Equilibrium Queueing Strategies in Random Environment

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Abstract. We study equilibrium queueing strategies in an M/M/1-type queueing system with strategic customers operating in a two-phase random environment described as a continuous-time Markov process. Strategic customers, upon arrival, choose either to join or to balk based on available information and anticipated utility, considering the trade-off between reward from service and waiting cost. Four observational scenarios are analyzed: fully observable (both queue length and environment phase are disclosed to a customer upon arrival), queue-only observable, environment-only observable, and fully unobservable. In each case, equilibrium joining strategies are analysed. In environment-only observable and fully unobservable cases, explicit solutions and equilibrium conditions are derived under rapid oscillations and very slow transitions between environment phases.

Keywords: Strategic Queueing · Equilibrium Strategies · Random Environment · Partial Observations

1 Introduction

We consider an M/M/1-type queue in a random environment modeled by a two-phase continuous-time Markov chain. Specifically, when the environment is in phase $i = 1, 2$ the arrival rate is λ_i, the service rate is μ_i, and the environment remains in that phase for an exponential time with parameter η_i, after which it jumps to the other phase. Thus, the rate matrix of the underlying environment process is

$$A = \begin{bmatrix} -\eta_1 & \eta_1 \\ \eta_2 & -\eta_2 \end{bmatrix}.$$

This queueing system has been extensively analysed (see e.g., [18, 16, 13, 2, 6] and follow up publications). The focus of our work is on the analysis of equilibrium strategies of individual customers. We assume that all the parameters of the model constitute common knowledge. We consider various cases of available information regarding the environment phase and the queue length for arriving customers. The following four cases will be investigated: (a) fully observable (FO) case, when both the environment phase and the queue length are know

upon arrival; (b) the case when only the queue length is observable (QO) upon arrival; (c) the case when only the environment phase is observable (EO) upon arrival and, finally, (d) no observation (NO) case when nether the environment phase nor the queue length are observable.

Inspired by the seminal article of Naor [12] (see also [19, 20, 5]), we would like to analyse equilibrium joining strategies in an evolving environment. Upon arrival, a customer can decide whether to join or to balk. If a customer balks, they leave the system forever. If a customer joins, they gain an eventual reward R for the service and accumulates waiting cost at a rate c_w per unit of time.

In [3] equilibrium joining strategies have been studied in a system with random environment when the server removes all present customers. In [4], strategic queueing has been studied for fluid queues in random environment. For background on strategic queueing we recommend the books [7, 8]. In [9] strategic queueing with arrival process with an i.i.d. random rate has been analysed.

2 FO case: fully observable system

We start our analysis with the fully observable (FO) case. Let $S_i(n)$ denote the sojourn time of a customer joining a queue of length n, when the environment is in phase $i = 1, 2$. Let $s_i(n)$ be its expected value. In particular, $S_i(0)$ is the customer's service time given the service starts in environment $i = 1, 2$. Using the memoryless property of the exponential distribution, we can write the following equation:

$$s_i(0) = \frac{1}{\mu_i + \eta_i} + \frac{\eta_i}{\mu_i + \eta_i} s_j(0), \quad i \neq j, \quad i, j \in \{1, 2\}. \tag{1}$$

That is, $1/(\mu_i + \eta_i)$ is the expected time until either the service is concluded or the environment changes phase, and then, with probability $\eta_i/(\mu_i + \eta_i)$ the environment switches to phase j, and the remaining expected service time is $s_j(0)$. Solving the system (1), we obtain

$$s_i(0) = \frac{\mu_j + \eta_1 + \eta_2}{\mu_1\mu_2 + \mu_1\eta_2 + \mu_2\eta_1}, \quad i \neq j, \quad i, j \in \{1, 2\}. \tag{2}$$

We also define the probability that a service initiated in environment i is completed in environment j:

$$q_{ij} = \mathbb{P}[\text{service completion occurs in environment } j \mid \text{service has started in } i].$$

