of the draining time, which is the interval starting at a moment the arrival process stops until the first moment thereafter where the system is completely empty. These results are compared with those obtained for the classical single-environment ASIP.

Future investigations may be directed into extending the research in several different directions: (i) ASIP models in random environment for $k > 2$ phases with synchronized disasters; (ii) G-ASIP systems in random environment for $k \geq 2$ phases;

(iii) ASIP models in random environment with disasters, which are not synchronized with phase changes.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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