Reasoning as for the derivation of (1) gives:

$$q_{ii} = \frac{\mu_i}{\mu_i + \eta_i} + \frac{\eta_i}{\mu_i + \eta_i} q_{ji}, \quad i \neq j, \quad i, j \in \{1, 2\},$$

The solution of the above linear system results in

$$q_{ii} = \frac{\mu_i(\mu_j + \eta_j)}{\mu_1\mu_2 + \mu_1\eta_2 + \mu_2\eta_1}, \quad i \neq j, \quad i, j \in \{1, 2\}, \tag{3}$$

and we have $q_{ij} = 1 - q_{ii}$ for $j \neq i$. The latter can be interpreted as follows:

$$q_{ii} = \sum_{k=0}^{\infty} \left(\frac{\eta_1 \eta_2}{(\mu_1 + \eta_1)(\mu_2 + \eta_2)} \right)^k \frac{\mu_i}{\mu_i + \eta_i}, \quad i = 1, 2,$$

that is, the underlying process performs k transitions back and forth, and then the service completion occurs before the phase transition.

When a customer arrives and sees n customers ahead, the first service (of the customer in service) will take an expected time $s_i(0)$ if the environment is in phase i. At the end of this service, the environment is in phase i with probability q_{ii} and in phase j with probability q_{ij} (where $j \neq i$). Hence, the recursion for the conditional expected sojourn time is:

$$s_i(n) = s_i(0) + q_{ii}\, s_i(n-1) + q_{ij}\, s_j(n-1), \quad i \neq j.$$

Define the vectors

$$s(n) = \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix}, \quad n = 0, 1, ...$$

and the matrix

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} q_{11} & 1 - q_{11} \\ 1 - q_{22} & q_{22} \end{bmatrix}.$$

Then, the recurrence becomes

$$s(n) = s(0) + Q\, s(n-1).$$

By iterating these relations, we obtain the closed-form expression

$$s(n) = \sum_{k=0}^{n} Q^k s(0). \tag{4}$$

We can also obtain another expression using the deviation matrix

$$H = (I - \underline{1}\pi)(I - Q + \underline{1}\pi)^{-1},$$

where π is the invariant probability measure of Q, i.e., $\pi = \pi Q$ with $\pi \underline{1} = 1$. Note that π is a row vector and $\underline{1}$ is a column vector of 1s. We can write

$$(I - Q) \sum_{k=0}^{n} Q^k = I - Q^{n+1}.$$

Premultiplying the above equality by matrix H and using the property of the deviation matrix $H(I - Q) = I - \underline{1}\pi$ (see e.g., the Appendix in [10]), we obtain

$$(I - \underline{1}\pi) \sum_{k=0}^{n} Q^k = H(I - Q^{n+1}).$$

Then, using $\pi = \pi Q$, yields

$$\sum_{k=0}^{n} Q^k = [(n+1)\underline{1}\pi + H(I - Q^{n+1})].$$

Postmultiplication of the above equation by $s(0)$, leads to an expression equivalent to (4)

$$s(n) = [(n+1)\underline{1}\pi + H(I - Q^{n+1})]s(0). \tag{5}$$

Thus, when an arriving customer sees the environment in phase i and n customers in the queue, the customer does not incur a loss and joins the queue if

$$R \geq c_w\, s_i(n), \quad i = \{1, 2\}. \tag{6}$$

We assume that for someone who already in front of the queue, it makes sense to join the queue even if the expected utility is zero.

3 QO case: only the queue length is observable

Now we consider the QO case when the queue length is observable upon arrival but not the phase of the environment. An arriving customer joins the queue if their expected utility is non-negative. In the QO case, the expected utility is given by

$$U_{L=n} = R - c_w \mathbb{E}^0[S|L = n],$$

where S is the sojourn time, L is the queue length observed upon arrival, and $\mathbb{E}^0[\cdot]$ denotes Palm expectation, i.e., the expected value given an arrival event. Then, conditioning on the environment phase E, we can write

$$U_{L=n} =$$

$$R - c_w(\mathbb{E}^0[S|E = 1, L = n]\mathbb{P}^0[E = 1|L = n] + \mathbb{E}^0[S|E = 2, L = n]\mathbb{P}^0[E = 2|L = n])$$

where $s_i(n)$ is given by (5) and the operations with zero superscript correspond to Palm versions. Specifically, the Palm probabilities are given by

$$\mathbb{P}^0[E = i, L = n] = \frac{\lambda_i p_{in}}{\lambda_1 p_{1\cdot} + \lambda_2 p_{2\cdot}}, \quad i = 1, 2, \quad n = 0, 1, ... \tag{7}$$

where

$$p_{1\cdot} = \frac{\eta_2}{\eta_1 + \eta_2}, \quad p_{2\cdot} = \frac{\eta_1}{\eta_1 + \eta_2}.$$

And hence

$$\mathbb{P}^0[E = i|L = n] = \frac{\lambda_i p_{in}}{\lambda_1 p_{1n} + \lambda_2 p_{2n}}, \quad i = 1, 2, \quad n = 0, 1, ...$$

Thus, the expected utility can be written as follows:

$$U_{L=n} = R - c_w \frac{1}{\lambda_1 p_{1n} + \lambda_2 p_{2n}}(\lambda_1 p_{1n} s_1(n) + \lambda_2 p_{2n} s_2(n)). \tag{8}$$

We have an explicit expression for $s_i(n)$, equation (5). However, it is unfortunate that there is no explicit expression for the stationary probability p_{in}. Therefore, as in [18], we consider two limiting cases: the case of rapid oscillations of the environment (Case C in [18]) and the case of very slow transitions between the environment phases (Case D in [18]). In these cases, the stationary distribution has an explicit product-form.

Based on [20] and [3], we conjecture in this study that equilibrium strategies in the QO case are of threshold type. We leave a thorough investigation of this conjecture as a future research.

3.1 QO case: rapid oscillations of the environment

Here, as in [18], we assume that the environment oscillates rapidly between two phases 1 and 2. Specifically, η_1 and η_2 tend to infinity simultaneously and their ratio η_1/η_2 tends to a finite constant C. In the limit, as the rates go to infinity, the stationary probabilities converge to

$$p_{in} = p_{i\cdot}\frac{1-\bar{\rho}}{1-\bar{\rho}^{\bar{n}+1}}\bar{\rho}^n, \quad n = 0, ..., \bar{n}, \tag{9}$$

where $\bar{n}$ is the equilibrium threshold and $\bar{\rho} = \bar{\lambda}/\bar{\mu}$ with

$$\bar{\lambda} = p_{1\cdot}.\lambda_1 + p_{2\cdot}.\lambda_2 \quad \text{and} \quad \bar{\mu} = p_{1\cdot}.\mu_1 + p_{2\cdot}.\mu_2,$$

$$p_{1\cdot} = \frac{1}{1+C} \quad \text{and} \quad p_{2\cdot} = \frac{C}{1+C}.$$

This product-form can be formally justified in the framework of singularly perturbed Markov chains [1].

Using (2), we conclude that

$$s_i(0) \to \frac{1+C}{\mu_1 + C\mu_2} = \frac{1}{\bar{\mu}}, \quad i = 1, 2,$$

as $\eta_1, \eta_2 \to \infty$. And hence,

$$s(n) = \sum_{k=0}^{n} Q^n s(0) = \sum_{k=0}^{n} Q^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\bar{\mu}} = \frac{(n+1)}{\bar{\mu}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{10}$$

Substituting (9) and (10) into (8), yields

$$U_{L=n} = R - c_w \frac{n+1}{\bar{\mu}},$$

which means that this case corresponds to the classical Naor's case with the averaged service rate $\bar{\mu}$.

3.2 QO case: very slow transitions between the environment phases

Now, let us consider the opposite situation when the transition rates between the phases of the environment are very slow. Technically, this means to let the rates η_1 and η_2 go to zero assuming that their ratio η_1/η_2 goes to a finite constant C.

In this case, as η_1 and η_2 go to zero, the stationary distribution converges to the product-form (the justification again follows from the theory of singularly perturbed Markov chains [1]):

$$p_{in} = p_{i\cdot}\frac{1-\rho_i}{1-\rho_i^{\bar{n}+1}}\rho_i^n, \quad n = 0, ..., \bar{n}, \tag{11}$$

where $\rho_i = \lambda_i/\mu_i$, $i = 1, 2$.

From the expression (2), we conclude that

$$s_i(0) \to \frac{1}{\mu_i}, \quad i = 1, 2,$$

as $\eta_1, \eta_2 \to 0$. And from (4), we conclude that $Q \to I$ (identity matrix), as $\eta_1, \eta_2 \to 0$. Hence,

$$s(n) = \sum_{k=0}^{n} Q^k s(0) = (n+1)\begin{bmatrix} 1/\mu_1 \\ 1/\mu_2 \end{bmatrix}. \tag{12}$$

Combining the above expression with (11) and (8), we obtain the following condition

$$U_{L=n} = R - c_w \frac{\lambda_1/\mu_1 p_{1\cdot}\tilde{p}_{1n} + \lambda_2/\mu_2 p_{2\cdot}\tilde{p}_{2n}}{\lambda_1 p_{1\cdot}\tilde{p}_{1n} + \lambda_2 p_{2\cdot}\tilde{p}_{2n}}(n+1) \geq 0.$$

where

$$\tilde{p}_{in} = \frac{1-\rho_i}{1-\rho_i^{\bar{n}+1}}\rho_i^n, \quad n = 0, ..., \bar{n}.$$

This condition can be solved numerically for a specific choice of the parameters. If $\mu_1 = \mu_2$, then we again retrieve the Naor's case. Another interesting particular case is when $\rho_1 = \rho_2$ and we have

$$U_{L=n} = R - c_w \frac{\lambda_1/\mu_1 p_{1\cdot}\tilde{p}_{1n} + \lambda_2/\mu_2 p_{2\cdot}\tilde{p}_{2n}}{\lambda_1 p_{1\cdot}\tilde{p}_{1n} + \lambda_2 p_{2\cdot}\tilde{p}_{2n}}(n+1) \geq 0,$$

which corresponds to the Naor's model with a modified mean service time, where the expectation is conditioned on arrival event.

4 EO case: only the environment is observable

Now, we consider the EO case when an arriving customer observes the current phase of the environment but does not see the queue length. Then, the equilibrium strategy of customers can be described by two probabilities a_1 and a_2. It is a randomized strategy [5, 7]. A customer takes the decision to join the queue

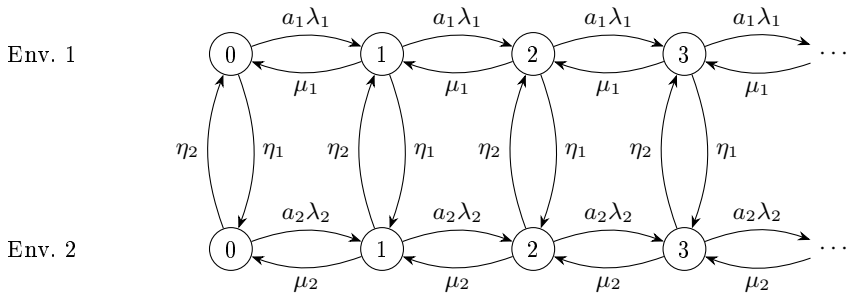


Fig. 1. Transition diagram when only the environment is observable.

with probability a_i, if they observe the environment in phase $i = 1, 2$. Under this strategy, the dynamics of the system is a random walk in random environment (for the state transitions see Fig. 1).

The expected utility can be written in the following form:

$$U_{E=i} = R - c_w \sum_{n=0}^{\infty} \mathbb{E}^0[S|L = n, E = i] P^0[L = n|E = i]$$

$$= R - c_w \sum_{n=0}^{\infty} s_i(n) \frac{p^0_{in}}{p^0_{i\cdot}} = R - c_w \sum_{n=0}^{\infty} s_i(n) \frac{p_{in}}{p_{i\cdot}}, \quad i = 1, 2, \tag{13}$$

We observe that in this case the factors with the arrival rates are cancelled. This is consistent with the literature [17, 11], where it was established that the PASTA property holds conditionally on the environment.

The equilibrium probabilities $a_i, i = 1, 2$, are solutions of the inequalities

$$\begin{cases} U_{E=1}(a_1, a_2) \geq 0, \\ U_{E=2}(a_1, a_2) \geq 0. \end{cases} \tag{14}$$

Again, we have an explicit expression for $s_i(n)$ but not for the stationary probabilities p_{in}. Therefore, as in the QO case, we consider two limiting cases: the case of rapid oscillations of the environment and the case of very slow transitions between the environment phases. In these cases, the stationary distribution has an explicit product-form.

4.1 EO case: rapid oscillations of the environment

Here we assume that the environment oscillates rapidly between two phases 1 and 2. Specifically, η_1 and η_2 tend to infinity simultaneously and their ratio η_1/η_2 tends to a finite constant C. In the limit, as the rates go to infinity, the stationary probabilities converge to

$$p_{in} = p_{i\cdot}(1 - \bar{\lambda}/\bar{\mu})(\bar{\lambda}/\bar{\mu})^n, \tag{15}$$

where

$$\bar{\lambda} = p_{1.}a_1\lambda_1 + p_{2.}a_2\lambda_2 \quad \text{and} \quad \bar{\mu} = p_{1.}\mu_1 + p_{2.}\mu_2,$$

$$p_{1.} = \frac{1}{1+C} \quad \text{and} \quad p_{2.} = \frac{C}{1+C}.$$

In addition to the justification in [18], this can also be formally justified in the framework of singularly perturbed Markov chains [1].

Using (10), we conclude that

$$U_{E=i}(a_1,a_2) = U_{E=1}(a_1,a_2) = R - \frac{c_w}{\bar{\mu}}\left(1-\frac{\bar{\lambda}}{\bar{\mu}}\right)\sum_{n=0}^{\infty}(n+1)\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^n = R - \frac{c_w}{\bar{\mu}-\bar{\lambda}}$$

and the utility is independent of the phase when a customer arrives. Intuitively, this follows from the fact that the environment oscillates very rapidly.

Define $\lambda_{max} = p_{1.}\lambda_1 + p_{2.}\lambda_2$. Similarly to Chapter 3 of [7], we can classify the equilibrium strategies into three cases:

1. If $\lambda_{max} \le \bar{\mu} - c_w/R$, then $a_1^* = a_2^* = 1$;
2. If $0 \le \bar{\mu} - c_w/R < \bar{\lambda}_{max}$, choose a_1^*, a_2^* such that $p_{1.}a_1\lambda_1 + p_{2.}a_2\lambda_2 = \bar{\mu} - c_w/R$;
3. If $\bar{\mu} - c_w/R < 0$ (i.e., $R < c_w/\bar{\mu}$), then $a_1^* = a_2^* = 0$.

We note that there is a freedom in the choice of a_1^* and a_2^* in case 2. One natural option can be to choose the same $a^* = (\bar{\mu} - c_w/R)/\bar{\lambda}_{max}$ for the two phases of the environment.

4.2 EO case: very slow transitions between the environment phases

Now, let us consider the opposite situation when the transition rates between the phases of the environment are very slow. Technically, this means to let the rates η_1 and η_2 go to zero assuming that their ratio η_1/η_2 goes to a finite constant C.

In this case, as η_1 and η_2 go to zero, the stationary distribution converges to the product-form [18]:

$$p_{in} = p_{i.}(1 - a_i\lambda_i/\mu_i)(a_i\lambda_i/\mu_i)^n, \quad i = 1,2. \tag{16}$$

Due to strategic customers, we can assume that $a_i\lambda_i < \mu_i$, $i = 1,2$, which implies that the system is stable in equilibrium. (The justification of the product form can be provided using the theory of singularly perturbed Markov chains [1].)

Combining (13) with (12) and (16), we obtain

$$U_{E=i} = R - \frac{c_w}{\mu_i - a_i\lambda_i}.$$

This leads to the following classification of the equilibrium strategies:

1. If $\lambda_i \le \mu_i - c_w/R$, $i = 1,2$, then $a_1^* = a_2^* = 1$;
2. If $\lambda_i \le \mu_i - c_w/R$ and $0 \le \mu_j - c_w/R < \lambda_j$, $j \ne i$, then $a_i^* = 1$ and $a_j^* = (\mu_j - c_w/R)/\lambda_j$;

3. If $0 \le \mu_i - c_w/R < \lambda_i$, $i = 1, 2$, then $a_i^* = (\mu_i - c_w/R)/\lambda_i$, $i = 1, 2$;
4. If $\mu_i - c_w/R < 0$ and $0 \le \mu_j - c_w/R < \lambda_j$, $j \ne i$, then $a_i^* = 0$ and $a_j^* = (\mu_j - c_w/R)/\lambda_j$;
5. If $\mu_i - c_w/R < 0$, $i = 1, 2$, then $a_1^* = a_2^* = 0$.

In fact, in the case of very slow transitions between the environment phases, we have a combination of two models from [5].

5 NO case: fully unobservable system

Finally, let us consider the NO case when no information is available to an arriving customer except the knowledge of the system parameters. In this case, the equilibrium strategy is again randomized [7], given by the joining probability a. Under this strategy, the dynamics of the system is again a random walk in random environment (for the state transitions see Fig. 2).

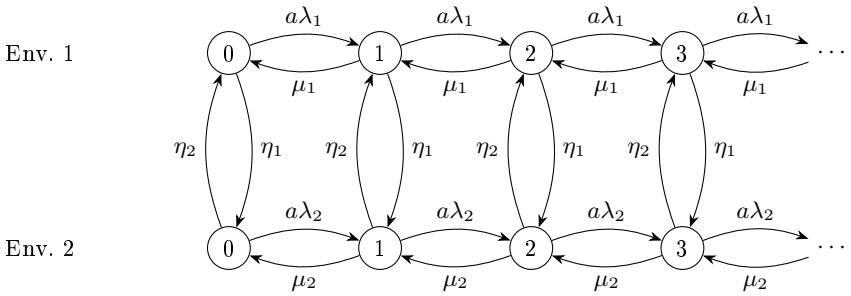


Fig. 2. Transition diagram for the fully unobservable case.

The expected utility is given by

$$U = R - c_w \sum_{n=0}^{\infty} \sum_{i=1,2} \mathbb{E}^0[S|E = i, L = n]\mathbb{P}^0[E = i, L = n], \tag{17}$$

where the Palm distribution is given by

$$\mathbb{P}^0[E = i, L = n] = p_{in}^0 = \frac{\lambda_i p_{in}}{\lambda_1 p_{1\cdot} + \lambda_2 p_{2\cdot}}. \tag{18}$$

Thus, we have

$$U = R - c_w \left(\sum_{n=0}^{\infty} s_1(n) \frac{\lambda_1 p_{1n}}{\lambda_1 p_{1\cdot} + \lambda_2 p_{2\cdot}} + \sum_{n=0}^{\infty} s_2(n) \frac{\lambda_2 p_{2n}}{\lambda_1 p_{1\cdot} + \lambda_2 p_{2\cdot}} \right). \tag{19}$$

Since in general we do not have an explicit expression for the stationary distribution, we proceed to the analysis of the two limiting cases.

5.1 NO case: rapid oscillations of the environment

As in the EO scenario, in the limit as the rates go to infinity, the stationary probabilities converge to [18]

$$p_{in} = p_{i\cdot}(1 - \bar{\lambda}/\bar{\mu})(\bar{\lambda}/\bar{\mu})^n, \tag{20}$$

where

$$\bar{\lambda} = a(p_{1\cdot}\lambda_1 + p_{2\cdot}\lambda_2) \quad \text{and} \quad \bar{\mu} = p_{1\cdot}\mu_1 + p_{2\cdot}\mu_2,$$
$$p_{1\cdot} = \frac{1}{1+C} \quad \text{and} \quad p_{2\cdot} = \frac{C}{1+C}.$$

Combining (19) with (20) and (10), we obtain

$$U = R - \frac{c_w}{\bar{\mu} - \bar{\lambda}}.$$

Hence this case is equivalent to the model [5] with the arrival and service rates averaged with respect to the rapidly oscillating environment. Consequently, we have the following classification of the equilibrium strategies:

1. If $\bar{\lambda}_{max} \le \bar{\mu} - c_w/R$, then $a^* = 1$;
2. If $0 \le \bar{\mu} - c_w/R < \bar{\lambda}_{max}$, then $a^* = (\bar{\mu} - c_w/R)/\bar{\lambda}_{max}$;
3. If $\bar{\mu} - c_w/R < 0$ (i.e., $R < c_w/\bar{\mu}$), then $a^* = 0$;

where $\bar{\lambda}_{max} = p_{1\cdot}\lambda_1 + p_{2\cdot}\lambda_2 = \frac{\lambda_1 + C\lambda_2}{1+C}$.

5.2 NO case: very slow transitions between the environment phases

Also, as in the EO case (Section 4.2), in the limit as the rates go to zero, the stationary probabilities converge to [18]

$$p_{in} = p_{i\cdot}(1 - a\lambda_i/\mu_i)(a\lambda_i/\mu_i)^n, \quad i = 1, 2. \tag{21}$$

Combining (19) with (21), (18) and (12), we can write

$$U = R - c_w\left(\frac{p^0_{1\cdot}}{\mu_1 - a\lambda_1} + \frac{p^0_{2\cdot}}{\mu_2 - a\lambda_2}\right) = R - c_w\left(\frac{p^0_{1\cdot}/\mu_1}{1 - a\rho_1} + \frac{p^0_{2\cdot}/\mu_2}{1 - a\rho_2}\right), \tag{22}$$

with $\rho_i = \lambda_i/\mu_i, i = 1, 2$. Without loss of generality, assume that $\rho_1 < \rho_2$. Then, it is easy to see that the function

$$f(a) = \left(\frac{p^0_{1\cdot}/\mu_1}{1 - a\rho_1} + \frac{p^0_{2\cdot}/\mu_2}{1 - a\rho_2}\right)$$

has two vertical asymptotes at $a = 1/\rho_2$ and $a = 1/\rho_1$ and, except these two points, it has positive derivative. We need to consider only the interval $[0, 1/\rho_2)$. Note also that

$$f(0) = \frac{p^0_{1\cdot}}{\mu_1} + \frac{p^0_{2\cdot}}{\mu_2}.$$

Thus, we have the following classification of the equilibrium strategies:

1. If $\frac{R}{c_w} \leq \frac{p^0_{1\cdot}}{\mu_1} + \frac{p^0_{2\cdot}}{\mu_2}$, then $a^* = 0$;
2. If $\rho_2 < 1$, there are two sub-cases:
   (a) If $\frac{R}{c_w} > \frac{p^0_{1\cdot}/\mu_1}{1-\rho_1} + \frac{p^0_{2\cdot}/\mu_2}{1-\rho_2}$, then $a^* = 1$;
   (b) If $\frac{R}{c_w} \leq \frac{p^0_{1\cdot}/\mu_1}{1-\rho_1} + \frac{p^0_{2\cdot}/\mu_2}{1-\rho_2}$, then

$$a^* = \frac{(\rho_1+\rho_2)R/c_w - \rho_2 p^0_{1\cdot}/\mu_1 - \rho_1 p^0_{2\cdot}/\mu_2 - \sqrt{D}}{2\rho_1\rho_2 R/c_w}, \tag{23}$$

   where

$$D = \left((\rho_1+\rho_2)\frac{R}{c_w} - \frac{\rho_2 p^0_{1\cdot}}{\mu_1} - \frac{\rho_1 p^0_{2\cdot}}{\mu_2}\right)^2 - 4\rho_1\rho_2\frac{R}{c_w}\left(\frac{R}{c_w} - \frac{p^0_{1\cdot}}{\mu_1} - \frac{p^0_{2\cdot}}{\mu_2}\right).$$

3. If $\rho_2 \geq 1$ and $\frac{R}{c_w} > \frac{p^0_{1\cdot}}{\mu_1} + \frac{p^0_{2\cdot}}{\mu_2}$, then a^* is given by (23).

It is interesting to observe that in comparison with the EO case with slow transitions between the environment phases, in the NO case we do not have a straightforward modification of the basic model in [5].

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References

1. Altman, E., Avrachenkov, K. E., and Núnez-Queija, R. (2004). Perturbation analysis for denumerable Markov chains with application to queueing models. *Advances in Applied Probability*, 36(3), 839-853.
2. Baccelli, F., and Makowski, A. M. (1986). Stability and bounds for single server queues in random environment. *Stochastic Models*, 2(2), 281-291.
3. Economou, A., and Manou, A. (2013). Equilibrium balking strategies for a clearing queueing system in alternating environment. *Annals of Operations Research*, 208, 489-514.
4. Economou, A., and Manou, A. (2016). Strategic behavior in an observable fluid queue with an alternating service process. *European Journal of Operational Research*, 254(1), 148-160.
5. Edelson, N. M., and Hilderbrand, D. K. (1975). Congestion tolls for Poisson queuing processes. *Econometrica: Journal of the Econometric Society*, 81-92.
6. Gupta, V., Harchol-Balter, M., Wolf, A. S., and Yechiali, U. (2006). Fundamental characteristics of queues with fluctuating load. In Proceedings of ACM Sigmetrics. 203-215.
7. Hassin, R., and Haviv, M. (2003). *To queue or not to queue: Equilibrium behavior in queueing systems*. Springer.
8. Hassin, R. (2016). *Rational Queueing*. CRC Press.

9. Hassin, R., Haviv, M., and Oz, B. (2023). Strategic behavior in queues with arrival rate uncertainty. *European Journal of Operational Research*, 309(1), 217-224.
10. Kemeny, J. G., and Snell, J. L. (1976). *Finite Markov chains*. 2nd edition. Springer.
11. Melamed, B., and Whitt, W. (1990). On arrivals that see time averages: A martingale approach. *Journal of Applied Probability*, 27(2), 376-384.
12. Naor, P. (1969). The regulation of queue size by levying tolls. *Econometrica*, 37(1), 15-24.
13. Neuts, M. F. (1978). The M/M/1 queue with randomly varying arrival and service rates. *Opsearch*, 15, 139-157.
14. Paz, N., and Yechiali, U. (2007). A Note on the $M/M/\infty$ Queue in Random Environment. *Technical report, Department of Statistics and Operations Research, Tel Aviv University* `https://www.math.tau.ac.il/ uriy/Publications.html`
15. Paz, N., and Yechiali, U. (2014). An M/M/1 queue in random environment with disasters. *Asia-Pacific Journal of Operational Research*, 31(03), 1450016.
16. Purdue, P. (1974). The M/M/1 queue in a Markovian environment. *Operations Research*, 22(3), 562-569.
17. Van Doorn, E.A., and Regterschot, G.J.K. (1988). Conditional PASTA. *Operations Research Letters*, 7(5), 229-232.
18. Yechiali, U., and Naor, P. (1971). Queuing problems with heterogeneous arrivals and service. *Operations Research*, 19(3), 722-734.
19. Yechiali, U. (1971). On optimal balking rules and toll charges in the GI/M/1 queuing process. *Operations Research*, 19(2), 349-370.
20. Yechiali, U. (1972). Customers' optimal joining rules for the GI/M/s queue. *Management Science*, 18(7), 434-443.